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USE OF MARKOV CHAIN MONTE CARLO METHODS IN A BAYESIAN ANALYSIS OF THE BLOCK AND BASU BIVARIATE EXPONENTIAL DISTRIBUTION

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Abstract. Metropolis algorithms along with Gibbs steps are proposed to perform a Bayesian analysis for the Block and Basu (ACBVE) bivariate exponential distribution. We also consider the use of Gibbs sampling to develop Bayesian inference for accelerated life tests assuming a power rule model and the ACBVE distribution. The methodology developed in this paper is exemplified with two examples.

Key words and phrases: Bivariate exponential distribution, Gibbs sampling, Metropolis algorithm, accelerated life testing.

1. Introduction

In many applications of life testing, we usually have two lifetimes X and Y associated to each unit. Among the different existing bivariate lifetime models to be used in these applications (Freund (1961), Marshall and Olkin (1967), Sarkar (1987), Block and Basu (1974), Downton (1972), Gumbel (1960), Hawkes (1972)) one model has been very well explored in the literature: the Block and Basu (1974) exponential distribution.

The bivariate exponential distribution of Block and Basu (ACBVE) with parameters λ_1, λ_2 and λ_3, for the lifetimes X and Y has a joint density function given by

$$f(x,y) = \begin{cases} f_1(x,y) = \dfrac{\lambda\lambda_1\lambda_{23}}{\lambda_{12}} \exp\{-\lambda_1 x - \lambda_{23} y\} & \text{if} \quad x < y \\ f_2(x,y) = \dfrac{\lambda\lambda_2\lambda_{13}}{\lambda_{12}} \exp\{-\lambda_{13} x - \lambda_2 y\} & \text{if} \quad x \geq y \end{cases} \tag{1.1}$$

where $\lambda_{12} = \lambda_1 + \lambda_2$, $\lambda_{13} = \lambda_1 + \lambda_3$, $\lambda_{23} = \lambda_2 + \lambda_3$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

The joint generating function for the ACBVE is given by

$$(1.2)\qquad m(s,t) = E(e^{sX+tY}) = \frac{\lambda}{\lambda_{12}(\lambda - t - s)}\left\{\frac{\lambda_1\lambda_{23}}{\lambda_{23}-t} + \frac{\lambda_2\lambda_{13}}{\lambda_{13}-s}\right\}.$$

From (1.2), we get the moments of interest for X and Y; thus, the means and variances for X and Y are given by,

$$(1.3)\qquad \begin{aligned} E(X) &= \frac{1}{\lambda_{13}} + \frac{\lambda_2\lambda_3}{\lambda\lambda_{12}\lambda_{13}} \\ E(Y) &= \frac{1}{\lambda_{23}} + \frac{\lambda_1\lambda_3}{\lambda\lambda_{12}\lambda_{23}} \\ \sigma_X^2 = \operatorname{var}(X) &= \frac{1}{\lambda_{13}^2} + \frac{\lambda_2\lambda_3(2\lambda_1\lambda + \lambda_2\lambda_3)}{\lambda^2\lambda_{12}^2\lambda_{13}^2} \\ \sigma_Y^2 = \operatorname{var}(Y) &= \frac{1}{\lambda_{23}^2} + \frac{\lambda_1\lambda_3(2\lambda_2\lambda + \lambda_1\lambda_3)}{\lambda^2\lambda_{12}^2\lambda_{23}^2}. \end{aligned}$$

The correlation coefficient for X and Y is given by

$$(1.4)\qquad \rho_{XY} = \frac{\lambda_3[(\lambda_1^2+\lambda_2^2)\lambda + \lambda_1\lambda_2\lambda_3]}{\phi_1\phi_2}$$

where

- $\phi_1 = [\lambda_{12}^2\lambda_{13}^2 + \lambda_2(\lambda_2 + 2\lambda_1)\lambda^2]^{1/2}$ and
- $\phi_2 = [\lambda_{12}^2\lambda_{23}^2 + \lambda_1(\lambda_1 + 2\lambda_2)\lambda^2]^{1/2}$.

Observe that $0 \le \rho_{XY} \le 1$ and $\rho_{XY} = 0$ only for the trivial cases $\lambda_3 = 0$ or $\lambda_1 = \lambda_2 = 0$.

Usually, researchers consider the use of standard asymptotic results based on the normality of the maximum likelihood estimators to get inferences for the parameters of the ACBVE distribution (1.1), but in general these asymptotic results can be very poor for small or moderate sample sizes.

A Bayesian analysis of the ACBVE model (1.1) is introduced by Achcar and Santander (1993) using non-informative prior densities for the parameters and Laplace's method of approximation for integrals (Tierney and Kadane (1986)) to get the posterior summaries of interest.

In this paper, we present Bayesian inferences for the ACBVE distribution (1.1) using Metropolis-within-Gibbs algorithms (Gelfand and Smith (1990)). We also consider the use of the Markov Chain Monte Carlo methods to get Bayesian inferences for accelerated life test problems using the ACBVE model (Achcar (1995)).

2. Bayesian inference for the ACBVE model

Considering a random sample of size n, $(X_1, Y_1), \ldots, (X_n, Y_n)$ of the ACBVE model (1.1), the likelihood function for λ_1, λ_2 and λ_3 is given by

$$L(\lambda_1,\lambda_2,\lambda_3) = \prod_{i=1}^{n} f_1^{\delta_i}(x_i,y_i) f_2^{1-\delta_i}(x_i,y_i)$$

where $\delta_i = 1$ if $X_i < Y_i$ and $\delta_i = 0$ if $X_i \geq Y_i$.

That is,

$$L(\lambda_1, \lambda_2, \lambda_3) = \frac{\lambda_1^r \lambda^n \lambda_2^{n-r} \lambda_{23}^r \lambda_{13}^{n-r}}{\lambda_{12}^n} \exp\{-\lambda_1 n\bar{x} - \lambda_2 n\bar{y} - \lambda_3 R\} \tag{2.1}$$

where $n\bar{x} = \sum_{i=1}^n x_i$, $n\bar{y} = \sum_{i=1}^n y_i$, $r = \sum_{i=1}^n \delta_i$ and $R = \sum_{i=1}^n [\delta_i y_i + (1 - \delta_i) x_i]$.

For Bayesian inference, considering the introduction of a latent variable N_1 representing the number of observations such that $X_i < Y_i$, we assume the following prior densities for N_1, λ_1, λ_2 and λ_3:

$$\begin{aligned} &N_1 \sim b\left(n, \frac{\lambda_1}{\lambda_{12}}\right) \\ &\lambda_1 \sim \Gamma(a_1, b_1), \quad a_1 \text{ and } b_1 \text{ known} \\ &\lambda_2 \sim \Gamma(a_2, b_2), \quad a_2 \text{ and } b_2 \text{ known} \\ &\lambda_3 \sim \Gamma(a_3, b_3), \quad a_3 \text{ and } b_3 \text{ known.} \end{aligned} \tag{2.2}$$

Here, $b(n, \lambda_1/\lambda_{12})$ denotes a binomial distribution with mean $n\lambda_1/\lambda_{12}$, $\lambda_1/\lambda_{12} = P(X < Y)$; $\Gamma(a_i, b_i)$ denotes a gamma distribution with mean a_i/b_i and variance a_i/b_i^2. We further assume independence among the parameters N_1, λ_1, λ_2 and λ_3.

The joint posterior density is,

$$\begin{aligned} \pi(N_1, \lambda_1, \lambda_2, \lambda_3 \mid \mathcal{D}) \propto &\binom{n}{N_1} \left(\frac{\lambda_1}{\lambda_{12}}\right)^{N_1} \left(\frac{\lambda_2}{\lambda_{12}}\right)^{n-N_1} \frac{\lambda^n}{\lambda_{12}^n} \\ &\cdot \lambda_{23}^r \lambda_{13}^{n-r} \lambda_1^{r+a_1-1} \lambda_2^{n-r+a_2-1} \lambda_3^{a_3-1} \\ &\cdot \exp\{-(n\bar{x} + b_1)\lambda_1 - (n\bar{y} + b_2)\lambda_2 - (R + b_3)\lambda_3\} \end{aligned} \tag{2.3}$$

where $\mathcal{D}$ denotes the data set.

The conditional posterior densities for the Gibbs algorithm are given by,

$$\begin{aligned} &N_1 \mid \lambda_1, \lambda_2, \lambda_3, \mathcal{D} \sim b\left(n, \frac{\lambda_1}{\lambda_{12}}\right) \\ &\pi(\lambda_1 \mid N_1, \lambda_2, \lambda_3, \mathcal{D}) \propto \lambda_1^{a_1-1} e^{-b_1\lambda_1} \psi_1(N_1, \lambda_1, \lambda_2, \lambda_3) \\ &\pi(\lambda_2 \mid N_1, \lambda_1, \lambda_3, \mathcal{D}) \propto \lambda_2^{a_2-1} e^{-b_2\lambda_2} \psi_2(N_1, \lambda_1, \lambda_2, \lambda_3) \\ &\pi(\lambda_3 \mid N_1, \lambda_1, \lambda_2, \mathcal{D}) \propto \lambda_3^{a_3-1} e^{-b_3\lambda_3} \psi_3(N_1, \lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &\psi_1(N_1, \lambda_1, \lambda_2, \lambda_3) = \lambda^n \lambda_{12}^{-2n} \lambda_{13}^{n-r} \lambda_1^{N_1+r} \exp\{-n\bar{x}\lambda_1\}, \\ &\psi_2(N_1, \lambda_1, \lambda_2, \lambda_3) = \lambda^n \lambda_{12}^{-2n} \lambda_{23}^r \lambda_2^{2n-N_1-r} \exp\{-n\bar{y}\lambda_2\}, \\ &\psi_3(N_1, \lambda_1, \lambda_2, \lambda_3) = \lambda^n \lambda_{23}^r \lambda_{13}^{n-r} \exp\{-R\lambda_3\} \end{aligned} \tag{2.5}$$

A sample of draws from the joint posterior density (2.3) can now be obtained by successively sampling N_1, λ_1, λ_2 and λ_3 from the conditional posterior densities given in (2.4). Observe that, we need to use the Metropolis-Hastings algorithm to generate the variables λ_1, λ_2 and λ_3 (Chib and Greenberg (1995)). In this way, the value of λ_1 is simulated as: at the r-th iteration (given the current value $N_1^{(r)}$, $\lambda_2^{(r)}$, $\lambda_3^{(r)}$), draw a candidate $\lambda_1^{(r)}$ from a gamma density $\Gamma(a_1, b_1)$; if it satisfies stationarity, move to this with probability

$$\min\left\{\frac{(\psi_1(N_1^{(r)}, \lambda_1^{(r)}, \lambda_2^{(r)}, \lambda_3^{(r)}))}{(\psi_1(N_1^{(r)}, \lambda_1^{(r-1)}, \lambda_2^{(r)}, \lambda_3^{(r)}))}, 1\right\},$$

and otherwise set $\lambda_1^{(r)} = \lambda_1^{(r-1)}$, where $\psi_1(N_1, \lambda_1, \lambda_2, \lambda_3)$ is defined in (2.5).

Similarly, we could drawn candidates $\lambda_2^{(r)}$ and $\lambda_3^{(r)}$ from the gamma densities $\Gamma(a_2, b_2)$, $\Gamma(a_3, b_3)$, respectively to be considered in the Metropolis-Hastings algorithm. For the choice of values for the parameters a_i and b_i, $i = 1, 2, 3$ for the gamma functions used as prior distributions and as candidate generating distributions for the Metropolis-Hastings algorithm, we should get some information on these parameters based on a preliminary analysis. As a special case, we could use some preliminary estimates λ_i, from which we get values for a_i and b_i observing that the gamma prior density for λ_i has mean a_i/b_i and variance a_i/b_i^2. Other distributions also could be considered as candidate generating distributions for the Metropolis-Hastings algorithm. We could monitor the convergence of the Gibbs samples using the Gelman and Rubin (1992) method that uses the analysis of variance technique to determine if further iterations are needed.

2.1 *Bayes estimators for the mean lifetimes*

We can use the Gibbs samplers to get inferences on the parameters of the ACBVE distribution (1.1) or functions of these parameters. In this case, we could approximate posterior moments of interest. As a special case, consider the mean lifetimes,

$$\begin{aligned} \mu_1 &= E[X] = \frac{\lambda\lambda_{12} + \lambda_2\lambda_3}{\lambda_{12}\lambda_{13}\lambda} \\ \mu_2 &= E[Y] = \frac{\lambda\lambda_{12} + \lambda_1\lambda_3}{\lambda_{12}\lambda_{23}\lambda}. \end{aligned} \tag{2.6}$$

Bayes estimators for μ_i, $i = 1, 2$ with respect to the squared error loss function are given by $E(\mu_i \mid \mathcal{D})$ which can be approximated by its Monte Carlo estimate,

$$\begin{aligned} \hat{\mu_1} &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R}\frac{\lambda^{(r,s)}\lambda_{12}^{(r,s)} + \lambda_2^{(r,s)}\lambda_3^{(r,s)}}{\lambda_{12}^{(r,s)}\lambda_{13}^{(r,s)}\lambda^{(r,s)}} \\ \hat{\mu_2} &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R}\frac{\lambda^{(r,s)}\lambda_{12}^{(r,s)} + \lambda_1^{(r,s)}\lambda_3^{(r,s)}}{\lambda_{12}^{(r,s)}\lambda_{23}^{(r,s)}\lambda^{(r,s)}} \end{aligned} \tag{2.7}$$

where $\lambda^{(r,s)} = \lambda_1^{(r,s)} + \lambda_2^{(r,s)} + \lambda_3^{(r,s)}$, $\lambda_{12}^{(r,s)} = \lambda_1^{(r,s)} + \lambda_2^{(r,s)}$, $\lambda_{13}^{(r,s)} = \lambda_1^{(r,s)} + \lambda_3^{(r,s)}$, $\lambda_{23}^{(r,s)} = \lambda_2^{(r,s)} + \lambda_3^{(r,s)}$ and $\lambda_1^{(r,s)}$, $\lambda_2^{(r,s)}$ and $\lambda_3^{(r,s)}$ denote the variates for λ_1, λ_2 and λ_3 drawn in the r-th iteration and the s-th replication where R and S are respectively, the total number of iterations and simulations of the Gibbs sampler.

2.2 Bayes estimators for the reliability function

Assuming the ACBVE model (1.1), the reliability function for a two-component system at specified time t_0 is given by,

$$R_S(t_0) = \exp\{-\lambda t_0\} \quad \text{for a series system, and}$$

$$(2.8) \qquad R_P(t_0) = \frac{\{\lambda(e^{\lambda_1 t_0} + e^{\lambda_2 t_0} - 1) - \lambda_3\}}{\lambda_{12} e^{\lambda t_0}} \quad \text{for a parallel system.}$$

Bayes estimators for $R_S(t_0)$ and $R_P(t_0)$ with respect to the squared error loss function are given by $E\{R_S(t_0) \mid \mathcal{D}\}$ and $E\{R_P(t_0) \mid \mathcal{D}\}$. Based on the Gibbs samplers, with R iterations and S simulations, Monte Carlo estimates for these posterior moments are given by

$$\hat{R_S}(t_0) = \frac{2}{RS} \sum_{s=1}^{S} \sum_{r=R/2+1}^{R} \exp\{-\lambda^{(r,s)} t_0\} \quad \text{for a series system, and}$$

$$(2.9) \qquad \hat{R_P}(t_0) = \frac{2}{RS} \sum_{s=1}^{S} \sum_{r=R/2+1}^{R} \frac{\{\lambda^{(r,s)}(e^{\lambda_1^{(r,s)} t_0} + e^{\lambda_2^{(r,s)} t_0} - 1) - \lambda_3^{(r,s)}\}}{\lambda_{12}^{(r,s)} e^{\lambda^{(r,s)} t_0}}$$

for a parallel system.

3. Bayesian inference for accelerated life tests (Alt) with ACBVE distribution

Consider a two-component lifetimes X and Y, J stress levels $V_1, V_2, \ldots, V_J$ and assume that life tests are conducted at constant application of the selected stresses. Using this information, we get inferences about the component lifetimes under normal stress condition given by V_0.

At a normal stress level V_0 assume that (X, Y) has ACBVE distribution (1.1) with parameters λ_{10}, λ_{20} and λ_{30}. Also assume that under a stress level V_j, $j = 1, 2, \ldots, J$, (X, Y) has the ACBVE model (1.1) with parameters λ_{1j}, λ_{2j} and λ_{3j}, $j = 1, 2, \ldots, J$ and consider the power rule model (Mann *et al.* (1974), Ebrahimi (1987)) given by

$$(3.1) \qquad \lambda_{ij} = c_i V_j^{\mathcal{P}}$$

where $i = 1, 2, 3$, $j = 0, 1, 2, \ldots, J$; c_1, c_2, c_3 and $\mathcal{P}$ are constants. The model (3.1) is also considered by Basu and Ebrahimi (1987).

Considering n_j units $(X_{1j}, Y_{1j}), \ldots, (X_{n_j j}, Y_{n_j j})$ at the beginning of each test with stress V_j, the likelihood function for c_1, c_2, c_3 and $\mathcal{P}$ is given by

$$L_j(c_1, c_2, c_3, \mathcal{P}) = \prod_{i=1}^{nj} f_1^{\delta_{ij}}(X_{ij}, Y_{ij}) f_2^{1-\delta_{ij}}(X_{ij}, Y_{ij})$$

where $\delta_{ij} = 1$ if $X_{ij} < Y_{ij}$ and $\delta_{ij} = 0$ if $X_{ij} \geq Y_{ij}$ and

$$f_1(X_{ij}, Y_{ij}) = \frac{c_1 c_{23} c_{123}}{c_{12}} V_j^{2\mathcal{P}} \exp\{-[c_1 X_{ij} + c_{23} Y_{ij}] V_j^{\mathcal{P}}\},$$

$$f_2(X_{ij}, Y_{ij}) = \frac{c_2 c_{13} c_{123}}{c_{12}} V_j^{2\mathcal{P}} \exp\{-[c_{13} X_{ij} + c_2 Y_{ij}] V_j^{\mathcal{P}}\},$$

$c_{12} = c_1 + c_2$; $c_{13} = c_1 + c_3$; $c_{23} = c_2 + c_3$ and $c_{123} = c_1 + c_2 + c_3$.

That is,

$$L_j(c_1, c_2, c_3, \mathcal{P}) = \frac{c_{123}^{n_j} c_1^{r_j} c_2^{n_j - r_j} c_{23}^{r_j} c_{13}^{n_j - r_j}}{c_{12}^{n_j}} (V_j^{2\mathcal{P}})^{n_j} \cdot \exp\{-[c_1 n_j \bar{X}_j + c_2 n_j \bar{Y}_j + c_3 R_j] V_j^{\mathcal{P}}\} \tag{3.2}$$

where $n_j \bar{X}_j = \sum_{i=1}^{n_j} X_{ij}$, $n_j \bar{Y}_j = \sum_{i=1}^{n_j} Y_{ij}$, $r_j = \sum_{i=1}^{n_j} \delta_{ij}$ and $R_j = \sum_{i=1}^{n_j} [\delta_{ij} Y_{ij} + (1 - \delta_{ij}) X_{ij}]$.

Assuming that the data obtained for the J stress levels $V_1, V_2, \ldots, V_J$ are independent, the likelihood function for c_1, c_2, c_3 and $\mathcal{P}$ is given by

$$L(c_1, c_2, c_3, \mathcal{P}) = \prod_{j=1}^{J} L_j(c_1, c_2, c_3, \mathcal{P}).$$

That is,

$$L(c_1, c_2, c_3, \mathcal{P}) = \frac{c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} \prod_{j=1}^{J} (V_j^{2\mathcal{P}})^{n_j} \cdot \exp\{-[c_1 S_X(\mathcal{P}) + c_2 S_Y(\mathcal{P}) + c_3 T(\mathcal{P})]\} \tag{3.3}$$

where $r = \sum_{j=1}^{J} r_j$, $n = \sum_{j=1}^{J} n_j$, $S_X(\mathcal{P}) = \sum_{j=1}^{J} n_j \bar{X}_j V_j^{\mathcal{P}}$, $S_Y(\mathcal{P}) = \sum_{j=1}^{J} n_j \bar{Y}_j V_j^{\mathcal{P}}$, $T(\mathcal{P}) = \sum_{j=1}^{J} R_j V_j^{\mathcal{P}}$; c_{12}, c_{13}, c_{23} and c_{123} are given in (3.2).

Also, considering the introduction of a latent variable N_1 representing the number of observations such that $X_{i1} < Y_{i1}$, we assume the following prior densities for N_1, c_1, c_2, c_3 and $\mathcal{P}$,

$$\begin{aligned} &N_1 \sim b(n_1, c_1/c_{12}) \\ &c_1 \sim \Gamma(a_1, b_1), \quad a_1 \text{ and } b_1 \text{ known} \\ &c_2 \sim \Gamma(a_2, b_2), \quad a_2 \text{ and } b_2 \text{ known} \\ &c_3 \sim \Gamma(a_3, b_3), \quad a_3 \text{ and } b_3 \text{ known} \\ &\mathcal{P} \sim N(\mu_0, \sigma_0^2), \quad \mu_0,\ \sigma_0^2 \text{ known.} \end{aligned} \tag{3.4}$$

Observe that N_1 has a binomial distribution with probability of success,

$$P(X_{i1} < Y_{i1}) = \frac{\lambda_1}{\lambda_{12}} = \frac{c_1 V_1^{\mathcal{P}}}{c_1 V_1^{\mathcal{P}} + c_2 V_2^{\mathcal{P}}} = \frac{c_1}{c_{12}}$$

$N(\mu_0, \sigma_0^2)$ denotes a normal distribution with mean μ_0 and variance σ_0^2. We also assume independence among the parameters N_1, c_1, c_2, c_3 and $\mathcal{P}$.

The joint posterior density is,

$$\pi(N_1, c_1, c_2, c_3, \mathcal{P} \mid \mathcal{D})$$

$$
\begin{aligned}
&\propto \binom{n}{N_1}\left(\frac{c_1}{c_{12}}\right)^{N_1}\left(\frac{c_2}{c_{12}}\right)^{n_1 \quad N_1}\frac{1}{c_{12}^n}\\
&\cdot c_1^{r+a_1-1}c_2^{n-r+a_2-1}c_3^{a_3-1}c_{23}^{r}c_{13}^{n-r}c_{123}^{n}\\
&\cdot\left\{\prod_{j=1}^{J}(V_j^{2\mathcal{P}})^{n_j}\right\}\exp\left\{-\frac{1}{2\sigma_0^2}(\mathcal{P}-\mu_0)^2\right\}\\
&\cdot\exp\{-[b_1+S_X(\mathcal{P})]c_1-[b_2+S_Y(\mathcal{P})]c_2-[b_3+T(\mathcal{P})]c_3\}
\end{aligned}
$$

where $\mathcal{D}$ denotes the data set.

The conditional posterior densities for the Gibbs algorithm are given by,

$$
\begin{aligned}
&N_1 \mid c_1,c_2,c_3,\mathcal{D}\propto b\left(n_1,\frac{c_1}{c_{12}}\right)\\
&\pi(c_1 \mid N_1,c_2,c_3,\mathcal{P},\mathcal{D})\propto c_1^{a_1-1}\exp\{\ \ b_1c_1\}\psi_1(N_1,c_1,c_2,c_3,\mathcal{P})\\
&\pi(c_2 \mid N_1,c_1,c_3,\mathcal{P},\mathcal{D})\propto c_2^{a_2-1}\exp\{-b_2c_2\}\psi_2(N_1,c_1,c_2,c_3,\mathcal{P})\\
&\pi(c_3 \mid N_1,c_1,c_2,\mathcal{P},\mathcal{D})\propto c_3^{a_3-1}\exp\{-b_3c_3\}\psi_3(N_1,c_1,c_2,c_3,\mathcal{P})\\
&\pi(\mathcal{P} \mid N_1,c_1,c_2,c_3,\mathcal{D})\propto \exp\left\{-\frac{1}{2\sigma_0^2}(\mathcal{P}-\mu_0)^2\right\}\psi_4(N_1,c_1,c_2,c_3,\mathcal{P})
\end{aligned}
\tag{3.5}
$$

where,

$$
\begin{aligned}
\psi_1(N_1,c_1,c_2,c_3,\mathcal{P})&=\frac{c_1^{N_1+r}c_{13}^{n-r}c_{123}^{n}}{c_{12}^{n+n_1}}e^{-S_X(\mathcal{P})c_1},\\
\psi_2(N_1,c_1,c_2,c_3,\mathcal{P})&=\frac{c_2^{n_1-N_1+n-r}c_{23}^{r}c_{123}^{n}}{c_{12}^{n+n_1}}e^{-S_Y(\mathcal{P})c_2},\\
\psi_3(N_1,c_1,c_2,c_3,\mathcal{P})&=c_{23}^{r}c_{13}^{n-r}c_{123}^{n}e^{-T(\mathcal{P})c_3}\qquad\text{and}\\
\psi_4(N_1,c_1,c_2,c_3,\mathcal{P})&=\left\{\prod_{j=1}^{J}(V_j^{2\mathcal{P}})^{n_j}\right\}\exp\{-c_1S_X(\mathcal{P})-c_2S_Y(\mathcal{P})-c_3T(\mathcal{P})\}.
\end{aligned}
$$

Observe that, we need to use the Metropolis-Hastings algorithm to generate the variables c_1, c_2, c_3 and $\mathcal{P}$. In this way, we could draw candidates $c_i^{(r)}$, $i=1,2,3$ from gamma densities $\Gamma(a_i,b_i)$, and draw candidates $\mathcal{P}^{(r)}$ from a normal density $N(\mu_0,\sigma_0^2)$ to be considered in the Metropolis-Hastings algorithm.

3.1 *Bayes estimators for the mean lifetimes*

Considering the Gibbs samples generated from (3.5) for N_1, c_1, c_2, c_3 and $\mathcal{P}$ we can get Monte Carlo estimates for posterior moments of interest. A special case is given by the mean lifetimes in a specified stress level j,

$$
\begin{aligned}
\mu_1^j=E(X)&=\frac{(c_{123}c_{12}+c_2c_3)}{c_{123}c_{12}c_{13}}V_j^{-\mathcal{P}}\\
\mu_2^j=E(Y)&=\frac{(c_{123}c_{12}+c_1c_3)}{c_{123}c_{12}c_{23}}V_j^{-\mathcal{P}}.
\end{aligned}
\tag{3.6}
$$

Bayes estimators for μ_i^j, $i = 1, 2$, $j = 1, \ldots, J$ with respect to the squared error loss function, in a specified stress j are given by $E(\mu_i^j \mid \mathcal{D})$ which can be approximated by its Monte Carlo estimate,

$$(3.7)\qquad \begin{aligned} \hat{\mu_1}^j &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R} \frac{c_{123}^{(r,s)}c_{12}^{(r,s)} + c_2^{(r,s)}c_3^{(r,s)}}{c_{12}^{(r,s)}c_{13}^{(r,s)}c_{123}^{(r,s)}} V_j^{-\mathcal{P}^{(r,s)}} \\ \hat{\mu_2}^j &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R} \frac{c_{123}^{(r,s)}c_{12}^{(r,s)} + c_1^{(r,s)}c_3^{(r,s)}}{c_{12}^{(r,s)}c_{23}^{(r,s)}c_{123}^{(r,s)}} V_j^{-\mathcal{P}^{(r,s)}} \end{aligned}$$

where $c_{123}^{(r,s)} = c_1^{(r,s)} + c_2^{(r,s)} + c_3^{(r,s)}$, $c_{12}^{(r,s)} = c_1^{(r,s)} + c_2^{(r,s)}$, $c_{13}^{(r,s)} = c_1^{(r,s)} + c_3^{(r,s)}$, $c_{23}^{(r,s)} = c_2^{(r,s)} + c_3^{(r,s)}$ and $c_1^{(r,s)}$, $c_2^{(r,s)}$, $c_3^{(r,s)}$ and $\mathcal{P}^{(r,s)}$ denote the variates for c_1, c_2, c_3 and $\mathcal{P}$ drawn in the r-th iteration and the s-th replication where R and S are respectively, the total number of iterations and simulations of the Gibbs sampler.

3.2 *Bayes estimators for the reliability function*

Assuming the ACBVE model (1.1) with parameters λ_{1j}, λ_{2j} and λ_{3j}, $j = 1, \ldots, J$ and considering the power rule model (3.1), the reliability function for a two-component system in a specified stress level j at specified time t_0 is given by,

$$(3.8)\qquad \begin{aligned} R_S^j(t_0) &= \exp\{-c_{123}V_j^{\mathcal{P}}t_0\} \quad \text{for a series system.} \\ R_P^j(t_0) &= \frac{\{c_{123}(e^{c_1 V_j^{\mathcal{P}} t_0} + e^{c_2 V_j^{\mathcal{P}} t_0} - 1) - c_3\}}{c_{12}e^{c_{123}V_j^{\mathcal{P}}t_0}} \quad \text{for a parallel system.} \end{aligned}$$

Bayes estimators for $R_S^j(t_0)$ and $R_P^j(t_0)$ with respect to the squared to the squared error loss function are given by $E\{R_S^j(t_0) \mid \mathcal{D}\}$ and $E\{R_P^j(t_0) \mid \mathcal{D}\}$. Based on the Gibbs samplers, with R iterations and S simulations, Monte Carlo estimates for these posterior moments are given by

$$(3.9)\qquad \begin{aligned} \hat{R_S}^j(t_0) &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R} \exp\{-c_{123}^{(r,s)}V_j^{\mathcal{P}^{(r,s)}}t_0\} \quad \text{for a series system,} \\ \hat{R_P}^j(t_0) &= \frac{2}{RS}\sum_{s=1}^{S}\sum_{r=R/2+1}^{R} \\ &\quad \cdot \frac{\{c_{123}^{(r,s)}(e^{c_1^{(r,s)}V_j^{\mathcal{P}^{(r,s)}}t_0} + e^{c_2^{(r,s)}V_j^{\mathcal{P}^{(r,s)}}t_0} - 1) - c_3^{(r,s)}\}}{c_{12}^{(r,s)}e^{c_{123}^{(r,s)}V_j^{\mathcal{P}^{(r,s)}}t_0}} \\ &\quad \text{for a parallel system.} \end{aligned}$$

4. Numerical illustrations

4.1 *Example* 1

In Table 1, we have 30 bivariate observations (X, Y) generated from a ACBVE with density (1.1) and parameters $\lambda_1 = 0.25$, $\lambda_2 = 0.16$ and $\lambda_3 = 0$. From the data of Table 1, we have $r = 16$, $n-r = 14$, $n = 30$, $\sum_{i=1}^{30} x_i = 114.51$, $\sum_{i=1}^{30} y_i = 165.67$ and $R = 207.77$ (see (2.1)).

Considering (from (2.2)) the priors $N_1 \sim b(30, \lambda_1/\lambda_{12})$, $\lambda_1 \sim \Gamma(a_1, b_1)$, $\lambda_2 \sim \Gamma(a_2, b_2)$ and $\lambda_3 \sim \Gamma(a_3, b_3)$, where $a_1 = 115$, $b_1 = 480$, $a_2 = 56$, $b_2 = 338$, $a_3 = 20$ and $b_3 = 1400$, we generated 10 separate Gibbs chains each of which ran for 2000 iterations, and we monitored the convergence of the Gibbs samplers using the Gelman and Rubin (1992) method that uses the analysis of variance technique to determine if further iterations are needed.

Table 1. ACBVE generated bivariate life time data with $\lambda_1 = 0.25$, $\lambda_2 = 0.16$ and $\lambda_3 = 0$.

i	x	y	i	x	y
1	3.73	2.54	16	3.42	1.09
2	5.83	7.74	17	7.71	0.33
3	8.44	9.89	18	6.92	2.59
4	7.95	2.47	19	7.76	3.77
5	7.66	8.77	20	0.16	6.07
6	3.47	1.86	21	7.79	6.98
7	2.75	1.30	22	0.66	0.49
8	0.57	5.04	23	10.83	4.03
9	3.48	1.13	24	4.23	2.71
10	4.12	7.24	25	3.23	18.74
11	2.08	9.40	26	1.00	9.10
12	4.19	1.50	27	3.08	12.43
13	0.82	6.29	28	0.55	13.50
14	1.14	2.61	29	0.37	5.52
15	0.18	8.17	30	0.39	2.37

For each parameter we considered the 1010th, 1020th, …, 2000th, iteration, which for 10 chains yields a sample of size 1000. In Table 2, we have the obtained posterior summaries for the parameters λ_1, λ_2 and λ_3, and in Fig. 1, we have the approximate marginal posterior densities considering the $S = 1000$ Gibbs samples. We also have in Table 2 the estimated potential scale reductions $\hat{R}$ (Gelman and Rubin (1992)) for all parameters. In this case 1000 iterations were sufficient for approximate convergence ($\sqrt{\hat{R}} < 1.1$ for all parameters).

It is interesting to observe that the maximum likelihood estimates for λ_1, λ_2 and λ_3 are given by $\hat{\lambda_1} = 0.2485$, $\hat{\lambda_2} = 0.1698$ and $\hat{\lambda_3} = 0.0164$. Usually, we can have very poor accuracy for the inferences based on the usual normal limiting

Table 2. Posterior summaries for the ACBVE model.

	Mean	Median	S. D	95% credible interval	$\hat{R}$
λ_1	0.2527	0.2517	0.0223	(0.2051; 0.2942)	1.0503
λ_2	0.1651	0.1646	0.0168	(0.1342; 0.2010)	1.0424
λ_3	0.0163	0.0161	0.0036	(0.0098; 0.0249)	1.0109

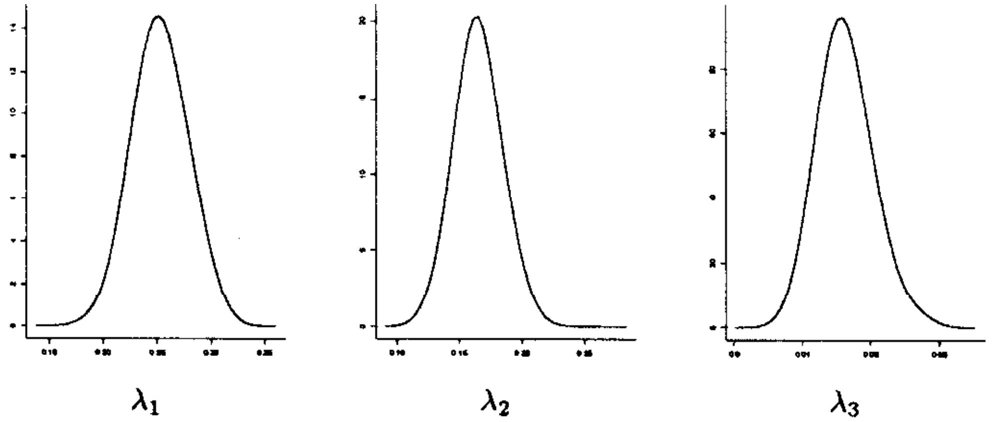

Fig. 1. Approximate marginal posterior densities for λ_1, λ_2 and λ_3.

Table 3. Bayes estimators for $R_s(t_0)$ and $R_p(t_0)$.

	True values		Monte Carlo estimates	
t_0	$R_S(t_0)$	$R_P(t_0)$	$R_S(t_0)$	$R_P(t_0)$
1	0.6636	0.9673	0.6481	0.9622
2	0.4404	0.8922	0.4204	0.8774
3	0.2923	0.7989	0.2730	0.7742
4	0.1940	0.7012	0.1774	0.6686
5	0.1287	0.6071	0.1154	0.5691
10	0.0166	0.2674	0.0136	0.2296
15	0.0021	0.1121	0.0016	0.0884

distribution for the maximum likelihood estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$. The approximate standard errors for $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ based on the Fisher information matrix are given by 0.0906, 0.0820 and 0.1017, respectively.

From (2.7), we get Monte Carlo estimates for μ_1 and μ_2 given by $\hat{\mu_1} = 3.8009$ and $\hat{\mu_2} = 5.6903$. In Table 3, we have Monte Carlo estimates (from (2.9)) for the Bayes estimators with respect to squared error for the reliability function of two-component systems at some values of t_0 considering series and parallel systems.

Table 4. Generated bivariate life time data.

$V_1 = 1$	$V_2 = 2$	$V_3 = 3$
(X_{1i}, Y_{1i})	(X_{2i}, Y_{2i})	(X_{3i}, Y_{3i})
(7.65; 2.18)	(2.29; 0.02)	(0.34; 0.20)
(16.67; 9.26)	(0.10; 0.38)	(1.50; 1.30)
(30.30; 6.72)	(0.88; 0.27)	(0.63; 0.60)
(1.30; 3.22)	(0.45; 0.04)	(0.68; 0.12)
(9.04; 2.23)	(1.66; 1.60)	(3.22; 0.09)
(5.15; 0.41)	(0.74; 1.67)	(1.91; 0.91)
(5.20; 5.91)	(2.50; 0.37)	(0.52; 0.58)
(5.00; 0.84)	(3.50; 0.03)	(0.30; 0.01)
(5.66; 0.42)	(8.45; 0.71)	(1.30; 0.02)
(11.80; 0.15)	(4.60; 0.83)	(0.52; 0.10)
(17.08; 10.37)	(2.66; 1.06)	(2.08; 0.30)
(17.92; 0.76)	(1.46; 1.04)	(0.95; 0.91)
(1.62; 2.73)	(1.03; 0.41)	(0.43; 0.02)
(1.42; 1.85)	(4.36; 1.34)	(0.25; 0.08)
(3.60; 1.50)	(0.76; 0.77)	(1.39; 0.08)

Table 5. Posterior summaries for ALT with ACBVE model.

	Mean	Median	S. D	95% credible interval	$\hat{R}$
c_1	0.0599	0.0589	0.0518	(0.03116; 0.09542)	1.0139
c_2	0.2669	0.2640	0.1447	(0.18867; 0.36606)	1.0151
c_3	0.0588	0.0571	0.0682	(0.00288; 0.10454)	1.0078
$\mathcal{P}$	2.0080	1.9999	0.6967	(1.60068; 2.46957)	0.9989

4.2 *Example* 2

In Table 4, we have the data of an accelerated life test considering three stress levels $V_1 = 1$, $V_2 = 2$ and $V_3 = 3$. At each stress level, 15 bivariate observations (X, Y) were generated from a ACBVE distribution and the power rule model (3.1) with $\mathcal{P} = 2$, $c_1 = 0.05$, $c_2 = 0.25$ and $c_3 = 0.00$. From Table 4, we get $n_1 = n_2 = n_3 = 15$, $n = 45$, $r_1 = 4$, $r_2 = 3$, $r_3 = 2$ and $r = 9$. (See (3.2) and (3.3).)

Considering from (3.4) the priors $N_1 \sim b(15, c_1/c_{12})$, $c_1 \sim \Gamma(a_1, b_1)$, $c_2 \sim \Gamma(a_2, b_2)$, $c_3 \sim \Gamma(a_3, b_3)$, and $\mathcal{P} \sim N(\mu_0, (\sigma_0^2))$, where $a_1 = 5$, $b_1 = 72$, $a_2 = 15$, $b_2 = 54$, $a_3 = 3$, $b_3 = 41$, $\mu_0 = 2$ and $\sigma_0^2 = 0.25$, we generated 10 separate Gibbs chains each of which ran for 2000 iterations. For each parameter, we considered the 1010th, 1020th, ..., 2000th iteration, which for 10 chains yields a sample of size 1000. In Table 5, we have the obtained posterior summaries for the parameters c_1, c_2, c_3 and $\mathcal{P}$, and the estimated potential scale reductions $\hat{R}$ for all parameters.

Monte Carlo estimates for the mean lifetimes under the normal stress level $V_1 = 1$ (see (3.7)) using the 1000 Gibbs samplers are given by $\hat{\mu}_1 = 9.76075$ and

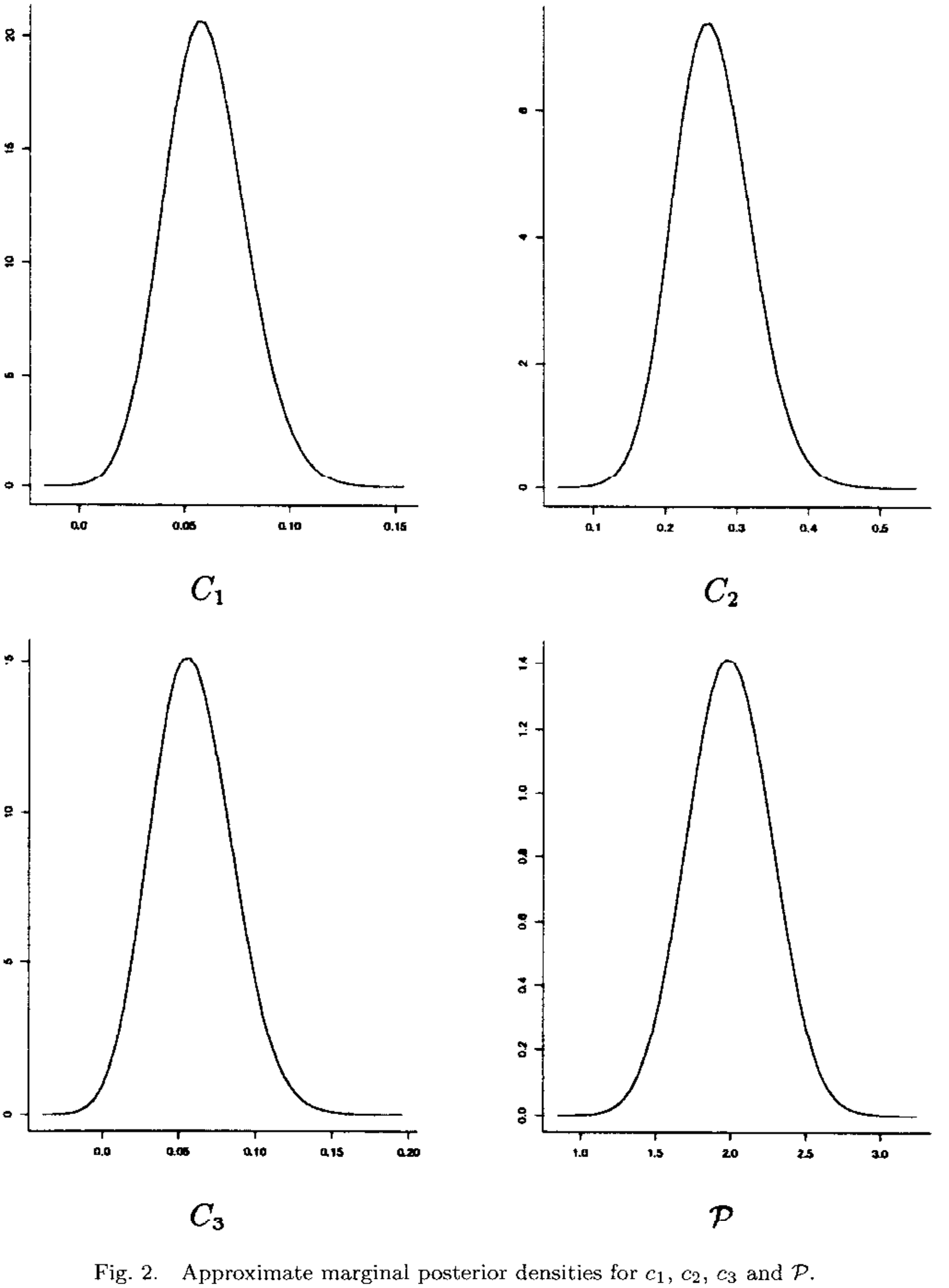

Fig. 2. Approximate marginal posterior densities for c_1, c_2, c_3 and $\mathcal{P}$.

$\hat{\mu_2} = 3.22346$.

In Table 6, we have Monte Carlo estimates (from (3.9)) for the Bayes estimators with respect to squared error for the reliability function of two-component systems at some values t_0 considering series and parallel systems in a specified

Table 6. Bayes estimators for $R_S(t_0)$ and $R_P(t_0)$.

	True values		Monte Carlo estimates	
t_0	$R_S(t_0)$	$R_P(t_0)$	$R_S(t_0)$	$R_P(t_0)$
1	0.6828	0.9756	0.6809	0.9750
2	0.4602	0.9186	0.4648	0.9171
3	0.3184	0.8460	0.3180	0.8439
4	0.2174	0.7681	0.2182	0.7658
5	0.1484	0.6908	0.1500	0.6827
10	0.0220	0.3832	0.0239	0.3849

stress level j.

It is interesting to observe that the maximum likelihood estimators for c_1, c_2, c_3 and $\mathcal{P}$ are given by $\hat{c_1} = 0.0571$, $\hat{c_2} = 0.2643$, $\hat{c_3} = 0.0602$ and $\hat{\mathcal{P}} = 2.03$. The approximate standard errors for $\hat{c_1}$, $\hat{c_2}$, $\hat{c_3}$ and $\hat{\mathcal{P}}$ based on the observed information matrix are given by 0.0245, 0.0671, 0.0332 and 0.2300, respectively.

In Fig. 2, we have the approximate marginal posterior densities for c_1, c_2, c_3 and $\mathcal{P}$ considering the $S = 1000$ Gibbs samples.

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