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ON NUMBER OF OCCURRENCES OF SUCCESS RUNS OF SPECIFIED LENGTH IN A HIGHER-ORDER TWO-STATE MARKOV CHAIN

MASAYUKI UCHIDA

The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan

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Abstract. Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots, X_n$ be a time-homogeneous $\{0,1\}$-valued m-th order Markov chain. The probability distributions of numbers of runs of "1" of length k ($k \geq m$) and of "1" of length k ($k < m$) in the sequence of a $\{0,1\}$-valued m-th order Markov chain are studied. There are some ways of counting numbers of runs with length k. This paper studies the distributions based on four ways of counting numbers of runs, i.e., the number of non-overlapping runs of length k, the number of runs with length greater than or equal to k, the number of overlapping runs of length k and the number of runs of length exactly k.

Key words and phrases: Probability generating function, discrete distribution, binomial distribution, binomial distribution of order k, higher order Markov chain.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of $\{0,1\}$-valued random variables. We often call X_n the n-th trial and we say S (success) and F (failure) for the outcomes "1" and "0", respectively. Let $n > k \geq 1$ be two fixed positive integer. For a given sequence of n letters S and F, there are many ways of counting the number of runs of S of length k. Four of the best-known type are:

I. a run of S of length k means a string of exact length k, where recounting starts immediately after a run occurs;

II. a run of S of length k means a string of S of length k or more;

III. a run of S of length k means a string of S of exact length k, allowing overlapping runs;

IV. a run of S of length k means a string of S of exact length k followed by an F.

For example, consider a realization of a sequence of S and F such as

$$SSSSFSSSFFFSSSSSSFFFFSS.$$

In this sequence $n = 23$. If we take $k = 3$, then there is only one run of three S of Type IV; there are four runs of three S of Type I, three runs of three S of Type II and seven runs of three S of Type III (see Uchida (1996*b*)).

The above distributions are called the binomial distributions of order k which are the distributions of the number of occurrences of consecutive k successes until the n-th trial. The system called a consecutive-k-out-of-n:F system is an interesting example of succession events in a sequence of $\{0,1\}$-valued random variables (cf. Aki and Hirano (1996), Chao *et al.* (1995), Fu (1996), Fu and Koutras (1994), Hirano (1994), Koutras and Alexandrou (1995) and references therein).

Hirano (1986) and Philippou and Makri (1986) obtained the distribution of the number of success-runs of Type I until the n-th trial, i.e., the binomial distribution of order k. Goldstein (1990) studied the Poisson approximation for the distribution of the number of success-runs of Type II until the n-th trial. Ling (1988, 1989) derived the distribution related to number of success-runs in independent trials in the way of counting of Type III.

Aki and Hirano (1993) studied the exact distribution of the number of the success-runs of Type I until the n-th trial in a $\{0,1\}$-valued Markov chain. Hirano and Aki (1993) obtained the exact distributions of numbers of success-runs of Type II and III until the n-th trial in a $\{0,1\}$-valued Markov chain. Uchida (1996*a*) obtained the exact distribution of the number of success-runs of Type IV in the sense of Mood's (1940) counting, i.e., success-runs of exact length k until the n-th trial in a $\{0,1\}$-valued Markov chain.

In this paper, we investigate the distributions of number of outcomes such as successes and numbers of success-runs with length k of Type I, II, III and IV among $X_1, X_2, \ldots, X_n$ in the following higher order two-state Markov chain.

Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots, X_n$ be a time-homogeneous $\{0,1\}$-valued m-th order Markov chain with

$$\begin{aligned}
\pi_{x_1,\ldots,x_m} &= P(X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m),\\
p_{x_1,\ldots,x_m} &= P(X_i = 1 \mid X_{i-m} = x_1, X_{i-m+1} = x_2, \ldots, X_{i-1} = x_m),\\
&= 1 - q_{x_1,\ldots,x_m},
\end{aligned}$$

for $x_1, \ldots, x_m = 0, 1$ and $= 1, 2, \ldots, n$. For $x_1, \ldots, x_m = 0, 1$, we assume that $0 < p_{x_1,\ldots,x_m} < 1$.

In Section 2, we study the distribution of the number of successes until the n-th trial. In Section 3, we consider the distributions of the numbers of success-runs of length k $(m \le k)$ of Type I, II, III and IV until the n-th trial, i.e., the number of non-overlapping success-runs with length k, the number of success-runs with length greater than or equal to k, the number of overlapping success-runs with length k and the number of success-runs with length exactly k. We also investigate the distributions of the numbers of success-runs with length k $(k < m)$ until the n-th trial.

Throughout the paper, we define that for $\alpha > \beta$

$$\prod_{i=\alpha}^{\beta} g(i) = 1 \quad \text{and} \quad \sum_{i=\alpha}^{\beta} g(i) = 0,$$

where $g(i)$ is a function.

The results in this paper are not only general and new but also available to numerical and symbolic calculations by using a computer algebra system, for example, the REDUCE system ver. 3.5 (Hearn (1993)). In particular, when k, n and m are given, we can obtain the generating functions (g.f.'s) of the probability generating functions (p.g.f.'s) of the distributions of the numbers of successes and the numbers of success-runs until the n-th trial, expectations and variances of the corresponding distributions by using the computer algebra system (cf. Hirano *et al.* (1997), Uchida (1994, 1996*b*) and Uchida and Aki (1995)).

2. Numbers of successes in a higher-order Markov chain

In this section, we consider the distribution of the number of successes until the n-th trial in the m-th order Markov chain. We give a method for deriving the g.f. of the p.g.f. of the conditional distribution of the number of successes until the n-th trial in the m-th order Markov chain.

A sequence which follows the m-th order Markov chain depends on the past occurrences of length m. A set of $\{0,1\}$-sequence of length m consists of 2^m elements, and can be uniquely regarded as a binary number. Further we translate it into a decimal number. For example, when $m = 3$, $p_{100} = p_4$ and when $m = 4$, $p_{1001} = p_9$ (see Hirano *et al.* (1997) and Uchida (1996*b*)).

Let $N_m = \{0, 1, 2, \dots, 2^m - 1\}$ and let f_i $(i = 0, 1)$ be the mapping from N_m to N_m such that

$$f_i(x) = 2x + i \pmod{2^m}, \qquad \text{for} \quad i = 0, 1.$$

Let $m \le k \le n$. Let ε_1 be the number of occurrences of "1" among $X_1, X_2, \dots, X_n$. We denote by $\phi_n^{(x)}(s)$ (for each $x \in N_m$) the p.g.f. of the conditional distribution of ε_1 given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_m$.

For $i = 0, 1, \dots, k-1$, let A_i be the event that we start with a "1"-run of length i and "0" occurs just after the "1"-run. Let C be the event that we start with a "1"-run of length k. For $x \in N_m$, let $\phi_n^{(x)}(s \mid A_i)$ and $\phi_n^{(x)}(s \mid C)$ be the p.g.f.'s of the conditional distributions of ε_1 given that the event $A_i \cap \{X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_0\}$ occurs and given that the event $C \cap \{X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_0\}$ occurs, respectively. Since A_i, $i = 0, 1, \dots, k-1$ and C construct a partition of the sample space, we have the following system of 2^m equations of conditional p.g.f.'s.

For each $x \in N_m$,

$$\begin{aligned}
\phi_0^{(x)}(s) &= 1 \\
\phi_n^{(x)}(s) &= q_x \phi_{n-1}^{(f_0(x))}(s) \\
&\quad + p_x s \sum_{i=1}^{n-1} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} s) \right] q_{f_1^i(x)} \phi_{n-i-1}^{(f_0 \circ f_1^i(x))}(s) \\
&\quad + p_x s \left[\prod_{j-1}^{n-1} (p_{f_1^j(x)} s) \right] \qquad \text{if} \quad 1 \le n < k.
\end{aligned}$$

For each $x \in N_m$,

$$\begin{aligned}\phi_n^{(x)}(s) &= \sum_{i=0}^{k-1} P(A_i)\phi_n^{(x)}(s \mid A_i) + P(C)\phi_n^{(x)}(s \mid C)\\ &= q_x \phi_{n-1}^{(f_0(x))}(s)\\ &\quad + p_x s \sum_{i=1}^{m-2}\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)}s)\right] q_{f_1^i(x)}\phi_{n-i-1}^{(f_0\circ f_1^i(x))}(s)\\ &\quad + p_x s\left[\prod_{j=1}^{m-2}(p_{f_1^j(x)}s)\right] q_{f_1^{m-1}(x)}\phi_{n-m}^{(2^m-2)}(s)\\ &\quad + p_x s\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}s)\right]\sum_{i=m}^{k-1}(p_{2^m-1}s)^{i-m}q_{2^m-1}\phi_{n-i-1}^{(2^m-2)}(s)\\ &\quad + p_x s\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}s)\right](p_{2^m-1}s)^{k-m}\phi_{n-k}^{(2^m-1)}(s) \qquad \text{if} \quad n \geq k,\end{aligned}$$

where we define that $\prod_{j=1}^{0}(p_{f_1^j(x)}s) = 1$.

Define

$$\Phi^{(x)}(s,z) \equiv \sum_{n=0}^{\infty}\phi_n^{(x)}(s)z^n.$$

For each $x \in N_m$,

$$\begin{aligned}\Phi^{(x)}(s,z) &= q_x z\Phi^{(f_0(x))}(s,z) + \sum_{i=1}^{m-2}\alpha_{x,i}(s)\Phi^{(f_0\circ f_1^i(x))}(s,z)\\ &\quad + \beta_x(s)\Phi^{(2^m-2)}(s,z) + \gamma_x(s)\Phi^{(2^m-1)}(s,z) + \delta_x(s),\end{aligned}$$

where for each $x, y \in N_m$ and $i \in L_m = \{1, 2, \ldots, m-2\}$,

$$\alpha_{x,i}(s) = p_x s\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)}s)\right] q_{f_1^i(x)}z^{i+1},$$

$$\begin{aligned}\beta_x(s) &= p_x s\left[\prod_{j=1}^{m-2}(p_{f_1^j(x)}s)\right] q_{f_1^{m-1}(x)}z^m\\ &\quad + p_x s\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}s)\right]\sum_{i=m}^{k-1}(p_{2^m-1}s)^{i-m}q_{2^m-1}z^{i+1},\end{aligned}$$

$$\gamma_x(s) = p_x s\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}s)\right](p_{2^m-1}s)^{k-m}z^k,$$

$$
\begin{aligned}
\delta_x(s) = & \sum_{n=0}^{k-1} \phi_n^{(x)}(s) z^n - q_x z \sum_{n=0}^{k-2} \phi_n^{(f_0(x))}(s) z^n \\
& - p_x s \sum_{i=1}^{m-2} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} s) \right] q_{f_1^i(x)} z^{i+1} \sum_{n=0}^{k-i-2} \phi_n^{(f_0 \circ f_1^i(x))}(s) z^n \\
& - p_x s \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} s) \right] q_{f_1^{m-1}(x)} z^m \sum_{n=0}^{k-m-1} \phi_n^{(2^m-2)}(s) z^n \\
& - p_x s \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} s) \right] \sum_{i=m}^{k-1} (p_{2^m-1} s)^{i-m} q_{2^m-1} z^{i+1} \sum_{n=0}^{k-i-2} \phi_n^{(2^m-2)}(s) z^n.
\end{aligned}
$$

For each $x, y \in N_m$,

$$
\begin{aligned}
a_{x,y}(s) &= q_x z 1\{f_0(x) = y\} + \sum_{i \in I_{x,y}} \alpha_{x,i}(s) \\
&\quad + \beta_x(s) 1\{y = 2^m - 2\} + \gamma_x(s) 1\{y = 2^m - 1\}, \\
b_{x,y}(s) &= 1\{x = y\} - a_{x,y}(s),
\end{aligned}
$$

where

$$
1\{x = y\} = \begin{cases} 1, & \text{if} \quad x = y, \\ 0, & \text{if} \quad x \neq y, \end{cases}
$$

$$
I_{x,y} = \{i \in L_m \mid f_0 \circ f_1^i(x) = y\}.
$$

Define

$$
\begin{aligned}
B &= (b_{x,y}(s))_{x,y \in N_m}, \\
B^{-1} &= (c_{x,y})_{x,y \in N_m}, \\
c_{x,y} &= \frac{\Delta_{y,x}}{|B|},
\end{aligned}
$$

where $\Delta_{x,y}$ is the (x, y)-cofactor of the matrix B.

Then, we have

$$
\Phi^{(x)}(s, z) = \sum_{i=0}^{2^m-1} \frac{\Delta_{i,x}}{|B|} \gamma_i(s) = \frac{1}{|B|} \sum_{i=0}^{2^m-1} \Delta_{i,x} \gamma_i(s).
$$

Here, we set that for each $x \in N_m$,

$$
\begin{aligned}
\Psi(s, z) &= |B|, \\
\Psi^{(x)}(s, z) &= \sum_{i=0}^{2^m-1} \Delta_{i,x} \gamma_i(s).
\end{aligned}
$$

We note that

$$\Psi^{(x)}(s,z) = \begin{vmatrix} b_{00}(s) & \cdots & b_{0,x-1}(s) & \delta_0(s) & b_{0,x+1}(s) & \cdots & b_{0,2^m-1}(s) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{2^m-1,0}(s) & \cdots & b_{2^m-1,x-1}(s) & \delta_{2^m-1}(s) & b_{2^m-1,x-1}(s) & \cdots & b_{2^m-1,2^m-1}(s) \end{vmatrix}$$

and

$$b_{x,2^m-1}(s) = \Xi_x - \Gamma_x(s)(p_{2^m-1}sz)^{k-m},$$

where

$$\Gamma_x(s) = p_x s \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} s)\right] z^m,$$

$$\Xi_x = 1\{x = 2^m - 1\} - q_x z 1\{f_0(x) = 2^m - 1\} - \sum_{i \in I_{x,2^m-1}} \alpha_{x,i}(s)$$

$$= 1\{x = 2^m - 1\}.$$

When $x = 2^m - 1$, we have

$$\Psi^{(2^m-1)}(s,z) = \begin{vmatrix} b_{00}(s) & \cdots & b_{0,2^m-3}(s) & b_{0,2^m-2}(s) & \delta_0(s) \\ \vdots & & \vdots & \vdots & \vdots \\ b_{2^m-1,0}(s) & \cdots & b_{2^m-1,2^m-3}(s) & b_{2^m-1,2^m-2}(s) & \delta_{2^m-1}(s) \end{vmatrix}.$$

On the other hand, when $x \neq 2^m - 1$,

$$\Psi^{(x)}(s,z) = \Lambda_1^{(x)}(s,z) + \Lambda_2^{(x)}(s,z),$$

where

$$\Lambda_1^{(x)}(s,z) = \begin{vmatrix} b_{00}(s) & \cdots & \delta_0(s) & \cdots & b_{0,2^m-2}(s) & \Xi_0 \\ \vdots & & \vdots & & \vdots & \vdots \\ b_{2^m-1,0}(s) & \cdots & \delta_{2^m-1}(s) & \cdots & b_{2^m-1,2^m-2}(s) & \Xi_{2^m-1} \end{vmatrix},$$

$$\Lambda_2^{(x)}(s,z) = -(p_{2^m-1}sz)^{k-m} \cdot \begin{vmatrix} b_{00}(s) & \cdots & \delta_0(s) & \cdots & b_{0,2^m-2}(s) & \Gamma_0(s) \\ \vdots & & \vdots & & \vdots & \vdots \\ b_{2^m-1,0}(s) & \cdots & \delta_{2^m-1}(s) & \cdots & b_{2^m-1,2^m-2}(s) & \Gamma_{2^m-1}(s) \end{vmatrix}.$$

Then, we obtain

PROPOSITION 2.1.

$$\Phi^{(x)}(s,z) = \frac{\Lambda_1^{(x)}(s,z) + \Lambda_2^{(x)}(s,z)}{\Psi(s,z)}, \qquad \textit{for} \quad x \in N_m \backslash \{2^m - 1\},$$

$$\Phi^{(2^m-1)}(s,z) = \frac{\Psi^{(2^m-1)}(s,z)}{\Psi(s,z)}.$$

3. Numbers of success-runs in a higher-order Markov chain

In this section, we investigate the distributions of the numbers of success-runs with length k of Type I, II, III and IV until the n-th trial in the m-th order Markov chain.

3.1 *Case* $m \leq k$

In this subsection, we consider the distributions of the numbers of success-runs with length k ($m \leq k$) of Type I, II, III and IV until the n-th trial. Let μ, ν, ξ and η be the numbers of "1"-runs of length k of Type I, II, III and IV until the n-th trial, respectively. For $x \in N_m$, we denote by $\phi_{n1}^{(x)}(t)$, $\phi_{n2}^{(x)}(t)$, $\phi_{n3}^{(x)}(t)$ and $\phi_{n4}^{(x)}(t)$ the p.g.f.'s of the conditional distributions of μ, ν, ζ and η given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m$, respectively. Define

$$\Phi_j^{(x)}(t,z) \equiv \sum_{n=0}^{\infty} \phi_{nj}^{(x)}(t) z^n.$$

3.1.1 *Case Type* I

Let A_i, C and I_{xy} be as in the proof of Proposition 2.1. For each $x \in N_m$, we denote by $\Phi_1^{(x)}(t,z \mid A_i)$, $\Phi_1^{(x)}(t,z \mid C)$ the respective g.f.'s of conditional p.g.f.'s. Since A_i, $i = 0, 1, \ldots, k-1$ and C construct a partition of the sample space, we have the following system of 2^m equations of the g.f.'s of conditional p.g.f.'s.

For each $x \in N_m$,

$$\begin{aligned}\Phi_1^{(x)}(t,z) &= \sum_{i=0}^{k-1} P(A_i)\Phi_1^{(x)}(t,z \mid A_i) + P(C)\Phi_1^{(x)}(t,z \mid C) \\ &= q_x z \Phi_1^{(f_0(x))}(t,z) + \sum_{i=1}^{m-2} \alpha_{x,i}(1)\Phi_1^{(f_0 \circ f_1^i(x))}(t,z) \\ &\quad + \beta_x(1)\Phi_1^{(2^m-2)}(t,z) + \gamma_x(1)t\Phi_1^{(2^m-1)}(t,z) + \delta_x(1).\end{aligned}$$

For each $x, y \in N_m$,

$$\begin{aligned} a'_{x,y}(t) &= q_x z 1\{f_0(x) - y\} + \sum_{i \in I_{x,y}} \alpha_{x,i}(1) \\ &\quad + \beta_x(1)1\{y = 2^m - 2\} + \gamma_x(1)t1\{y = 2^m - 1\}, \\ b'_{x,y}(t) &= 1\{x = y\} - a'_{x,y}(t).\end{aligned}$$

Define

$$\begin{aligned} B' &= (b'_{x,y}(t))_{x,y \in N_m}, \\ (B')^{-1} &- (c'_{x,y})_{x,y \in N_m}, \\ c'_{x,y} &= \frac{\Delta'_{y,x}}{|B'|},\end{aligned}$$

where $\Delta'_{x,y}$ is the (x,y)-cofactor of the matrix B'.

From the above system of equations, we have

$$\Phi_1^{(x)}(t,z) = \sum_{i=0}^{2^m-1} \frac{\Delta'_{i,x}}{|B'|}\delta_i(1) = \frac{1}{|B'|}\sum_{i=0}^{2^m-1} \Delta'_{i,x}\delta_i(1).$$

Here, we set that

$$\Psi_1(t,z) = |B'|,$$
$$\Psi_1^{(x)}(t,z) = \sum_{i=0}^{2^m-1} \Delta'_{i,x}\delta_i(1),$$

for each $x \in N_m$.

We note that

$$b'_{x,2^m-1}(t) = \Xi_x - \Gamma_x(1)(p_{2^m-1}z)^{k-m}t$$

and when $y \neq 2^m - 1$,

$$b'_{x,y}(t) = b_{x,y}(1).$$

When $x = 2^m - 1$, we have

$$\Psi_1^{(2^m-1)}(t,z) = \begin{vmatrix} b_{00}(1) & \cdots & b_{0,2^m-3}(1) & b_{0,2^m-2}(1) & \delta_0(1) \\ \vdots & & \vdots & \vdots & \vdots \\ b_{2^m-1,0}(1) & \cdots & b_{2^m-1,2^m-3}(1) & b_{2^m-1,2^m-2}(1) & \delta_{2^m-1}(1) \end{vmatrix}$$
$$= \Psi^{(2^m-1)}(1,z).$$

On the other hand, when $x \neq 2^m - 1$,

$$\Psi_1^{(x)}(t,z) = \begin{vmatrix} b_{00}(1) & \cdots & \delta_0(1) & \cdots & b_{0,2^m-2}(1) & \Xi_0 \\ \vdots & & \vdots & & \vdots & \vdots \\ b_{2^m-1,0}(1) & \cdots & \delta_{2^m-1}(1) & \cdots & b_{2^m-1,2^m-2}(1) & \Xi_{2^m-1} \end{vmatrix}$$
$$- (p_{2^m-1}z)^{k-m}t$$
$$\cdot \begin{vmatrix} b_{00}(1) & \cdots & \delta_0(1) & \cdots & b_{0,2^m-2}(1) & \Gamma_0(1) \\ \vdots & & \vdots & & \vdots & \vdots \\ b_{2^m-1,0}(1) & \cdots & \delta_{2^m-1}(1) & \cdots & b_{2^m-1,2^m-2}(1) & \Gamma_{2^m-1}(1) \end{vmatrix}$$
$$= \Lambda_1^{(x)}(1,z) + \Lambda_2^{(x)}(1,z)t.$$

Then, we have

THEOREM 3.1.

$$\Phi_1^{(x)}(t,z) = \frac{\Lambda_1^{(x)}(1,z) + \Lambda_2^{(x)}(1,z)t}{\Psi_1(t,z)}, \qquad \textit{for} \quad x \in N_m \backslash \{2^m - 1\},$$
$$\Phi_1^{(2^m-1)}(t,z) = \frac{\Psi^{(2^m-1)}(1,z)}{\Psi_1(t,z)}.$$

3.1.2 *Case Type* II, III, IV

For $i = 0, 1, \ldots, n-1$, let A_i be the event that we start with a "1"-run of length i and "0" occurs just after the "1"-run. Let C be the event that we start with a "1"-run of length n. For each $x \in N_m$ and $j = 2, 3, 4$, we denote by $\Phi_j^{(x)}(t, z \mid A_i)$, $\Phi_j^{(x)}(t, z \mid C)$ the respective g.f.'s of conditional p.g.f.'s. Since A_i, $i = 0, 1, \ldots, n-1$ and C construct a partition of the sample space, we have the following system of 2^m equations of the g.f.'s of conditional p.g.f.'s.

For each $x \in N_m$,

$$\begin{aligned}\Phi_j^{(x)}(t,z) &= \sum_{i=0}^{n-1} P(A_i)\Phi_j^{(x)}(t,z \mid A_i) + P(C)\Phi_j^{(x)}(t,z \mid C)\\ &= q_x z \Phi_j^{(f_0(x))}(t,z) + \sum_{i=1}^{m-2} \alpha''_{x,i}\Phi_j^{(f_0 \circ f_1^i(x))}(t,z)\\ &\quad + \beta''_x \Phi_j^{(2^m-2)}(t,z) + \gamma''_x,\end{aligned}$$

where for each $x \in N_m$ and $i \in L_m$,

$$\alpha''_{x,i} = p_x \left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1},$$

$$\begin{aligned}\beta''_x &= p_x \left[\prod_{j=1}^{m-2}(p_{f_1^j(x)})\right] q_{f_1^{m-1}(x)} z^m\\ &\quad + p_x \left[\prod_{j=1}^{m-1}(p_{f_1^j(x)})\right] \sum_{i=m}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} z^{i+1}\\ &\quad + p_x \left[\prod_{j=1}^{m-1}(p_{f_1^j(x)})\right] \sum_{i=k}^{\infty} (p_{2^m-1})^{i-m} q_{2^m-1} z^{i+1}\varphi_{ij}\end{aligned}$$

$$\begin{aligned}\gamma''_x &= p_x \left[\prod_{j=1}^{m-1}(p_{f_1^j(x)})\right] z^m \sum_{n=k}^{\infty} (p_{2^m-1})^{n-m} z^{n-m}\varphi_{nj}\\ &\quad + \sum_{n=0}^{k-1} z^n - q_x z \sum_{n=0}^{k-2} z^n\\ &\quad - p_x \sum_{i=1}^{m-2}\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1} \sum_{n=0}^{k-i-2} z^n\\ &\quad - p_x \left[\prod_{j=1}^{m-2}(p_{f_1^j(x)})\right] q_{f_1^{m-1}(x)} z^m \sum_{n=0}^{k-m-1} z^n\\ &\quad - p_x \left[\prod_{j=1}^{m-1}(p_{f_1^j(x)})\right] \sum_{i=m}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} z^{i+1} \sum_{n=0}^{k-i-2} z^n,\end{aligned}$$

$$\varphi_{ij} = \begin{cases} t, & \text{if} \quad j = 2, \\ t^{i-k+1}, & \text{if} \quad j = 3, \\ t1\{i = k\} + 1\{i \neq k\}, & \text{if} \quad j = 4. \end{cases}$$

For each $x, y \in N_m$,

$$a''_{x,y}(t) = q_x z 1\{f_0(x) = y\} + \sum_{i \in I_{x,y}} \alpha''_{x,i} + \beta''_x 1\{y = 2^m - 2\},$$
$$b''_{x,y}(t) = 1\{x = y\} - a''_{x,y}(t).$$

Define

$$B'' = (b''_{x,y}(t))_{x,y \in N_m},$$
$$(B'')^{-1} = (c''_{x,y})_{x,y \in N_m},$$
$$c''_{x,y} = \frac{\Delta''_{y,x}}{|B''|},$$

where $\Delta''_{x,y}$ is the (x, y)-cofactor of the matrix B''.

From the above system of equations, we have

$$\Phi_j^{(x)}(t, z) = \sum_{i=0}^{2^m-1} \frac{\Delta''_{i,x}}{|B''|} \gamma''_i = \frac{1}{|B''|} \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i.$$

Here, we set that

$$\Psi_j(t, z) = |B''|,$$
$$\Psi_j^{(x)}(t, z) = \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i,$$

for each $x \in N_m$.

Then, we have

THEOREM 3.2.

$$\Phi_j^{(x)}(t, z) = \frac{\Psi_j^{(x)}(t, z)}{\Psi_j(t, z)}, \qquad \textit{for} \quad x \in N_m.$$

Remark 1. By the above result, we can easily obtain the g.f. of the p.g.f. of the distribution of the number of success-runs of length k until the n-th trial by means of computer algebra. Here we use a computer algebra, REDUCE ver. 3.5 for generating the system of equations of g.f.'s of conditional p.g.f.'s. As a matter of fact, we can also compute the expectation and the variance of the distribution, i.e., the binomial distribution of order k in a higher order Markov chain. Moreover, we can obtain the probability mass function (p.m.f.) of the distribution by using the fact which the p.g.f. of the distribution is a rational function (cf. Stanley (1986), Uchida and Aki (1995) and Uchida (1996*b*)).

3.2 *Case* $k < m$

In this subsection, we consider the g.f.'s of the p.g.f.'s of the distributions of the numbers of success-runs with length k ($k < m$) until the n-th trial.

3.2.1 *Case Type* I

Let $\Phi_1^{(x)}(t, z \mid A_i)$, $\Phi_1^{(x)}(t, z \mid C)$ be as in the proof of Theorem 3.1. Then, we have the following system of 2^m equations of the g.f.'s of conditional p.g.f.'s by considering all possibilities of the first occurrence of 0.

For each $x \in N_m$,

$$\begin{aligned}\Phi_1^{(x)}(t,z) &= \sum_{i=0}^{k-1} P(A_i)\Phi_1^{(x)}(t,z \mid A_i) + P(C)\Phi_1^{(x)}(t,z \mid C) \\ &= q_x z \Phi_1^{(f_0(x))}(t,z) + \sum_{i=1}^{k-1} \alpha'_{x,i}\Phi_1^{(f_0 \circ f_1^i(x))}(t,z) \\ &\quad + \beta'_x \Phi_1^{(f_1^k(x))}(t,z) + \gamma'_x,\end{aligned}$$

where for each $x \in N_m$ and $i \in L_m$,

$$\begin{aligned}\alpha'_{x,i} &= p_x \left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1}, \\ \beta'_x &= p_x \left[\prod_{j=1}^{k-1}(p_{f_1^j(x)})\right] z^k t, \\ \gamma'_x &= \sum_{n=0}^{k-1} z^n - q_x z \sum_{n=0}^{k-2} z^n - p_x \sum_{i=1}^{k-1}\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1} \sum_{n=0}^{k-i-2} z^n.\end{aligned}$$

For each $x, y \in N_m$,

$$\begin{aligned}a'_{x,y}(t) &= q_x z 1\{y = 2x\} + \sum_{i \in I_{x,y}} \alpha'_{x,i} + \beta'_x 1\{y = f_1^k(x)\}, \\ b'_{x,y}(t) &= 1\{x = y\} - a'_{x,y}(t).\end{aligned}$$

Define

$$\begin{aligned}B' &= (b'_{x,y}(t))_{x,y \in N_m}, \\ (B')^{-1} &= (c'_{x,y})_{x,y \in N_m}, \\ c'_{x,y} &= \frac{\Delta'_{y,x}}{|B'|},\end{aligned}$$

where $\Delta'_{x,y}$ is the (x, y)-cofactor of the matrix B'.

From the above system of linear equations, we have

$$\Phi_1^{(x)}(t,z) = \sum_{i=0}^{2^m-1} \frac{\Delta''_{i,x}}{|B''|}\gamma''_i = \frac{1}{|B''|}\sum_{i=0}^{2^m-1} \Delta''_{i,x}\gamma''_i.$$

Here, we set that for each $x \in N_m$,

$$\Psi_1(t,z) = |B'|,$$
$$\Psi_1^{(x)}(t,z) = \sum_{i=0}^{2^m-1} \Delta'_{i,x}\gamma'_i.$$

Then, we have

THEOREM 3.3.

$$\Phi_1^{(x)}(t,z) = \frac{\Psi_1^{(x)}(t,z)}{\Psi_1(t,z)}, \qquad for \quad x \in N_m.$$

3.2.2 *Case Type* II, III, IV

Let $\Phi_j^{(x)}(t,z \mid A_i)$, $\Phi_j^{(x)}(t,z \mid C)$ be as in the proof of Theorem 3.2. Then, we have the following system of 2^m equations of the g.f.'s of conditional p.g.f.'s by considering all possibilities of the first occurrence of 0.

For each $x \in N_m$,

$$\begin{aligned}\Phi_j^{(x)}(t,z) &= \sum_{i=0}^{n-1} P(A_i)\Phi_j^{(x)}(t,z \mid A_i) + P(C)\Phi_j^{(x)}(t,z \mid C)\\ &= q_x z \Phi_j^{(f_0(x))}(t,z) + \sum_{i=1}^{m-1} \alpha''_{x,i}\Phi_j^{(f_0\circ f_1^i(x))}(t,z)\\ &\quad + \beta''_x \Phi_j^{(2^m-2)}(t,z) + \gamma''_x,\end{aligned}$$

where for each $x \in N_m$ and $i \in L_m$,

$$\alpha''_{x,i} = \begin{cases} p_x \left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1}, & \text{if} \quad 1 < i < k-1,\\ p_x \left[\prod_{j=1}^{i-1}(p_{f_1^j(x)})\right] q_{f_1^i(x)} z^{i+1}\varphi_{ij}, & \text{if} \quad k \le i \le m-2,\end{cases}$$

$$\begin{aligned}\beta''_x &= p_x \left[\prod_{j=1}^{m-2}(p_{f_1^j(x)})\right] q_{f_1^{m-1}(x)} z^m \varphi_{m-1,j}\\ &\quad + p_x \left[\prod_{j=1}^{m-1}(p_{f_1^j(x)})\right] \sum_{i=m}^{\infty}(p_{2^m-1})^{i-m} q_{2^m-1} z^{i+1}\varphi_{ij},\end{aligned}$$

$$\gamma''_x = \sum_{n=k}^{\infty} p_x \left[\prod_{j=1}^{n-1}(p_{f_1^j(x)})\right] z^n \varphi_{nj}$$

$$+\sum_{n=0}^{k-1} z^n - q_x z \sum_{n=0}^{k-2} z^n$$
$$- p_x \sum_{i=1}^{k-1} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)}) \right] q_{f_1^i(x)} z^{i+1} \sum_{n=0}^{k-i-2} z^n,$$

$$\varphi_{ij} = \begin{cases} t, & \text{if} \quad j = 2, \\ t^{i-k+1}, & \text{if} \quad j = 3, \\ t1\{i = k\} + 1\{i \neq k\}, & \text{if} \quad j = 4. \end{cases}$$

For each $x, y \in N_m$,

$$a''_{x,y}(t) = q_x z 1\{y = 2x\} + \sum_{i \in I_{x,y}} \alpha''_{x,i} + \beta''_x 1\{y = 2^m - 2\},$$
$$b''_{x,y}(t) = 1\{x = y\} - a''_{x,y}(t).$$

Define

$$B'' = (b''_{x,y})_{x,y \in N_m},$$
$$(B'')^{-1} = (c''_{x,y})_{x,y \in N_m},$$
$$c''_{x,y} = \frac{\Delta''_{y,x}}{|B''|},$$

where $\Delta''_{x,y}$ is the (x, y)-cofactor of the matrix B''.

From the above system of linear equations, we have

$$\Phi_j^{(x)}(t, z) = \sum_{i=0}^{2^m-1} \frac{\Delta''_{i,x}}{|B''|} \gamma''_i = \frac{1}{|B''|} \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i.$$

Here, we set that for each $x \in N_m$,

$$\Psi_j(t, z) = |B''|,$$
$$\Psi_j^{(x)}(t, z) = \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i.$$

Then, we have

THEOREM 3.4.

$$\Phi_j^{(x)}(t, z) = \frac{\Psi_j^{(x)}(t, z)}{\Psi_j(t, z)}, \qquad \textit{for} \quad x \in N_m.$$

Remark 2. In general, when $k < m$, the corresponding distributions depend on the initial condition of the m-th order Markov chain and are not necessarily as simple as the case $m \leq k$.

Remark 3. By using these results, we can easily obtain the g.f. of the p.g.f. of the distribution of the number of success-runs of length k until the n-th trial by means of a computer algebra. In the same way as Remark 1, we use a computer algebra, REDUCE ver. 3.5 for generating the system of equations of g.f.'s of conditional p.g.f.'s (see Uchida (1996c)).

References

Aki, S. and Hirano, K. (1993). Discrete distributions related to succession events in a two-state Markov chain, *Statistical Sciences and Data Analysis*; *Proceedings of the Third Pacific Area Statistical Conference* (eds. K. Matusita, M. L. Puri and T. Hayakawa), 467–474, VSP International Science Publishers, Zeist.

Aki, S. and Hirano, K. (1996). Lifetime distribution and estimation problems of consecutive-k-out-of-n:F systems, *Ann. Inst. Statist. Math.*, **48**, 185–199.

Chao, M. T., Fu, J. C. and Koutras, M. V. (1995). A survey of the reliability studies of consecutive-k-out-of-n:F systems and its related systems, *IEEE Transactions on Reliability*, **44**, 120–127.

Fu, J. C. (1996). Distribution theory of runs and patterns associated with a sequence of multi-state trials, *Statist. Sinica*, **6**, 957–974.

Fu, J. C. and Koutras, M. V. (1994). Distribution theory of runs: A Markov chain approach, *J. Amer. Statist. Assoc.*, **89**, 1050–1058.

Goldstein, L. (1990). Poisson approximation and DNA sequence matching, *Comm. Statist. Theory Methods*, **19**, 4167–4179.

Hearn, A. C. (1993). *REDUCE User's Manual*, Ver. 3.5, Rand Publication CP78 (Rev. 10/93), Konrad-Zuse-Zentrum, Berlin.

Hirano, K. (1986). Some properties of the distributions of order k, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 43–53, Reidel, Dordrecht.

Hirano, K. (1994). Consecutive-k-out-of-n:F systems, *Proc. Inst. Statist. Math.*, **42**, 45–61 (in Japanese).

Hirano, K. and Aki, S. (1993). On number of occurrences of success runs of specified length in a two-state Markov chain, *Statist. Sinica*, **3**, 313–320.

Hirano, K., Aki, S. and Uchida, M. (1997). Distributions of numbers of success-runs until the first consecutive k successes in higher order Markov dependent trials, *Advances in Combinatorial Methods and Applications to Probability and Statistics* (ed. N. Balakrishnan), 401–410, Birkhauser, Boston.

Koutras, M. V. and Alexandrou, V. A. (1995). Runs, scans and urn model distributions: A unified Markov chain approach, *Ann. Inst. Statist. Math.*, **47**, 743–766.

Ling, K. D. (1988). On binomial distributions of order k, *Statist. Probab. Lett.*, **6**, 247–250.

Ling, K. D. (1989). A new class of negative binomial distributions of order k, *Statist. Probab. Lett.*, **7**, 371–376.

Mood, A. M. (1940). The distribution theory of runs, *Ann. Math. Statist.*, **11**, 367–392.

Philippou, A. N. and Makri, F. S. (1986). Successes, runs and longest runs, *Statist. Probab. Lett.*, **4**, 101–105.

Stanley, R. P. (1986). *Enumerative Combinatorics*, Vol. I, Wadsworth & Brooks/Cole, California.

Uchida, M. (1994). Order statistics with discrete distributions and probability that ties occur somewhere, *J. Japan Soc. Comput. Staist.*, **7**, 47–56.

Uchida, M. (1996a). On number of occurrences of success-runs of exact length k, Research Memo., No. 597, The Institute of Statistical Mathematics, Tokyo.

Uchida, M. (1996*b*). Joint distributions of numbers of success-runs until the first consecutive k successes in a higher-order two-state Markov chain, Research Memo., No. 619, The Institute of Statistical Mathematics, Tokyo.

Uchida, M. (1996*c*). On number of occurrences of success runs specified length in a higher-order two-state Markov chain, Research Memo., No. 625, The Institute of Statistical Mathematics, Tokyo.

Uchida, M. and Aki, S. (1995). Sooner and later waiting time problems in a two-state Markov chain, *Ann. Inst. Statist. Math.*, **47**, 415–433.