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SOME SIMPLE TEST PROCEDURES FOR NORMAL MEAN VECTOR WITH INCOMPLETE DATA

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Abstract. The problem of testing normal mean vector when the observations are missing from subsets of components is considered. For a data matrix with a monotone pattern, three simple exact tests are proposed as alternatives to the traditional likelihood ratio test. Numerical power comparisons between the proposed tests and the likelihood ratio test suggest that one of the proposed tests is indeed comparable to the likelihood ratio test and the other two tests perform better than the likelihood ratio test over a part of the parameter space. The results are extended to a nonmonotone pattern and illustrated using an example.

Key words and phrases: Fisher's method of combining independent tests, likelihood ratio test, missing data, monotone pattern, power, Tippett's test, union-intersection test.

1. Introduction

Inferences based on incomplete or missing data have aroused considerable amount of interest among statisticians in the past as well as present because of their frequent occurrence in practice. The reasons for missingness could be various which will not be discussed in this article. However, to ignore the process that causes missing data it is commonly assumed that the data are missing at random. That is, the missingness does not depend on the missing (unobserved) values of the response variable. For an interesting exposition of such issues we refer the readers to Rubin (1976) and Little (1988, 1995).

In this article we consider the problem of testing multivariate normal mean vector when the data are missing from subsets of components. A commonly used approach to this problem is based on likelihood method. In the past several researchers have considered various forms of missing patterns and suggested likelihood based procedures for estimation and testing. In particular, monotone (or triangular) pattern has received a special attention in the literature since its nested

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structure allows explicit derivation of maximum likelihood estimators and likelihood ratio test statistics for the mean μ and covariance matrix Σ. Anderson (1957) gives a simple and unified approach to derive maximum likelihood estimators for various patterns of incomplete data. Bhargava (1962) derived likelihood ratio tests and their approximate null distributions for several problems. Eaton and Kariya (1983) discuss the difficulties involved in making inferences based on incomplete data and show that no locally most powerful invariant test for the mean vector exists.

In general, exact probability density functions of many of the likelihood ratio criteria that can be derived using the inverse Mellin transform are quite complicated and difficult to use (Muirhead (1982), p. 303). So one needs to approximate the null distribution using methods such as the Box series and Edgeworth series approximations. These approximations either require the moments or cumulants of the likelihood ratio statistics which need to be computed as they are not explicitly available for the present problems in the literature. To avoid such problems it is desired to find some exact and easy to use tests for the mean vector when the data are incomplete.

In the following, we first consider the case where the data set has a monotone pattern. For easy reference, we present the likelihood ratio test for testing the mean vector and then propose three other tests. These tests are obtained by combining independent tests (one for testing a subset of components of the mean vector and another for the conditional mean vector) using union-intersection principle, Fisher's method, and Tippett's approach. These are the contents of Section 2. In Section 3 the results are extended to a nonmonotone pattern which is a multivariate generalization of Lord's (1955) pattern. Derivation of the power functions of the proposed tests seems to be involved. We therefore have estimated the powers of these tests using simulation in Section 4. Power comparisons indicate that the test based on Fisher's method is comparable to the likelihood ratio test. The tests based on union-intersection principle and Tippett's approach are preferable to other tests when one of the components of the mean vector is away from its specified value. An appealing feature of the proposed tests is that they are quite simple to use and do not necessitate new table values to implement. They require only p-values from F distributions which are provided by many standard statistical softwares and electronic calculators. The results of Section 2 are illustrated using a practical example in Section 5. Finally, in Section 6 we make some remarks regarding generalization of the results to some other situations.

2. Monotone pattern

Let x be a $p \times 1$ random vector which is distributed according to a multivariate normal distribution with unknown mean vector μ and unknown and arbitrary positive definite covariance matrix Σ. Let x be partitioned as $(x_1', x_2', \ldots, x_k')'$ such that x_i is a $p_i \times 1$ vector, $i = 1, \ldots, k$, and $p_1 + \cdots + p_k = p$. Partition the mean vector μ and the covariance matrix Σ accordingly. Consider a random sample of N_1 independent observations from the above distribution that has the following pattern

$$
\begin{array}{ll}
 & x_{11},\dots,x_{1N_k},\dots,x_{1N_2},\dots,x_{1N_1} \\
(2.1) & x_{21},\dots,x_{2N_k},\dots,x_{2N_2} \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & x_{k1},\dots,x_{kN_k}
\end{array}
$$

known as monotone or triangular pattern. That is, N_i observations are available on $p_1 + \cdots + p_i$ components, $i = 1, \dots, k$.

Let $X^{(l)}$ denote the submatrix of (2.1) formed by the first $p_1 + \cdots + p_l$ rows and the first N_l columns, $l = 1, \dots, k$. Let $\bar{x}^{(l)}$ and $S^{(l)}$ denote respectively the sample mean vector and the sums of squares and products matrix based on $X^{(l)}$, $l = 1, \dots, k$. For simplicity let us assume that $k = 2$ (for generalization, see Subsection 2.5). Note that

$$
\begin{aligned}
&\bar{x}^{(1)} = \bar{x}_{1,1} \sim N_{p_1}(\mu_1, \Sigma_{11}/N_1), \qquad S^{(1)} = S_{11,1} \sim W_{p_1}(N_1 - 1, \Sigma_{11}) \\
&x^{(2)} = \begin{pmatrix} \bar{x}_{1,2} \\ \bar{x}_{2,2} \end{pmatrix} \sim N_p(\mu, \Sigma/N_2) \qquad \text{and} \\
&S^{(2)} = \begin{pmatrix} S_{11,2} & S_{12,2} \\ S_{21,2} & S_{22,2} \end{pmatrix} \sim W_p(N_2 - 1, \Sigma).
\end{aligned}
$$

Let $\mu_{2.1} = \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1$ and $\Sigma_{2.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. The maximum likelihood estimators are given by (Anderson (1957)) $\hat{\mu}_1 = \bar{x}_{1,1}$, $\hat{\mu}_{2.1} = \bar{x}_{2,2} - S_{21,2}S_{11,2}^{-1}\bar{x}_{1,2}$, $\hat{\Sigma}_{11} = S_{11,1}/N_1$ and $\hat{\Sigma}_{2.1} = S_{2.1,2}/N_2 = (S_{22,2} - S_{21,2}S_{11,2}^{-1}S_{12,2})/N_2$. Define

$$
\begin{aligned}
&Q_1 = N_1[\hat{\mu}_1 - \mu_1]'\hat{\Sigma}_{11}^{-1}[\hat{\mu}_1 - \mu_1], \\
(2.2) \qquad &Q_{2d} = N_2[\bar{x}_{1,2} - \mu_1]'S_{11,2}^{-1}[\bar{x}_{1,2} - \mu_1] \qquad \text{and} \\
&Q_2 = N_2[\hat{\mu}_{2.1} - (\mu_2 - S_{21,2}S_{11,2}^{-1}\mu_1)]'\hat{\Sigma}_{2.1}^{-1}[\hat{\mu}_{2.1} - (\mu_2 - S_{21,2}S_{11,2}^{-1}\mu_1)].
\end{aligned}
$$

Write $R_2 = Q_2/(1 + Q_{2d})$. It is known that $R_2 \sim \frac{N_2 p_2}{N_2 - p} F_{p_2, N_2 - p}$ independently of $Q_1 \sim \frac{N_1 p_1}{N_1 - p_1} F_{p_1, N_1 - p_1}$ (Seber (1984), p. 52).

2.1 *Likelihood ratio test*

The likelihood ratio test statistic (Bhargava (1962)) for testing

$$
(2.3) \qquad H_0 : \mu = 0 \qquad \text{vs} \qquad H_a : \mu \neq 0
$$

is given by

$$
\begin{aligned}
(2.4) \qquad \Lambda &= (1 + Q_1/N_1)^{-N_1/2}(1 + R_2/N_2)^{-N_2/2} \\
&= \Lambda_1 \Lambda_2 \qquad \text{(say)}.
\end{aligned}
$$

The likelihood ratio test rejects H_0 when Λ is too small. Its approximate distribution under H_0 using the Box series approximation (for Box series approximation, for example, see Muirhead (1982), p. 303) is given by

$$P(-2\rho\ln\Lambda \le x) = (1-w_2)P(\chi^2_p \le x) + w_2 P(\chi^2_{p+4} \le x) + O(N_2^{-3}),$$

where

$$\rho = 1 - \frac{N_1 p(p+2) - (N_1 - N_2)p_1(p_1+2)}{2N_2N_1p},$$

and

$$\begin{aligned} w_2 &= p\{N_1^2p^2(p^2-4) - (N_1-N_2)p_1(p_1+2) \\ &\quad \times[3(N_1-N_2)(p_1+1)^2 - 6(p+1)^2N_1 + (N_1+N_2)(4p(p_1+1)+3)]\} \\ &\quad \div 12\{N_1p[2(N_2-1)-p] + (N_1-N_2)p_1(p_1+2)\}^2. \end{aligned}$$

Thus, for a given level of significance α and an observed value Λ_0 of Λ, the likelihood ratio test rejects $H_0 : \mu = 0$ when $P(-2\rho\ln\Lambda > -2\rho\ln\Lambda_0) < \alpha$.

Note that the likelihood ratio test for

$$H_{01} : \mu_1 = 0 \quad \text{vs} \quad H_{a1} : \mu_1 \ne 0 \tag{2.5}$$

rejects H_{01} if Λ_1 is too small and the likelihood ratio test of

$$H_{02} : \mu_2 = 0, \mu_1 = 0 \quad \text{vs} \quad H_{a2} : \mu_2 \ne 0, \mu_1 = 0 \tag{2.6}$$

rejects H_{02} for small values of Λ_2. Equivalently, we observe that H_{01} is rejected for large values of Q_1 and H_{02} is rejected for large values of R_2. Further, recall that $Q_1 \sim \frac{N_1p_1}{N_1-p_1}F_{p_1,N_1-p_1}$ independently of $R_2 \sim \frac{N_2p_2}{N_2-p}F_{p_2,N_2-p}$. Thus the testing problem in (2.3) can be decomposed into two independent testing problems and they can be combined, using some well-known methods, to get a single test for (2.3).

2.2 *Fisher's method of combining independent tests*

Let p_{v1} denote the p-value of the test (2.5) based on Q_1 and let p_{v2} denote the p-value of the test (2.6) based on R_2. Define $Z_i = -\ln(p_{vi})$, $i = 1, 2$. Note that Z_1 and Z_2 are independent exponential random variables with mean one. Let

$$W = Z_1 + Z_2.$$

For a given $0 < \alpha < 1$, the test based on Fisher's method of combining independent tests rejects H_0 if

$$2W > \chi^2_4(\alpha),$$

where $\chi^2_4(\alpha)$ denotes the 100(1-α)-th percentile point of a chi-squared distribution with 4 degrees of freedom.

2.3 *Tippett's test*

Let p_{v1} and p_{v2} be as defined in Subsection 2.2. Tippett's test rejects H_0 for large values of $\max\{Z_1, Z_2\}$. Since Z_1 and Z_2 are independent exponential random variables with mean one, the critical region can be easily identified. Indeed after some algebraic manipulations it can be shown that Tippett's test rejects H_0 if

$$\min\{p_{v1}, p_{v2}\} < 1 - (1 - \alpha)^{1/2}$$

for given α.

2.4 *Union-intersection test*

The test based on union-intersection principle rejects H_0 in (2.3) for large values of $\max\{Q_1, R_2\}$. Instead of $\max\{Q_1, R_2\}$, we want to use $\max\{Q_1^*, R_2^*\}$, where $Q_1^* = (N_1 - 1)Q_1/N_1$ and $R_2^* = (N_2 - p_1 - 1)R_2/N_2$. That is, $S_{11,1}/N_1$ in Q_1 is replaced by $S_{11,1}/(N_1 - 1)$ which is an unbiased estimator of Σ_{11}; similarly, $S_{2.1,2}/N_2$ in R_2 is replaced by $S_{2.1,2}/(N_2 - p_1 - 1)$ which is an unbiased estimator of $\Sigma_{2.1}$. Let M_0 be an observed value of $\max\{Q_1^*, R_2^*\}$. Then, for a given level of significance α, this test rejects H_0 if $P[\max\{Q_1^*, R_2^*\} > M_0] < \alpha$ or equivalently

$$1 - P\left(F_{p_1, N_1 - p_1} \le \frac{(N_1 - p_1)M_0}{(N_1 - 1)p_1}\right) P\left(F_{p_2, N_2 - p} \le \frac{(N_2 - p)M_0}{(N_2 - p_1 - 1)p_2}\right) < \alpha.$$

Remark 2.1. Although the above modification does not change the tests in Subsections 2.2 and 2.3, we found from preliminary simulation studies that, on an overall basis, the test based on $\max\{Q_1^*, R_2^*\}$ is better than the test based on $\max\{Q_1, R_1\}$. This type of modification was suggested for the likelihood ratio test in Section 2.1 by Bhargava (1962) and for testing equality of several normal covariance matrices by Perlman (1980). Bhargava suggested using $(N_1 - 1)/2$ instead of $N_1/2$ and $(N_2 - p_1 - 1)/2$ instead of $N_2/2$ in the exponent terms of (2.4); however, we observed from numerical studies (not reported here) that the power differences between the modified likelihood ratio test and the likelihood ratio test are minute.

2.5 *Generalization*

The proposed testing methods can be easily extended to the monotone pattern (2.1) with $k \ge 3$ in an obvious manner. For instance, when $k = 3$, we merely need to combine the test for $H_{03} : \mu_3 = 0, \mu_2 = 0, \mu_1 = 0$ vs $H_{a3} : \mu_3 \ne 0, \mu_2 = 0, \mu_1 = 0$ with the other two independent tests for (2.5) and (2.6) to get a single test for $H_0 : \mu = 0$. Let $\bar{x}^{(3)}$ and $S^{(3)}$ be as defined at the beginning of the section. Let

$$\hat{\mu}_{3.21} = \bar{x}_{3,3} - (S_{31,3}, S_{32,3}) \begin{pmatrix} S_{11,3} & S_{12,3} \\ S_{21,3} & S_{22,3} \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}_{1,3} \\ \bar{x}_{2,3} \end{pmatrix}$$

and

$$Q_{3d} = N_3(\bar{x}'_{1,3}, \bar{x}'_{2,3}) \begin{pmatrix} S_{11,3} & S_{12,3} \\ S_{21,3} & S_{22,3} \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}_{1,3} \\ \bar{x}_{2,3} \end{pmatrix},$$

where $\bar{x}_{i,3}$ denote subvectors of $\bar{x}^{(3)}$ and $S_{ij,3}$ denote submatrices of $S^{(3)}$. The test statistic for H_{03} is given by $R_3 = N_3^2\hat{\mu}'_{3.21}S^{-1}_{3.21,3}\hat{\mu}_{3.21}/(1+Q_{3d})$. When H_{03} is true, R_3 follows $\frac{N_3 p_3}{N_3-p}F_{p_3,N_3-p}$ independently of Q_1 and R_2. In this case, $R_3^* = (N_3-p_1-p_2-1)R_3/N_3$. These statistics Q_1, R_2 and R_3 (or Q_1^*, R_2^*, and R_3^*) can be combined to get a single test for $H_0 : \mu = 0$ as in the case of $k = 2$.

The expressions for the Q_1 and R_i's for a general k can be obtained using the MLEs (see Jinadasa and Tracy (1992)) of μ and Σ. Following the notations defined at the beginning of Section 2, the sample summary statistics can be written as

$$\bar{x}^{(l)} = \begin{pmatrix} \bar{x}_{1,l} \\ \bar{x}_{2,l} \\ \vdots \\ \bar{x}_{l,l} \end{pmatrix}, \qquad S^{(l)} = \begin{pmatrix} S_{11,l} & \dots & S_{1l,l} \\ \vdots & & \vdots \\ S_{l1,l} & \dots & S_{ll,l} \end{pmatrix}$$

for $l - 1, \dots, k$. Further, define

$$(B_{l1}, \dots, B_{l\overline{l-1}}) - (S_{l1,l} \cdots S_{l\overline{l-1},l}) \begin{pmatrix} S_{11,l} & \cdots & S_{1\overline{l-1},l} \\ \vdots & & \vdots \\ S_{\overline{l-1}1,l}, & \cdots & , S_{\overline{l-1}\,\overline{l-1},l} \end{pmatrix}^{-1}.$$

Using these notations, the MLEs can be expressed as

$$\hat{\mu}_1 = \bar{x}^{(1)} = \bar{x}_{1,1}, \qquad \hat{\mu}_l = \bar{x}_{l,l} - \sum_{j=1}^{l-1} B_{lj}(\bar{x}_{j,l} - \hat{\mu}_j),$$

$$\hat{\Sigma}_{11} = \frac{S^{(1)}}{N_1}, \qquad N_l\hat{\Sigma}_{l.\overline{l-1}\dots 1} = S_{ll,l} - \sum_{j=1}^{l-1} B_{lj}S_{jl,l},$$

$$(\hat{\Sigma}_{l1}, \dots, \hat{\Sigma}_{l\overline{l-1}}) = (B_{l1}, \dots, B_{l\overline{l-1}}) \begin{pmatrix} \hat{\Sigma}_{11} & \cdots & \hat{\Sigma}_{1\overline{l-1}} \\ \vdots & \ddots & \vdots \\ \hat{\Sigma}_{\overline{l-1}1} & \cdots & \hat{\Sigma}_{\overline{l-1}\,\overline{l-1}} \end{pmatrix},$$

and

$$\hat{\Sigma}_{ll} = \hat{\Sigma}_{l.\overline{l-1}\dots 1} + \sum_{j=1}^{l-1} B_{lj}\hat{\Sigma}_{jl}, \qquad l = 2, \dots, k.$$

In terms of these notations, we can write

$$Q_1 = N_1(\hat{\mu}_1 - \mu_1)'\hat{\Sigma}_{11}^{-1}(\hat{\mu}_1 - \mu_1)$$

and

$$Q_l = N_l \left(\hat{\mu}_{l.\overline{l-1}\dots 1} - \left(\mu_l - \sum_{j=1}^{l-1} B_{lj}\mu_j \right) \right)'$$

$$\cdot \hat{\Sigma}^{-1}_{l.\overline{l-1}\dots 1} \left(\hat{\mu}_{l.\overline{l-1}\dots 1} - \left(\mu_l - \sum_{j=1}^{l-1} B_{lj}\mu_j \right) \right)$$

where $\hat{\mu}_{l.\overline{l-1}\dots 1} = \bar{x}_{l,l} - \sum_{j=1}^{l-1} B_{lj}\bar{x}_{j,l}$, $l = 2, \dots, k$.

Let

$$Q_{ld} = N_l(\bar{x}'_{1,l}, \dots, \bar{x}'_{\overline{l-1},l}) \begin{pmatrix} S_{11,l}, & \cdots & , S_{1\overline{l-1},l} \\ \vdots & \ddots & \vdots \\ S_{\overline{l-1}1,l}, & \cdots & , S_{\overline{l-1}\,\overline{l-1},l} \end{pmatrix}^{1} \begin{pmatrix} \bar{x}_{1,l} \\ \vdots \\ \bar{x}_{\overline{l-1},l} \end{pmatrix}$$

$$\sim \frac{p_{(l-1)}}{N_l - p_{(l-1)}} F_{p_{(l-1)}, N_l - p_{(l-1)}}, \qquad l = 2, \dots, k,$$

where $p_{(l)} = \sum_{j=1}^{l} p_j$. Let $R_l = Q_l/(1 + Q_{ld})$, $l = 2, \dots, k$. It is well known that $Q_1, R_2, \dots, R_k$ are all statistically independent with $Q_1 \sim N_1 p_1 F_{p_1, N_1 - p_1}/(N_1 - p_1)$ and $R_l \sim N_l p_l F_{p_l, N_l - p_{(l)}}/(N_l - p_{(l)})$. Using these notations and the results, the LRT statistic can be written as

$$\Lambda = (1 + Q_1/N_1)^{-N_1/2}(1 + R_2/N_2)^{-N_2/2} \cdots (1 + R_k/N_k)^{\ \ N_k/2},$$

and its approximate null distribution can be obtained using the Box series approximation. Other test statistics and their null distribution can be derived along the lines of the case $k = 2$. We note that for a general k, $Q_1^* = (N_1 - 1)Q_1/N_1$ and $R_l^* = (N_l - p_{(l-1)} - 1)R_l/N_l$, where $p_{(l)} = \sum_{j=1}^{l} p_j$, $l = 2, \dots, k$.

3. A nonmonotone pattern

In this section we consider a pattern of data which is a multivariate generalization of Lord's (1955) pattern. The data matrix has the following form

$$\begin{array}{ll} x_{11}, \dots, x_{1N_2} & x_{1N_2+1}, \dots, x_{1N_1} \\ x_{21}, \dots, x_{2N_2} & \\ & x_{3N_2+1}, \dots, x_{3N_1}. \end{array}$$

That is, there are N_2 independent observations from $N_{p_1+p_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$ and $N_1 - N_2$ independent observations from $N_{p_1+p_3}\left(\begin{bmatrix} \mu_1 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{bmatrix}\right)$. This pattern is identical to the pattern in Section 2 if the observations on the third subset of components are ignored. The problem of testing the mean vector is zero can be decomposed into (2.5), (2.6) and

$$H_{03} : \mu_3 = 0, \mu_1 = 0 \quad \text{vs} \quad H_{a3} : \mu_3 \neq 0, \mu_1 = 0. \tag{3.1}$$

Let $(\bar{x}_1^*, \bar{x}_3^*)$ and V be the sample mean vector and the sums of squares and products matrix based on the last $(N_1 - N_2)$ observations on x_1 and x_3 components respectively. Partition V as $\begin{pmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{pmatrix}$ so that V_{13} is of order $p_1 \times p_3$. Let $\mu_{3.1} = \mu_3 - \Sigma_{31}\Sigma_{11}^{-1}\mu_1$ and $\Sigma_{3.1} = \Sigma_{33} - \Sigma_{31}\Sigma_{11}^{-1}\Sigma_{13}$. The maximum likelihood estimators are given by

$$\hat{\mu}_{3.1} = \bar{x}_3^* - V_{31}V_{11}^{-1}\bar{x}_1^* \quad \text{and} \quad \hat{\Sigma}_{3.1} = \frac{(V_{33} - V_{31}V_{11}^{-1}V_{13})}{(N_1 - N_2)}.$$

Define $Q_3 = (N_1 - N_2)[\hat{\mu}_{3.1} - (\mu_3 - V_{31}V_{11}^{-1}\mu_1)]'\hat{\Sigma}_{3.1}^{-1}[\hat{\mu}_{3.1} - (\mu_3 - V_{31}V_{11}^{-1}\mu_1)]$ and $Q_{3d} = (N_1 - N_2)[\bar{x}_1^* - \mu_1]'V_{11}^{-1}[\bar{x}_1^* - \mu_1]$. Let $R_3 = Q_3/(1 + Q_{3d})$. It can be easily seen that $R_3 \sim (N_1 - N_2)p_3F_{p_3,(N_1-N_2-p_1-p_3)}/(N_1 - N_2 - p_1 - p_3)$.

The likelihood ratio test statistic for this set up is

$$\begin{aligned}\Lambda &= \Lambda_1\Lambda_2(1 + R_3/(N_1 - N_2))^{-(N_1-N_2)/2}\\ &= \Lambda_1\Lambda_2\Lambda_3,\end{aligned}$$

where Λ_1 and Λ_2 are as given in Section 2. An approximate null distribution of Λ can be obtained using the Box series approximation.

Let p_{v3} denote the p value of the test (3.1) based on R_3. Let $Z_3 = -\ln(p_{v3})$, which follows an exponential distribution with mean 1 and independent of Z_1 and Z_2 defined in Section 2. Further define $W = \sum_{i=1}^{3} Z_i$. For a given level of significance α, the test based on Fisher's method rejects H_0 if $2W > \chi_6^2(\alpha)$, where $\chi_6^2(\alpha)$ denotes the $100(1-\alpha)$-th percentile point of a chi-squared distribution with 6 degree of freedom; Tippett's test rejects H_0 whenever $\min\{p_{v1}, p_{v2}, p_{v3}\} < 1 - (1-\alpha)^{1/3}$.

The union-intersection test rejects H_0 for large values of $\max\{Q_1^*, R_2^*, R_3^*\}$, where Q_1^* and Q_2^* are defined in Section 2, and $R_3^* = (N_1 - N_2 - p_1 - 1)R_3/(N_1 - N_2)$. Let $M_0 = \max\{Q_1^*, R_2^*, R_3^*\}$. For a given level α, the union-intersection test rejects H_0 if

$$\begin{aligned}1 - P\left(F_{p_1,N_1-p_1} \le \frac{(N_1 - p_1)M_0}{(N_1 - 1)p_1}\right) P\left(F_{p_2,N_2-p} \le \frac{(N_2 - p)M_0}{(N_2 - p_1 - 1)p_2}\right)\\ \times P\left(F_{p_3,N_1-N_2-p_1-p_3} \le \frac{(N_1 - N_2 - p_1 - p_3)M_0}{(N_1 - N_2 - p_1 - 1)p_3}\right) < \alpha.\end{aligned}$$

4. Power comparisons

As mentioned earlier, it is difficult to derive power functions of the proposed tests in Sections 2 and 3. Even though approximate power functions of likelihood ratio tests can be derived using the Box series approximation, in order to have fair comparisons we estimated the powers of all four tests using simulation (100000 runs). Wishart variates are generated using the Fortran subroutine by Smith and Hocking (1972) and normal variates are generated using the IMSL subroutine RNNOA. The powers are estimated for different values of $\delta_1 = \mu_1'\Sigma_{11}^{-1}\mu_1$, $\delta_2 = \mu_{2.1}'\Sigma_{2.1}^{-1}\mu_{2.1}$ and $\delta_3 = \mu_{3.21}'\Sigma_{3.21}^{-1}\mu_{3.21}$ as in Morrison and Bhoj (1973). For Lord's pattern $\delta_3 = \mu_{3.1}'\Sigma_{3.1}^{-1}\mu_{3.1}$. Since all the tests are lower triangular invariant, we take Σ to be an identity matrix for computing powers. The estimated powers of the likelihood ratio test (LRT), the test based on Fisher's method (FT), Tippett's test (TP) and the test based on union intersection method (UIT) are given in Tables 1 and 2 for monotone pattern and in Table 3 for Lord's pattern.

Table 1. Simulated powers of the LRT, FT, TP and UIT.
$N_1 = 30,\ N_2 = 20,\ p_1 = p_2 = 1,\ \alpha = 0.05$

δ_1	δ_2	LRT	FT	TP	UIT	δ_1	δ_2	LRT	FT	TP	UIT
0.00	0.00	0.05	0.05	0.05	0.05	0.10	0.30	0.70	0.71	0.62	0.63
0.00	0.20	0.36	0.34	0.35	0.37	0.10	0.50	0.86	0.85	0.80	0.80
0.00	0.35	0.58	0.55	0.58	0.59	0.10	0.70	0.94	0.93	0.90	0.90
0.00	0.50	0.74	0.71	0.74	0.75	0.20	0.10	0.66	0.67	0.62	0.61
0.00	1.00	0.96	0.95	0.96	0.97	0.30	0.10	0.80	0.81	0.78	0.77
0.10	0.00	0.29	0.29	0.30	0.29	0.50	0.10	0.94	0.95	0.95	0.94
0.20	0.00	0.53	0.53	0.53	0.54	0.20	0.20	0.76	0.77	0.69	0.69
0.30	0.00	0.71	0.71	0.74	0.73	0.20	0.30	0.83	0.84	0.86	0.86
0.40	0.00	0.84	0.84	0.86	0.85	0.20	0.50	0.93	0.93	0.87	0.87
0.50	0.00	0.91	0.91	0.93	0.93	0.30	0.20	0.87	0.88	0.82	0.81
0.10	0.10	0.45	0.46	0.40	0.40	0.50	0.20	0.96	0.97	0.95	0.95
0.10	0.20	0.59	0.60	0.52	0.52	0.40	0.40	0.97	0.97	0.94	0.94

Table 2. Simulated powers of the LRT, FT, TP and UIT.
$N_1 = 20,\ N_2 = 14,\ N_3 = 8,\ p_1 = p_2 = p_3 = 1,\ \alpha = 0.05$

δ_1	δ_2	δ_3	LRT	FT	TP	UIT	δ_1	δ_2	δ_3	LRT	FT	TP	UIT
0.0	0.0	0.0	0.05	0.05	0.05	0.05	0.5	0.0	0.5	0.72	0.77	0.75	0.67
0.0	0.0	1.0	0.39	0.27	0.30	0.41	0.5	0.0	1.0	0.82	0.85	0.78	0.73
0.0	0.0	2.0	0.65	0.48	0.55	0.68	0.5	0.0	2.0	0.92	0.92	0.85	0.83
0.0	0.0	3.0	0.80	0.63	0.73	0.83	0.5	0.0	3.0	0.96	0.96	0.90	0.90
0.0	0.0	4.0	0.89	0.74	0.84	0.91	0.4	0.0	0.5	0.64	0.69	0.65	0.57
0.0	0.5	0.0	0.39	0.42	0.47	0.41	0.7	0.0	0.5	0.84	0.88	0.88	0.81
0.0	1.0	0.0	0.69	0.72	0.79	0.74	1.0	0.0	0.5	0.93	0.96	0.96	0.93
0.0	1.5	0.0	0.87	0.88	0.93	0.90	0.5	0.3	0.0	0.73	0.81	0.77	0.67
0.5	0.0	0.0	0.56	0.65	0.71	0.60	0.5	0.5	0.0	0.81	0.88	0.82	0.72
0.7	0.0	0.0	0.73	0.80	0.86	0.78	0.5	1.0	0.0	0.93	0.96	0.91	0.85
1.0	0.0	0.0	0.88	0.92	0.96	0.92	0.4	0.5	0.0	0.76	0.83	0.76	0.65
0.0	0.5	0.5	0.56	0.58	0.52	0.50	0.7	0.5	0.0	0.90	0.94	0.91	0.84
0.0	0.5	1.0	0.70	0.69	0.58	0.59	1.0	0.5	0.0	0.96	0.98	0.98	0.94
0.0	0.5	2.0	0.85	0.82	0.70	0.74	0.5	0.5	0.5	0.89	0.93	0.84	0.76
0.0	0.5	3.0	0.92	0.90	0.80	0.84	0.5	0.5	1.0	0.93	0.96	0.86	0.80
0.0	0.7	0.5	0.68	0.69	0.66	0.62	0.5	1.0	0.5	0.96	0.98	0.92	0.87
0.0	1.0	0.5	0.80	0.82	0.81	0.77	0.7	0.5	0.5	0.94	0.97	0.92	0.86
0.0	1.5	0.5	0.92	0.93	0.94	0.91	0.7	1.0	1.0	0.99	0.99	0.97	0.93

Table 3. Simulated powers of the LRT, FT, TP and UIT.

$N_1 = 20,\ N_2 = 10,\ p_1 = p_2 = p_3 = 1,\ \alpha = 0.05$

δ_1	δ_2	δ_3	LRT	FT	TP	UIT	δ_1	δ_2	δ_3	LRT	FT	TP	UIT
0.00	0.00	0.00	0.05	0.05	0.05	0.05	0.25	0.00	2.00	0.93	0.93	0.88	0.88
0.00	0.00	0.50	0.30	0.27	0.29	0.31	0.50	0.00	0.50	0.80	0.83	0.78	0.72
0.00	0.00	1.00	0.54	0.48	0.55	0.58	0.50	0.00	0.75	0.84	0.88	0.82	0.78
0.00	0.00	2.00	0.84	0.78	0.86	0.88	0.50	0.00	1.00	0.90	0.91	0.85	0.81
0.00	0.00	2.50	0.91	0.86	0.92	0.94	0.00	0.50	0.50	0.61	0.59	0.47	0.51
0.25	0.00	0.00	0.33	0.36	0.40	0.32	0.00	1.00	0.50	0.79	0.77	0.67	0.70
0.50	0.00	0.00	0.61	0.65	0.72	0.63	0.00	2.00	0.50	0.94	0.93	0.89	0.91
0.75	0.00	0.00	0.80	0.83	0.89	0.83	0.00	1.00	1.00	0.91	0.89	0.78	0.82
1.00	0.00	0.00	0.91	0.92	0.96	0.93	0.25	0.50	0.50	0.80	0.83	0.64	0.62
0.25	0.00	0.50	0.59	0.62	0.53	0.49	0.25	1.00	0.50	0.91	0.92	0.76	0.75
0.25	0.00	1.00	0.77	0.78	0.69	0.67	0.50	0.50	0.50	0.91	0.93	0.83	0.78

We see from Tables 1, 2 and 3 that the differences between the powers of the likelihood ratio test and the test based on Fisher's method are appreciable when one of the components of μ is away from its specified value compared to others; otherwise they are comparable. On average, these two tests are equally efficient. Between the test based on Fisher's method and Tippett's test, the latter is preferable to the former if only one of the components of μ is different from its specified value. The union-intersection test is preferable to others when only one of the components of μ is away from its specified value. Note that for Lord's pattern with $N_2 = N_1/2$, the power at $(\delta_1, \delta_2, \delta_3)$ is equal to the power at $(\delta_1, \delta_3, \delta_2)$ for all tests. Furthermore, Tippett's test and the union-intersection test are useful to identify the components that caused the rejection of H_0. Preliminary simulation studies for the case $k = 4$ (not reported here) indicate that the power comparisons of the tests are similar to the cases $k = 2$ and 3 and so we expect that the power comparison results given above will hold for any $k \geq 2$.

5. An example

For the sake of illustration of the results we consider an example given in Johnson and Wichern ((1992), p. 183). The data set consists of measurements on perspiration from 20 healthy females, and satisfy the normality assumption. Each observation has three components, namely, x_1 = sweat rate, x_2 = sodium content and x_3 = potassium content. We created a monotone pattern in the data set by deleting the observations on (x_2, x_3) from randomly selected units 4, 7, 12 and 17, on x_3 from randomly selected units 3, 8 and 10 and then rearranging the data to have pattern in (2.1). In the notations of Section 2, we have $p_1 = 1$, $p_2 = 1$, $p_3 = 1$, $N_1 = 20$, $N_2 = 16$, and $N_3 = 13$. The hypotheses considered in the example are $H_0 : \mu' = (4, 50, 10)$ and $H_1 : \mu' \neq (4, 50, 10)$. The sample summary statistics are as follows:

$$\bar{x}^{(1)} = 4.64, \quad S^{(1)} = 54.71, \quad \bar{x}^{(2)} = \begin{pmatrix} 4.89 \\ 43.73 \end{pmatrix}, \quad S^{(2)} = \begin{pmatrix} 46.60 & 167.46 \\ . & 2399.89 \end{pmatrix},$$

$$\bar{x}^{(3)} = \begin{pmatrix} 4.75 \\ 43.48 \\ 9.46 \end{pmatrix} \quad \text{and} \quad S^{(3)} = \begin{pmatrix} 39.57 & 190.06 & -15.75 \\ . & 2166.48 & -79.93 \\ . & . & 28.39 \end{pmatrix}.$$

The value of the likelihood ratio test statistic is 4.02 with a p-value of 0.261. The computed values of $Q_1 = 2.995$ (p-value 0.108), $R_2 = 10.038$ (p-value 0.010), and $R_3 = 0.311$ (p-value 0.635). Further, $M_0 = \max\{Q_1^*, R_2^*, R_3^*\} = 8.783$ with p-value 0.032, the statistic W based on Fisher's method is 6.30 with p-value 0.390, and the p-value of Tippett's test is 0.031. So at 5% level of significance the union-intersection test and Tippett's test reject H_0. Note that the second mean is quite away from the hypothesized value as compared to the other two means and so as was noticed in the simulation study earlier, the union-intersection test and Tippett's test provide sufficient evidence against H_0. Further, both union-intersection and Tippett's tests indicate that the second component caused the rejection (the critical value of $\max\{Q_1^*, R_2^*, R_3^*\}$ at 5% level is 7.448, which is obtained by solving $P[\max\{Q_1^*, R_2^*, R_3^*\} < c] = 0.95$ for c).

6. Conclusions

The testing methods considered in this article are in general applicable to patterns of data for which Anderson's (1957) likelihood factorization method can be used to derive maximum likelihood estimators. Further, they can be extended to two-sample problems by partitioning the data matrices appropriately. Of course, in all these situations none of the tests which are considered in this article is expected to dominate others uniformly since they are different functions of the same set of pivots obtained from likelihood ratio test statistics; however, the combined tests may be simpler to use.

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