Ann. Inst. Statist. Math.
Vol. 50, No. 2, 325–335 (1998)

THE NONEXISTENCE OF PROCEDURES WITH BOUNDED PERFORMANCE CHARACTERISTICS IN CERTAIN PARAMETRIC INFERENCE PROBLEMS

YOSHIKAZU TAKADA

Department of Mathematics, Faculty of Science, Kumamoto University, Kumamoto 860-0862, Japan

(Received November 21, 1996; revised May 29, 1997)

Abstract. This paper gives a condition which implies the nonexistence of parametric statistical procedures with bounded risk or error performance characteristics. Many examples for which such a condition is satisfied are considered.

Key words and phrases: Fixed-width confidence interval, location-scale family, scale family, inverse Gaussian distribution, errors-in-variables regression model, calibration problem.

1. Introduction

Let $\boldsymbol{X}$ be a random vector with distribution P_θ, $\theta \in \Theta$, where Θ is a parameter space. Consider the following two goals of making inferences about a function $\tau = \tau(\theta)$. We assume that τ is real-valued for simplicity, though an extension to a vector-valued function is easy.

(I) *Bounded risk.* Let $L(\theta, a) = \omega(|a - \tau(\theta)|)$ be a loss function, where $\omega(u)(\geq 0)$ is a given function nondecreasing in $[0, \infty)$ and $\sup_{u \geq 0} \omega(u) = M(\leq \infty)$. Then the goal is to find if there exists an estimator $\hat{\tau}(\boldsymbol{X})$ of τ such that

$$E_\theta\{L(\theta, \hat{\tau}(\boldsymbol{X}))\} \leq W, \tag{1.1}$$

where $W(0 < W < M)$ is a given constant.

(II) *Hypothesis testing.* Consider a hypothesis testing $H_0 : \tau = \tau_0$ against $H_1 : \tau = \tau_1$ $(\tau_0 \neq \tau_1)$. Then the goal is to find if there exists a critical function $\phi(\boldsymbol{X})$ such that the probability of error of first kind is less than α $(0 < \alpha < 1)$ and the power is greater than $1 - \beta$ $(0 < \beta < 1)$, where α and β are given constants.

For example, let $\omega(u) = 1$ for $u \geq \rho(> 0)$ and zero otherwise. Then the goal (I) is equivalent to finding if there exists an estimator $\hat{\tau}(\boldsymbol{X})$ of τ such that

$$P_\theta\{|\hat{\tau}(\boldsymbol{X}) - \tau| < \rho\} \geq 1 - \alpha,$$

where α $(0 < \alpha < 1)$ is a given constant (*fixed-width confidence interval*).

For the location parameter of the location-scale family, Lehmann (1951) showed that there do not exist estimators for the goal (I). Stein (1945) proved a result which implies the nonexistence of critical regions for the goal (II) with $\alpha < 1 - \beta$ for testing the mean of a normal population with unknown variance; but his arguments can be extended to the general location-scale family. For the details, see Chatterjee (1991) (cf. Takada (1986, 1988)). Blum and Rosenblatt (1969) got the similar result for the scale family. See also Hoeffding and Wolfowitz (1958), Singh (1963), and Gleser and Hwang (1987). The purpose of this paper is to give a condition for the nonexistence of statistical procedures in terms of the distance of distributions.

In Section 2 we consider multi-parameter distributions. Applications are considered in Section 3, in which we treat the location-scale family, errors-in-variables regression model, and calibration problem. Section 4 considers one-parameter distributions.

2. Multi-parameter distributions

We define a distance between probability distributions by

$$d(\theta, \theta') = \sup_{A \in \mathcal{B}} |P_\theta(A) - P_{\theta'}(A)| \tag{2.1}$$

for $\theta, \theta' \in \Theta$, where $\mathcal{B}$ denotes the Borel sigma field of the sample space of $\boldsymbol{X}$. Then it holds that

$$d(\theta, \theta') = \sup_{\phi \in \Phi} |E_\theta \phi(\boldsymbol{X}) - E_{\theta'} \phi(\boldsymbol{X})| \tag{2.2}$$

and

$$d(\theta, \theta') = \frac{1}{2} \int |g(\boldsymbol{x} \mid \theta) - g(\boldsymbol{x} \mid \theta')| \nu(d\boldsymbol{x}), \tag{2.3}$$

where Φ is the class of all functions ϕ with $0 \le \phi \le 1$ and $g(\boldsymbol{x} \mid \theta)$ is the density function of $\boldsymbol{X}$ with respect to a measure ν. See Hoeffding and Wolfowitz (1958), p. 709.

In this section we assume $\theta = (\tau, \lambda)$ with some vector λ and the range of τ is unbounded. Consider the following condition; for any τ and $\tau' (\tau \neq \tau')$

$$\lim_{\lambda \to a} d(\theta, \theta') = 0, \tag{2.4}$$

where $\theta = (\tau, \lambda)$, $\theta' = (\tau', \lambda)$ and a is some constant vector or infinity. Then we get the following two theorems. Though the essence of the proofs are almost the same as those of Stein (1945) and Lehmann (1951) (see Chatterjee (1991)), they are added for the sake of completeness.

THEOREM 2.1. *If* (2.4) *is satisfied, then there do not exist estimators for the goal* (I).

PROOF. Suppose that there exists an estimator $\hat{\tau}(\boldsymbol{X})$ of τ such that (1.1) is satisfied. By the *Markov inequality*, it holds that for any $\rho(>0)$

$$E_\theta\{L(\theta,\hat{\tau}(\boldsymbol{X}))\} \geq \omega(\rho)P_\theta\{|\hat{\tau}(\boldsymbol{X})-\tau| \geq \rho\}.$$

Hence for ρ with $\omega(\rho)>0$

$$P_\theta\{|\hat{\tau}(\boldsymbol{X})-\tau| < \rho\} \geq 1 - E_\theta\{L(\theta,\hat{\tau}(\boldsymbol{X}))\}/\omega(\rho) \geq 1-\alpha, \tag{2.5}$$

where $\alpha = W/\omega(\rho)$. Since $W < M$, there exists a $\rho(>0)$ such that $0<\alpha<1$.

Let r be any positive integer. Choose $\tau_1,\ldots,\tau_r$ such that $|\tau_i-\tau_j|>2\rho$ for $i\neq j$. Let

$$C_i = \{\boldsymbol{x}; |\hat{\tau}(\boldsymbol{x})-\tau_i| < \rho\}.$$

Then $C_i \cap C_j = \emptyset$ for $i \neq j$ and from (2.5)

$$P_{\theta_i}(C_i) \geq 1-\alpha, \tag{2.6}$$

where $\theta_i = (\tau_i, \lambda)$. It follows from (2.1) and (2.4) that there exists a λ such that

$$|P_{\theta_1}(C_i) - P_{\theta_i}(C_i)| < (1-\alpha)/2$$

for $i=2,\ldots,r$. So from (2.6) we get

$$P_{\theta_1}(C_i) > (1-\alpha)/2.$$

Because the C_i's are mutually exclusive,

$$P_{\theta_1}\left(\bigcup_{i=2}^{r} C_i\right) = \sum_{i=2}^{r} P_{\theta_1}(C_i) > (r-1)(1-\alpha)/2,$$

which is impossible since r is an arbitary integer. So the proof is completed.

THEOREM 2.2. *If* (2.4) *is satisfied, then there do not exist critical functions for the goal* (II) *if* $\alpha < 1-\beta$.

PROOF. Suppose that there exists a critical function ϕ such that

$$E_{\theta_0}\phi(\boldsymbol{X}) \leq \alpha \tag{2.7}$$

and

$$E_{\theta_1}\phi(\boldsymbol{X}) \geq 1-\beta \tag{2.8}$$

for $\theta_0 = (\tau_0,\lambda)$ and $\theta_1 = (\tau_1,\lambda)$ with arbitary λ. For any fixed λ, consider the hypothesis test of $H_o : \tau = \tau_0$ against $H_1 : \tau = \tau_1$ with significance level α. Then from the *Neyman-Pearson lemma* there exists the most powerful critical function ϕ_λ such that

$$E_{\theta_0}\phi_\lambda(\boldsymbol{X}) = \alpha \tag{2.9}$$

and from (2.7) and (2.8)

$$E_{\theta_1}\phi_\lambda(\boldsymbol{X}) \geq 1-\beta. \tag{2.10}$$

It follows from (2.2) and (2.4) that

$$\lim_{\lambda\to a} |E_{\theta_1}\phi_\lambda(\boldsymbol{X}) - E_{\theta_0}\phi_\lambda(\boldsymbol{X})| = 0,$$

which implies

$$\lim_{\lambda\to a} E_{\theta_1}\phi_\lambda(\boldsymbol{X}) = \alpha$$

by (2.9). Hence from (2.10) it is necessary for the existence of a critical function that $\alpha \geq 1-\beta$, which completes the proof.

3. Applications

3.1 *Independent and identical distributions*

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with a common density function $f(x \mid \theta)$, $\theta \in \Theta$ with respect to a measure η. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and

$$d^{(1)}(\theta, \theta') = \frac{1}{2}\int |f(x \mid \theta) - f(x \mid \theta')|\eta(dx). \tag{3.1}$$

Then it holds that

$$d(\theta, \theta') \leq nd^{(1)}(\theta, \theta').$$

See Hoeffding and Wolfowitz (1958), p. 709. Hence if

$$\lim_{\lambda\to a} d^{(1)}(\theta, \theta') = 0, \tag{3.2}$$

then the condition (2.4) is satisfied for any n, and hence there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

Example 1. (Location-scale family) Suppose that the common distribution belongs to a location-scale family with density

$$f(x \mid \theta) = \sigma^{-1}f((x-\mu)/\sigma),$$

where $\theta = (\mu, \sigma)$, $\sigma > 0$ and $f(x)$ is continuous (a.e.). We consider such a function $\gamma(\theta)$ that for each fixed σ the transformation $\gamma = \gamma(\theta)$ from μ to γ is one-to-one.

Let the inverse transformation be $\mu = \psi(\tau, \sigma)$ and let $\theta = (\tau, \sigma)$ and $\theta' = (\tau', \sigma)$. Then it follows from (3.1) that

$$d^{(1)}(\theta, \theta') = \frac{1}{2} \int \left| f(x) - f\left(x + \frac{\mu - \mu'}{\sigma}\right) \right| dx,$$

where $\mu' = \psi(\tau', \sigma)$. Hence if for any $\tau \neq \tau'$

$$\lim_{\sigma \to \infty} \frac{\psi(\tau, \sigma) - \psi(\tau', \sigma)}{\sigma} = 0, \tag{3.3}$$

then (3.2) is satisfied, so that there do not exist fixed sample size statistical procedures for the goals (I) or (II).

More generally, if the distribution of a transform $h(X_i)$ of X_i, where $h'(x) > 0$ exists for all x, belongs to a location-scale family for which (3.3) holds, then there do not exist fixed sample size statistical procedures for the goals (I) or (II).

Example 2. (Lognormal distribution) Suppose that the common distribution is lognormal, that is, $\log X_i$'s are distributed as i.i.d. normal random variables with mean μ and variance σ^2. Let $\tau = \exp(\mu + \sigma^2/2)$. Then τ is the mean of the lognormal distribution. It is easy to see that (3.3) is satisfied.

Example 3. (Weibull distribution) Suppose that the common distribution is a Weibull distribution with density

$$f(x \mid \beta, \eta) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta - 1} \exp\left(-\left(\frac{x}{\eta}\right)^{\beta}\right), \qquad x > 0,$$

where $\beta > 0$ and $\eta > 0$. Let $\tau = \eta\Gamma(\frac{\beta+1}{\beta})$. Then τ is the mean of the Weibull distribution. It is easy to see that the density of $Y_i = \log X_i$ is

$$f(y \mid \theta) = \frac{1}{\sigma} e^{(y-\mu)/\sigma} \exp[-e^{(y-\mu)/\sigma}],$$

where $\theta = (\mu, \sigma)$, $\sigma = 1/\beta$ and $\mu = \log \eta$. Hence the distribution belongs to the location-scale family. Since $\tau = e^{\mu}\Gamma(1 + \sigma)$, (3.3) is satisfied.

Example 4. (Pareto distribution) Suppose that the common distribution is a Pareto distribution with density

$$f(x \mid k, a) = ak^a x^{-(a+1)}, \qquad x > k,$$

where $k > 0$ and $a > 0$. Let $\tau = 2^{1/a}k$. Then τ is the median of the Pareto distribution. It is easy to see that the distribution of $Y_i = \log X_i$ is exponential with density

$$f(y \mid \theta) = \sigma^{-1} \exp(-(y - \mu)/\sigma), \qquad y > \mu,$$

where $\theta = (\mu, \sigma)$, $\mu = \log k$ and $\sigma = 1/a$. Hence the distribution belongs to the location-scale family. Since $\tau = 2^{\sigma} e^{\mu}$, (3.3) is satisfied.

Example 5. (Inverse Gaussian distribution) Suppose that the common distribution is an inverse Gaussian distribution with density

$$f(x \mid \theta) = \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \qquad x > 0,$$

where $\theta = (\mu, \lambda)$, $\mu > 0$ and $\lambda > 0$. We consider the parameter μ which is the mean of the distribution. Let $\theta = (\mu, \lambda)$ and $\theta' = (\mu', \lambda)$ $(\mu' > \mu)$. Note that $f(x \mid \theta') > f(x \mid \theta)$ if and only if $x > x_0$, where $x_0 = \frac{2}{1/\mu + 1/\mu'}$. So from (3.1) we get

$$\begin{aligned} d^{(1)}(\theta, \theta') &= \frac{1}{2}\left\{\int_0^{x_0} (f(x \mid \theta) - f(x \mid \theta'))dx \right. \\ &\qquad \left. + \int_{x_0}^{\infty} (f(x \mid \theta') - f(x \mid \theta))dx\right\} \\ &= \int_0^{x_0} (f(x \mid \theta) - f(x \mid \theta'))dx \\ &= \Phi\left\{\sqrt{\frac{\lambda}{x_0}}\left(\frac{x_0}{\mu} - 1\right)\right\} - \Phi\left\{\sqrt{\frac{\lambda}{x_0}}\left(\frac{x_0}{\mu'} - 1\right)\right\} \\ &\quad + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{x_0}}\left(\frac{x_0}{\mu} + 1\right)\right\} \\ &\quad - e^{2\lambda/\mu'}\Phi\left\{-\sqrt{\frac{\lambda}{x_0}}\left(\frac{x_0}{\mu'} + 1\right)\right\}. \end{aligned} \tag{3.4}$$

See Shuster (1968). Since the right hand side of (3.4) converges to zero as $\lambda \to 0$, there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

Example 6. (Trinomial distribution) Suppose that the random variables X_i's have three possible outcomes A, B, C with

$$P(X_i = A) = p_1, \qquad P(X_i = B) = p_2, \qquad P(X_i = C) = 1 - p_1 - p_2,$$

where $0 < p_i < 1$, $i = 1, 2$ and $0 < p_1 + p_2 < 1$. Let $\tau = p_2/p_1$ and $\theta = (\tau, p_1)$. Then from (3.1) we get

$$d^{(1)}(\theta, \theta') = p_1 |\tau - \tau'|,$$

where $\theta' = (\tau', p_1)$. Hence $\lim_{p_1 \to 0} d^{(1)}(\theta, \theta') = 0$, which implies that there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

3.2 *Errors-in-variables regression model*

Consider random variables X_{i1} and X_{i2} satisfying

$$X_{i1} = u_i + \epsilon_{i1},$$
$$X_{i2} = \gamma + \delta u_i + \epsilon_{i2},$$

$i = 1, \dots, n$, where γ, δ are unknown constants, u_i are unknown, and $(\epsilon_{i1}, \epsilon_{i2})$ are i.i.d. with density function $f(v_1, v_2)$ which is continuous (a.e.). The model is called an errors-in-variables regression model (or a measurement error model). The model with fixed constants u_i is called a functional model, while if u_i are i.i.d., the model is called a structural model. We consider inference for δ.

First we consider a functional model. Let $\boldsymbol{X} = (X_{11}, X_{12}, \dots, X_{n1}, X_{n2})$. Then the density function of $\boldsymbol{X}$ is given by

$$g(\boldsymbol{x} \mid \theta) = \prod_{i=1}^{n} f(x_{i1} - u_i, x_{i2} - \gamma - \delta u_i),$$

where $\theta = (\delta, \gamma, u_1, \dots, u_n)$. Since

$$\lim_{\substack{\gamma\to 0 \\ u_i \to 0, i=1,\dots,n}} g(\boldsymbol{x} \mid \theta) = \prod_{i=1}^{n} f(x_{i1}, x_{i2}) \qquad \text{(a.e.)},$$

from (2.3) we get

$$\lim_{\substack{\gamma\to 0 \\ u_i \to 0, i=1,\dots,n}} d(\theta, \theta') = 0,$$

where $\theta' = (\delta', \gamma, u_1, \dots, u_n)$ $(\delta \neq \delta')$. So there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

Next we consider a structural model. Assume that u_i are independent of $(\epsilon_{i1}, \epsilon_{i2})$ and the common distribution of u_i belongs to a location-scale family with density $\frac{1}{\sigma} h(\frac{u-\mu}{\sigma})$, where μ and $\sigma (> 0)$ are unknown. The density function of $\boldsymbol{X}$ becomes

$$\begin{aligned} g(\boldsymbol{x} \mid \theta) &= \prod_{i=1}^{n} \int f(x_{i1} - u, x_{i2} - \gamma - \delta u) \frac{1}{\sigma} h\left(\frac{u-\mu}{\sigma}\right) du \\ &= \prod_{i=1}^{n} \int f(x_{i1} - \mu - \sigma u, x_{i2} - \gamma - \delta(\mu + \sigma u)) h(u) du, \end{aligned}$$

where $\theta = (\delta, \gamma, \mu, \sigma)$. It is easy to see that

$$\lim_{\substack{\gamma\to 0 \\ \mu \to 0, \sigma \to 0}} g(\boldsymbol{x} \mid \theta) = \prod_{i=1}^{n} f(x_{i1}, x_{i2}) \qquad \text{(a.e.)}.$$

Then from (2.3) we get

$$\lim_{\substack{\gamma\to 0 \\ \mu \to 0, \sigma \to 0}} d(\theta, \theta') = 0,$$

where $\theta' = (\delta', \gamma, \mu, \sigma)$ $(\delta \neq \delta')$. Hence there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

Remark 1. In an errors-in-variable regression model Gleser and Hwang (1987) showed the nonexistence of confidence intervals of finite expected length for δ, which is stronger than our result for the goal (I).

3.3 *Calibration problem*

Consider the linear regression model

$$X_i = \gamma + \delta z_i + \epsilon_i, \qquad i = 1, \ldots, n$$
$$X_{n+1} = \gamma + \delta z_{n+1} + \epsilon_{n+1},$$

where γ, δ are unknown, z_i, $i = 1, \ldots, n$, are known constants and z_{n+1} is an unknown constant. We assume that ϵ_i are i.i.d. and the common distribution has a density $f(u)$ which is continuous (a.e.). The calibration problem is to determine the value of z_{n+1} by using $\boldsymbol{X} = (X_1, \ldots, X_n, X_{n+1})$.

The density of $\boldsymbol{X}$ is given by

$$g(\boldsymbol{x} \mid \theta) = \left\{ \prod_{i=1}^{n} f(x_i - \gamma - \delta z_i) \right\} f(x_{n+1} - \gamma - \delta z_{n+1}),$$

where $\theta = (z_{n+1}, \gamma, \delta)$. Since

$$\lim_{\substack{\gamma \to 0 \\ \delta \to 0}} g(\boldsymbol{x} \mid \theta) = \left\{ \prod_{i=1}^{n} f(x_i) \right\} f(x_{n+1}) \qquad \text{(a.e.)},$$

from (2.3) we get

$$\lim_{\substack{\gamma \to 0 \\ \delta \to 0}} d(\theta, \theta') = 0$$

for $\theta' = (z'_{n+1}, \gamma, \delta)$ $(z_{n+1} \neq z'_{n+1})$. Hence there do not exist fixed sample size statistical procedures for any of the goals (I) or (II).

4. One-parameter distributions

In this section we consider only the goal (I) for one-parameter distributions. Goal (II) is not considered because for many examples there exists a consistent estimator of τ from which it is possible to construct a critical function that satisfies goal (II) for a sufficiently large sample size.

We assume that $\Theta \supset (c, \infty)$ for some constant c and $\tau(\theta) = \theta$. Consider the following condition;

$$\text{(4.1)} \qquad \lim_{\lambda \to \infty} d(\theta + \lambda, \theta' + \lambda) = 0$$

for any $\theta \neq \theta' (> c)$.

THEOREM 4.1. *If* (4.1) *is satisfied, then there do not exist estimators for the goal* (I).

PROOF. Suppose that there exists an estimator $\hat{\theta}(\boldsymbol{X})$ of θ such that (1.1) is satisfied. Then as is in the proof of Theorem 2.1, it can be shown that there exist $\rho(>0)$ and $\alpha(0<\alpha<1)$ such that

$$P_\theta\{|\hat{\theta}(\boldsymbol{X})-\theta|<\rho\}\geq 1-\alpha. \tag{4.2}$$

For any positive integer $r>0$ choose $\theta_1,\dots,\theta_r(>c)$ such that

$$|\theta_i-\theta_j|>2\rho \qquad (i\neq j),$$

which implies that $C_i\cap C_j=\emptyset$ $(i\neq j)$ for any λ where

$$C_i=\{\boldsymbol{x};|\hat{\theta}(\boldsymbol{x})-(\theta_i+\lambda)|<\rho\}.$$

From (4.2) we get

$$P_{\theta_i+\lambda}(C_i)\geq 1-\alpha. \tag{4.3}$$

From (2.1) and (4.1), there exists a sufficiently large λ such that

$$|P_{\theta_1+\lambda}(C_i)-P_{\theta_i+\lambda}(C_i)|\leq(1-\alpha)/2$$

for $i=2,\dots,r$. So from (4.3) we get

$$P_{\theta_1+\lambda}(C_i)>(1-\alpha)/2.$$

Since the C_i's are mutually exclusive,

$$P_{\theta_1+\lambda}(\cup_{i=2}^r C_i)>(r-1)(1-\alpha)/2,$$

which is impossible since r is arbitary. So the proof is completed.

Example 7. (Binomial distribution) Suppose that the random variables X_i's are distributed as

$$P(X_i=1)=p, \qquad P(X_i=0)=1-p \qquad (0<p<1).$$

Let $\theta=p/(1-p)$. Then it follows from (3.1) that

$$d^{(1)}(\theta+\lambda,\theta'+\lambda)=\left|\frac{\theta+\lambda}{1+\theta+\lambda}-\frac{\theta'+\lambda}{1+\theta'+\lambda}\right| \tag{4.4}$$

for $\theta\neq\theta'(>0)$. The right hand side of (4.4) converges to zero as $\lambda\to\infty$, and hence (4.1) is satisfied. So there do not exist fixed sample size estimators for the goal (I). The same result also holds for $\theta=1/p$.

Example 8. (Scale family) Suppose that the common distribution belongs to a scale family with density

$$f(x \mid \sigma) = \sigma^{-1} f(x/\sigma),$$

where $\sigma > 0$ and $f(x)$ is continuous (a.e). Let $\theta = \psi(\sigma)$ where ψ is a one-to-one function and let $\phi(\theta)$ be the inverse function of ψ. Then it follows from (3.1) that

$$d^{(1)}(\theta + \lambda, \theta' + \lambda) = \frac{1}{2} \int \left| f(x) - \frac{\phi(\theta + \lambda)}{\phi(\theta' + \lambda)} f\left(\frac{\phi(\theta + \lambda)}{\phi(\theta' + \lambda)} x \right) \right| dx.$$

Hence if

$$\lim_{\lambda \to \infty} \frac{\phi(\theta + \lambda)}{\phi(\theta' + \lambda)} = 1, \tag{4.5}$$

then (4.1) is satisfied. So there do not exist fixed sample size estimators for the goal (I). The result of Blum and Rosenblatt (1969) for the goal (I) with $\theta = \sigma$ follows from (4.5).

Assume $\mu = 0$ in Example 2 and $k = 1$ in Example 4. Then the distribution of the $\log X_i$'s belongs to the scale-family, and it is easy to see that (4.5) is satisfied. So there do not exist fixed sample size estimators for the goal (I).

Remark 2. We have not discussed methods to obtain statistical procedures to satisfy the goals (I) or (II) when there do not exist fixed sample size statistical procedures. For some problems we can employ two-stage procedures to obtain solutions for the goals (I) or (II) (e.g. Section 10 of Zacks (1971) and Chatterjee (1991)).

Acknowledgements

The author is very grateful to the two referees for their helpful comments, which led to a greatly improved revision of his initial paper. Especially, he owes the proof of Theorem 4.1 under the condition (4.1) to the suggestion of the referee.

References

Blum, J. R. and Rosenblatt, J. (1969). Fixed precision estimation in the class of IFR distributions, *Ann. Inst. Statist. Math.*, **21**, 211–213.

Chatterjee, S. K. (1991). Two-stage and multistage procedures, *Handbook of Sequential Analysis* (ed. B. K. Ghosh and P. K. Sen), 21–45, Marcel Dekker, New York.

Gleser, L. J. and Hwang, J. T. (1987). The nonexistence of $100(1 - \alpha)\%$ confidence sets of finite expected diameter in errors-in-variables and related models, *Ann. Statist.*, **15**, 1351–1362.

Hoeffding, W. and Wolfowitz, J. (1958). Distinguishability of sets of distributions, *Ann. Math. Statist.*, **29**, 700–718.

Lehmann, E. L. (1951). *Notes on the Theory of Estimation*, University of California Press, Berkeley.

Shuster, J. (1968). On the inverse Gaussian distribution function, *J. Amer. Statist. Assoc.*, **63**, 1514–1516.

Singh, R. (1963). Existence of bounded length confidence intervals, *Ann. Math. Statist.*, **34**, 1474–1485.

Stein, C. (1945). A two-sample test for a linear hypothesis whose power is independent of the variance, *Ann Math. Statist.*, **16**, 243–258.

Takada, Y. (1986). Non-existence of fixed sample size procedures for scale families, *Sequential Anal.*, **5**, 93–101.

Takada, Y. (1988). Two-stage procedures for a multivariate normal distribution, *Kumamoto J. Math.*, **1**, 1–8.

Zacks, S. (1971). *The Theory of Statistical Inference*, Wiley, New York.