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JOINT DISTRIBUTIONS OF NUMBERS OF SUCCESS-RUNS UNTIL THE FIRST CONSECUTIVE k SUCCESSES IN A HIGHER-ORDER TWO-STATE MARKOV CHAIN

MASAYUKI UCHIDA

The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan

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Abstract. Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots$ be a time-homogeneous $\{0,1\}$-valued m-th order Markov chain. Joint distributions of the numbers of trials, failures and successes, of the numbers of trials and success-runs of length l ($m \leq l \leq k$) and of the numbers of trials and success-runs of length l ($l \leq m \leq k$) until the first consecutive k successes are obtained in the sequence $X_1, X_2, \ldots$. There are some ways of counting numbers of runs of length l. This paper studies the joint distributions based on four ways of counting numbers of runs, i.e., the number of non-overlapping runs of length l, the number of runs of length greater than or equal to l, the number of overlapping runs of length l and the number of runs of length exactly l. Marginal distributions of them can be obtained immediately, and surprisingly their distributions are very simple.

Key words and phrases: Probability generating function, discrete distribution, success and failure runs, geometric distribution, geometric distribution of order k, higher order Markov chain.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of $\{0,1\}$-valued random variables. We often call X_n the n-th trial and we say S (success) and F (failure) for the outcomes "1" and "0", respectively. The distribution of the waiting time for the first consecutive k successes in independent Bernoulli trials is called the geometric distribution of order k (cf. Feller (1968) and Phillippou *et al.* (1983)). Recently, exact distribution theory for so called discrete distributions of order k has been extensively developed and some of the results were applied to problems on the reliability of the consecutive-k-out-of-n:F system by many authors (cf. Aki *et al.* (1996), Aki and Hirano (1993, 1996), Chao *et al.* (1995), Fu and Koutras (1994), Hirano (1994), Hirano and Aki (1993), Hirano *et al.* (1991), Koutras (1997), Koutras and Alexandrou (1995) and references therein).

In the literature, numbers of success-runs of length l are counted in various ways. Four of the best-known types are Type I (non-overlapping success-runs with length l in the sense of Feller's (1968) counting), Type II (success-runs of length greater than or equal to l in the sense of Goldstein's (1990) counting), Type III (overlapping success-runs of length l in the sense of Ling's (1988, 1989) counting), Type IV (success-runs with length exactly l in the sense of Mood's (1940) counting).

For example, consider a realization of a sequence of S and F such as

$$SSSSFSSSFFFSSSSSSFFFFSS.$$

If we take $l = 3$, then there is only one success-run of Type IV; there are four success-runs of Type I, three success-runs of Type II and seven success-runs of Type III.

Throughout the paper, let k, l and m be fixed positive integers such that $l, m \leq k$. We denote by $G_k(p)$ the geometric distribution of order k, and by $G_k(p, a)$ the shifted geometric distribution of order k so that its support begins at a. Here the geometric distribution, to be denoted by $G(p)$, is defined as the distribution of the number of failures preceding the first success. Note that $G(p) = G_1(p, 0)$.

Aki and Hirano (1994) studied the exact marginal distributions of the numbers of failures, successes and success-runs of length less than k of Type III until the first consecutive k successes for some random sequences such as a sequence of independent and identically distributed $\{0, 1\}$-valued random variables, a time-homogeneous first-order $\{0, 1\}$-valued Markov chain and binary sequences of order k (cf. Aki (1985)). When the $\{0, 1\}$-sequence follows the first-order Markov chain, they pointed out that the distribution of the number of success-runs of length l of Type III until the first consecutive k successes is the shifted geometric distribution of order $k - l$ with the support $\{k - l + 1, k - l + 2, \dots\}$, i.e., $G_{k-l}(p, k - l + 1)$.

Aki and Hirano (1995) obtained the joint distributions of the numbers of failures, successes and success-runs of length less than k of Type I, II and III until the first consecutive k successes for some random sequences such as a sequence of independent and identically distributed integer valued random variables, a time-homogeneous first-order $\{0, 1\}$-valued Markov chain and binary sequences of order k. Further, Hirano *et al.* (1997) studied the distributions of the numbers of success-runs of length l of Type I, II, III and IV until the first occurrence of success-run of length k in the m-th order Markov dependent trials. They pointed out that when $m \leq l < k$, the resulting distributions coincide with the corresponding distributions in independent trials with a success probability whose value is given by the transition probability that a success occurs after a success-run of length m in the m-th order Markov chain.

In this paper, we investigate the joint distributions of waiting time and number of outcomes such as successes, failures and success runs with length less than k of Type I, II, III and IV until the first consecutive k successes in the following higher order two-state Markov chain.

Let $X_{-m+1}, X_{-m+2}, \dots, X_0, X_1, X_2, \dots$ be a time-homogeneous $\{0, 1\}$-valued m-th order Markov chain with

$$\pi_{x_1,\dots,x_m} = P(X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_m),$$

$$p_{x_1,\ldots,x_m} = P(X_i = 1 \mid X_{i-m} = x_1, X_{i-m+1} = x_2, \ldots, X_{i-1} = x_m),$$
$$= 1 - q_{x_1,\ldots,x_m},$$

for $x_1, \ldots, x_m = 0, 1$ and $i = 1, 2, \ldots$. For $x_1, \ldots, x_m = 0, 1$, we assume that $0 < p_{x_1,\ldots,x_m} < 1$.

We denote by τ the waiting time for the first consecutive k successes in the sequence $X_1, X_2, \ldots$. In Section 2, we derive the joint distributions of waiting time and number of outcomes such as successes, failures and success-runs for the first consecutive k successes in a second-order two-state Markov chain. In Section 3, we study the joint distributions of the waiting time and the numbers of failures and successes until τ. In Section 4, we consider the joint distributions of the numbers of trials and success-runs of length l ($m \le l \le k$) of Type I, II, III and IV until τ. We also investigate the joint distributions of the numbers of trials and success-runs of length l ($l < m < k$) until τ.

Throughout the paper, we define that for $\alpha > \beta$

$$\prod_{i=\alpha}^{\beta} g(i) = 1 \quad \text{and} \quad \sum_{i=\alpha}^{\beta} g(i) = 0,$$

where $g(i)$ is a function.

The results in this paper are not only general and new but also available to numerical and symbolic calculations by using a computer algebra system, for example, the REDUCE system ver. 3.5. In particular, when k, l and m are given, we can obtain the probability generating functions (p.g.f.'s) of the joint distributions of the numbers of successes, failures and trials and the numbers of success-runs and trials until τ, expectations and variances of the corresponding marginal distributions by using the computer algebra system (cf. Uchida (1994) and Uchida and Aki (1995)).

2. Preliminary

In this section, we study the joint distributions of waiting time and numbers of outcomes such as successes, failures and success-runs for the first consecutive k successes in the following second-order two-state Markov chain.

Let $X_{-1}, X_0, X_1, X_2, \ldots$ be a time-homogeneous second-order $\{0,1\}$-valued Markov chain with transition probabilities

$$p_{ij} = P(X_n = 1 \mid X_{n-2} = i, X_{n-1} = j) = 1 - q_{ij}$$

for $i \in \{0, 1\}$.

2.1 *Joint distribution of numbers of successes and failures in a second-order Markov chain*

In this subsection, we consider the joint distribution of the numbers of failures and successes until τ. Let ε_0 and ε_1 be the numbers of occurrences of "0" and "1" among $X_1, X_2, \ldots, X_\tau$. For $j_{-1} = 0, 1$ and $j_0 = 0, 1$, we denote by $\phi^{(j_{-1},j_0)}(t, u, s)$ the joint p.g.f. of the conditional distribution of $(\tau, \varepsilon_0, \varepsilon_1)$ given that $X_{-1} = j_{-1}$ and $X_0 = j_0$.

PROPOSITION 2.1.

$$\phi^{(j_{-1},j_0)}(t,u,s) = \frac{\psi^{(j_{-1},j_0)}(t,u,s)}{\psi(t,u,s)}, \qquad j_{-1}, j_0 \in \{0,1\},$$

where

$$\begin{aligned}
\psi(t,u,s) &= 1 - \psi_1(t,u,s)\left(q_{01}tu + \sum_{i=2}^{k-1} p_{01}ts(p_{11}ts)^{i-2}q_{11}tu\right),\\
\psi^{(0,0)}(t,u,s) &= \frac{p_{00}ts}{1-q_{00}tu}p_{01}ts(p_{11}ts)^{k-2},\\
\psi^{(0,1)}(t,u,s) &= [(q_{01}tu + p_{01}tsq_{11}tu - p_{11}tsq_{01}tu)\psi_1(t,u,s) + p_{11}ts]\\
&\quad \cdot p_{01}ts(p_{11}ts)^{k-2},\\
\psi^{(1,0)}(t,u,s) &= \psi_1(t,u,s)p_{01}ts(p_{11}ts)^{k-2},\\
\psi^{(1,1)}(t,u,s) &= [\{(1+p_{11}ts)q_{11}tup_{01}ts - (p_{11}ts)^2q_{01}tu\}\psi_1(t,u,s) + (p_{11}ts)^2]\\
&\quad \cdot (p_{11}ts)^{k-2},\\
\psi_1(t,u,s) &= \frac{q_{10}tup_{00}ts}{1-q_{00}tu} + p_{10}ts.
\end{aligned}$$

PROOF. For $i = 0, 1, \ldots, k-1$, let A_i be the event that we start with a "1"-run of length i and "0" occurs just after the "1"-run. Let C be the event that we start with a "1"-run of length k. For $j_{-1} = 0, 1$ and $j_0 = 0, 1$, let $\phi^{(j_{-1},j_0)}(t,u,s \mid A_i)$ and $\phi^{(j_{-1},j_0)}(t,u,s \mid C)$ be the p.g.f.'s of the conditional joint distributions of $(\tau, \varepsilon_0, \varepsilon_1)$ given that the event $A_i \cap \{X_{-1} = j_{-1}, X_0 = j_0\}$ occurs and given that the event $C \cap \{X_{-1} = j_{-1}, X_0 = j_0\}$ occurs, respectively. Since A_i, $i = 0, 1, \ldots, k-1$ and C construct a partition of the sample space, we have

$$\begin{aligned}
\phi^{(0,0)}(t,u,s) &= \sum_{i=0}^{k-1} P(A_i)\phi^{(0,0)}(t,u,s \mid A_i) + P(C)\phi^{(0,0)}(t,u,s \mid C)\\
&= q_{00}tu\phi^{(0,0)}(t,u,s)\\
&\quad + \left(p_{00}tsq_{01}tu + \sum_{i=2}^{k-1} p_{00}tsp_{01}ts(p_{11}ts)^{i-2}q_{11}tu\right)\phi^{(1,0)}(t,u,s)\\
&\quad + p_{00}tsp_{01}ts(p_{11}ts)^{k-2}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\phi^{(0,1)}(t,u,s) &= \left(q_{01}tu + \sum_{i=1}^{k-1} p_{01}ts(p_{11}ts)^{i-1}q_{11}tu\right)\phi^{(1,0)}(t,u,s) \\
&\quad + p_{01}ts(p_{11}ts)^{k-1}, \\
\phi^{(1,0)}(t,u,s) &= q_{10}tu\phi^{(0,0)}(t,u,s) \\
&\quad + \left(p_{10}tsq_{01}tu + \sum_{i=2}^{k-1} p_{10}tsp_{01}ts(p_{11}ts)^{i-2}q_{11}tu\right)\phi^{(1,0)}(t,u,s) \\
&\quad + p_{10}tsp_{01}ts(p_{11}ts)^{k-2},
\end{aligned}$$

and

$$\phi^{(1,1)}(t,u,s) = \sum_{i=0}^{k-1}(p_{11}ts)^i q_{11}tu\phi^{(1,0)}(t,u,s) + (p_{11}ts)^k.$$

From the above four equations, we have the desired results. This completes the proof.

Remark 1. By setting $t = u = 1$ in the formulas of Proposition 2.1, we see that

$$\phi^{(0,0)}(1,1,s) = \phi^{(1,0)}(1,1,s)$$

and

$$\phi^{(0,0)}(1,1,s) = \frac{s(1-p_{11}s)p_{01}s(p_{11}s)^{k-2}}{1-p_{11}s-q_{01}s+p_{11}sq_{01}s-p_{01}sq_{11}s+p_{01}s(p_{11}s)^{k-2}q_{11}s}.$$

Now, we consider the following first-order Markov chain. Let $Y_0, Y_1, Y_2, \ldots$ be a time-homogeneous first-order $\{0,1\}$-valued Markov chain with $P(Y_i = 1 \mid Y_{i-1} = 0) = p_0$, $P(Y_i = 0 \mid Y_{i-1} = 0) = q_0$, $P(Y_i = 1 \mid Y_{i-1} = 1) = p_1$, $P(Y_i = 0 \mid Y_{i-1} = 1) = q_1$, for $i = 1, 2, \ldots$. We denote by τ_0 the waiting time for the first consecutive k successes in the sequence $Y_1, Y_2, \ldots$. Let $\varphi(s)$ be the pgf of the conditional distribution of the waiting time τ_0 given that $Y_0 = 0$. Then, we have

$$\varphi(s) = \frac{(1-p_1s)p_0s(p_1s)^{k-1}}{1-p_1s-q_0s+p_1sq_0s-p_0sq_1s+p_0s(p_1s)^{k-1}q_1s}.$$

Hence, we see that the marginal distribution of ε_1 given that $X_0 = 0$, $X_{-1} = 0$ or 1 is the shifted geometric distribution of order $k-1$ given that $Y_0 = 0$ in the above first-order Markov chain $Y_1, Y_2, \ldots$.

2.2 *Joint distribution of numbers of success-runs in a second-order Markov chain*

In this subsection, we consider the joint distribution of the numbers of success-runs of length l ($l = 2, \ldots, k$) until τ. Let μ_l, ν_l, ξ_l and η_l be the numbers of "1"-runs of length l of Type I, II, III and IV until τ, respectively. For $j_{-1}, j_0 = 0, 1$, we denote by $\phi_1^{(j_{-1},j_0)}(t,t_2,\ldots,t_k)$, $\phi_2^{(j_{-1},j_0)}(t,t_2,\ldots,t_k)$, $\phi_3^{(j_{-1},j_0)}(t,t_2,\ldots,t_k)$ and $\phi_4^{(j_{-1},j_0)}(t,t_2,\ldots,t_k)$ the joint p.g.f.s of the conditional distributions of $(\tau, \mu_2, \ldots,$

$\mu_k)$, $(\tau, \nu_2, \ldots, \nu_k)$, $(\tau, \xi_2, \ldots, \xi_k)$ and $(\tau, \eta_2, \ldots, \eta_k)$ given that $X_{-1} = j_{-1}$ and $X_0 = j_0$, respectively.

THEOREM 2.2.

$$\phi_j^{(j_{-1},j_0)}(t, t_2, \ldots, t_k) = \frac{\psi^{(j_{-1},j_0)}(t,1,1)\varphi_{kj}}{\phi_j(t, t_2, \ldots, t_k)}, \qquad j_{-1}, j_0 = 0, 1,$$

where

$$\phi_j(t, t_2, \ldots, t_k) = 1 - \left(\frac{q_{10}tp_{00}t}{1 - q_{00}t} + p_{10}t\right)\left(q_{01}t + \sum_{i=2}^{k-1} p_{01}t(p_{11}t)^{i-2}q_{11}t\varphi_{ij}\right)$$

and

$$\varphi_{ij} = \begin{cases} t_2^{[i/2]} \cdots t_k^{[i/k]}, & \text{if } j = 1, \\ t_2 \cdots t_i, & \text{if } j = 2, \\ t_2^{i-1} t_3^{i-2} \cdots t_i, & \text{if } j = 3, \\ t_i, & \text{if } j = 4. \end{cases}$$

PROOF. Let A_i, C be as in the proof of Proposition 2.1 and for $j_{-1}, j_0 = 0, 1$, we denote by $\phi_j^{(j_{-1},j_0)}(t, t_2, \ldots, t_k \mid A_i)$, $\phi_j^{(j_{-1},j_0)}(t, t_2, \ldots, t_k \mid C)$ the respective conditional p.g.f.'s. Since A_i, $i = 0, 1, \ldots, k-1$ and C construct a partition of the sample space, we have

$$\begin{aligned}
&\phi_j^{(0,0)}(t, t_2, \ldots, t_k) \\
&\quad = \sum_{i=0}^{k-1} P(A_i)\phi_j^{(0,0)}(t, t_2, \ldots, t_k \mid A_i) + P(C)\phi_j^{(0,0)}(t, t_2, \ldots, t_k \mid C) \\
&\quad = q_{00}t\phi_j^{(0,0)}(t, t_2, \ldots, t_k) \\
&\qquad + \left(p_{00}tq_{01}t + \sum_{i=2}^{k-1} p_{00}tp_{01}t(p_{11}t)^{i-2}q_{11}t\varphi_{ij}\right)\phi_j^{(1,0)}(t, t_2, \ldots, t_k) \\
&\qquad + p_{00}tp_{01}t(p_{11}t)^{k-2}\varphi_{kj}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\phi_j^{(0,1)}(t, t_2, \ldots, t_k) \\
&\quad = \left(q_{01}t + p_{01}tq_{11}t + \sum_{i=2}^{k-1} p_{01}t(p_{11}t)^{i-1}q_{11}t\varphi_{ij}\right)\phi_j^{(1,0)}(t, t_2, \ldots, t_k) \\
&\qquad + p_{01}t(p_{11}t)^{k-1}\varphi_{kj}, \\
&\phi_j^{(1,0)}(t, t_2, \ldots, t_k) \\
&\quad = q_{10}t\phi_j^{(0,0)}(t, t_2, \ldots, t_k)
\end{aligned}$$

$$+\left(p_{10}tq_{01}t+\sum_{i=2}^{k-1}p_{10}tp_{01}t(p_{11}t)^{i-2}q_{11}t\varphi_{ij}\right)\phi_j^{(1,0)}(t,t_2,\dots,t_k)$$
$$+p_{10}tp_{01}t(p_{11}t)^{k-2}\varphi_{kj},$$

and

$$\phi_j^{(1,1)}(t,t_2,\dots,t_k)$$
$$=\left(q_{11}t+p_{11}tq_{11}t+\sum_{i=2}^{k-1}(p_{11}t)^i q_{11}t\varphi_{ij}\right)\phi_j^{(1,0)}(t,t_2,\dots,t_k)+(p_{11}t)^k\varphi_{kj}.$$

From the above four equations, we have the desired results. This completes the proof.

COROLLARY 2.3. *The conditional joint distribution of* $(\mu_2,\dots,\mu_k)$ *does not depend on the values* X_{-1} *and* X_0. *In particular, if* $k=l(m+1)$ *holds for a positive integer* m, *then for every* $l=2,\dots,k-1$, *the marginal distribution of* μ_l *is the geometric distribution of order* m, *i.e.*, $\mu_l\sim G_m(p_{11}^l,m+1)$.

PROOF. By setting $j=1$, $t=1$ in the formulas of Theorem 2.2, we see that

$$\phi_1^{(0,0)}=\phi_1^{(0,1)}=\phi_1^{(1,0)}=\phi_1^{(1,1)}.$$

The second statement is easy to prove by substituting 1 for all arguments except for t_l. This completes the proof.

COROLLARY 2.4. *The conditional joint distribution of* $(\nu_2,\dots,\nu_k)$ *does not depend on the values* X_{-1} *and* X_0. *Especially, for every* $l=2,\dots,k-1$, *the marginal distribution of* ν_l *is the geometric distribution of order* 1, *i.e.*, $\nu_l\sim G_1(p_{11}^{k-l})$.

COROLLARY 2.5. *The conditional joint distribution of* $(\xi_2,\dots,\xi_k)$ *does not depend on the values* X_{-1} *and* X_0. *Especially, for every* $l=2,\dots,k-1$, *the marginal distribution of* ξ_l *is the shifted geometric distribution of order* $k-l$, *i.e.*, $\xi_l\sim G_{k-l}(p_{11},k-l+1)$.

COROLLARY 2.6. *The conditional joint distribution of* $(\eta_2,\dots,\eta_k)$ *does not depend on the values* X_{-1} *and* X_0. *Especially, for every* $l=2,\dots,k-1$, *the marginal distribution of* η_l *is the geometric distribution, i.e.*, $\eta_l\sim G(\frac{p_{11}^{k-l}}{p_{11}^{k-l}+q_{11}})$.

3. Joint distribution of numbers of successes and failures in a higher-order Markov chain

In this section, we consider the joint distribution of the numbers of failures, successes and trials until τ. We give a method for deriving the p.g.f. of the conditional joint distribution of the numbers of failures, successes and trials until τ in the m-th order Markov chain.

A sequence which follows the m-th order Markov chain depends on the past occurrences of length m. A set of $\{0,1\}$-sequence of length m consists of 2^m elements, and can be uniquely regarded as a binary number. Further we translate it into a decimal number. For example, when $m=3$, $p_{100}=p_4$ and when $m=4$, $p_{1001}=p_9$. Let $N_m=\{0,1,2,\ldots,2^m-1\}$ and let f_i $(i=0,1)$ be the mapping from N_m to N_m such that

$$f_i(x)=2x+i \pmod{2^m}, \qquad \text{for} \quad i=0,1.$$

Let ε_0 and ε_1 be the numbers of occurrences of "0" and "1" among $X_1,X_2,\ldots,X_\tau$. We denote by $\phi^{(x)}(t,u,s)$ (for each $x\in N_m$) the joint p.g.f. of the conditional distribution of $(\tau,\varepsilon_0,\varepsilon_1)$ given that $X_{-m+1}=x_1,X_{-m+2}=x_2,\ldots,X_0=x_m$.

For $i=0,1,\ldots,k-1$, let A_i be the event that we start with a "1"-run of length i and "0" occurs just after the "1"-run. Let C be the event that we start with a "1"-run of length k. For $x\in N_m$, let $\phi^{(x)}(t,u,s\mid A_i)$ and $\phi^{(x)}(t,u,s\mid C)$ be the p.g.f.'s of the conditional joint distributions of $(\tau,\varepsilon_0,\varepsilon_1)$ given that the event $A_i\cap\{X_{-m+1}=x_1,X_{-m+2}=x_2,\ldots,X_0=x_0\}$ occurs and given that the event $C\cap\{X_{-m+1}=x_1,X_{-m+2}=x_2,\ldots,X_0=x_0\}$ occurs, respectively. Since A_i, $i=0,1,\ldots,k-1$ and C construct a partition of the sample space, we have the following system of 2^m equations of conditional p.g.f.'s.

For each $x\in N_m$,

$$\begin{aligned}
\phi^{(x)}(t,u,s) &= \sum_{i=0}^{k-1} P(A_i)\phi^{(x)}(t,u,s\mid A_i)+P(C)\phi^{(x)}(t,u,s\mid C)\\
&= q_x tu\phi^{(f_0(x))}(t,u,s)\\
&\quad + p_x ts\sum_{i=1}^{m-2}\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)}ts)\right] q_{f_1^i(x)}tu\phi^{(f_0\circ f_1^i(x))}(t,u,s)\\
&\quad + p_x ts\left[\prod_{j=1}^{m-2}(p_{f_1^j(x)}ts)\right] q_{f_1^{m-1}(x)}tu\phi^{(2^m-2)}(t,u,s)\\
&\quad + p_x ts\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}ts)\right]\sum_{i=m}^{k-1}(p_{2^m-1}ts)^{i-m}q_{2^m-1}tu\phi^{(2^m-2)}(t,u,s)\\
&\quad + p_x ts\left[\prod_{j=1}^{m-1}(p_{f_1^j(x)}ts)\right](p_{2^m-1}ts)^{k-m}.
\end{aligned}$$

For each $x,y\in N_m$ and $i\in L_m=\{1,2,\ldots,m-2\}$, we set

$$\alpha_{x,i}(t,u,s)=p_x ts\left[\prod_{j=1}^{i-1}(p_{f_1^j(x)}ts)\right] q_{f_1^i(x)}tu,$$

$$\beta_x(t,u,s)=p_x ts\left[\prod_{j=1}^{m-2}(p_{f_1^j(x)}ts)\right] q_{f_1^{m-1}(x)}tu$$

$$+ p_x ts \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} ts)\right] \sum_{i=m}^{k-1} (p_{2^m-1} ts)^{i-m} q_{2^m-1} tu,$$

$$\gamma_x(t,s) = p_x ts \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} ts)\right] (p_{2^m-1} ts)^{k-m},$$

$$I_{x,y} = \{i \in L_m \mid f_0 \circ f_1^i(x) = y\}.$$

For each $x, y \subset N_m$,

$$a_{x,y}(t,u,s) = q_x tu 1\{f_0(x) = y\} + \sum_{i \in I_{x,y}} \alpha_{x,i}(t,u,s) + \beta_x(t,u,s) 1\{y = 2^m - 2\},$$

$$b_{x,y}(t,u,s) = 1\{x = y\} - a_{x,y}(t,u,s),$$

where

$$1\{x = y\} = \begin{cases} 1, & \text{if} \quad x = y, \\ 0, & \text{if} \quad x \neq y. \end{cases}$$

In particular,

$$b_{x,2^m-2}(t,u,s) = \Xi_x(t,u,s) - \Gamma_x(t,s) \sum_{i=m}^{k-1} (p_{2^m-1} ts)^{i-m} q_{2^m-1} tu,$$

where

$$\Gamma_x(t,s) - p_x ts \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} ts)\right],$$

$$\Xi_x(t,u,s) = 1\{x = 2^m - 2\} - q_x tu 1\{f_0(x) = 2^m - 2\} - \sum_{i \in I_{x,2^m-2}} \alpha_{x,i}(t,u,s)$$

$$- p_x ts \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} ts)\right] q_{f_1^{m-1}(x)} tu.$$

Define

$$B = (b_{x,y}(t,u,s))_{x,y \in N_m},$$

$$B^{-1} = (c_{x,y})_{x,y \in N_m},$$

$$c_{x,y} = \frac{\Delta_{y,x}}{|B|},$$

where $\Delta_{x,y}$ is the (x,y)-cofactor of the matrix B.

Then, we have

$$\phi^{(x)}(t,u,s) = \sum_{i=0}^{2^m-1} \frac{\Delta_{i,x}}{|B|} \gamma_i(t,s) = \frac{1}{|B|} \sum_{i=0}^{2^m-1} \Delta_{i,x} \gamma_i(t,s).$$

Here, we set that for each $x \in N_m$,

$$\psi(t,u,s) = |B|,$$
$$\psi^{(x)}(t,u,s) = \sum_{i=0}^{2^m-1} \Delta_{i,x}\gamma_i(t,s).$$

Then, we obtain

PROPOSITION 3.1.

$$\phi^{(x)}(t,u,s) = \frac{\psi^{(x)}(t,u,s)}{\psi(t,u,s)}, \qquad for \quad x \in N_m.$$

Remark 2. By the above result, we can easily obtain the p.g.f. of the joint distribution of the numbers of successes, failures and trials until τ by means of computer algebra. Here we use a computer algebra, REDUCE ver. 3.5 for generating the system of equations of conditional p.g.f.'s. As a matter of fact, we can also compute the expectation and the variance of τ, i.e., the geometric distribution of order k in a higher order Markov chain. Moreover, we can obtain the probability mass function (p.m.f.) of the distribution of τ by using the fact which the p.g.f. of the distribution of τ is a rational function (cf. Stanley (1986) and Uchida and Aki (1995)).

4. Joint distribution of numbers of success-runs in a higher-order Markov chain

In this section, we investigate the joint distributions of the numbers of success-runs of length l of Type I, II, III and IV until τ in the m-th order Markov chain.

4.1 *Case $m \leq l$*

In this subsection, we consider the joint distributions of the numbers of trials and success-runs of length l $(m \leq l \leq k)$ of Type I, II, III and IV until τ. Let μ_l, ν_l, ξ_l and η_l be the numbers of "1"-runs of length l of Type I, II, III and IV until τ, respectively. For $x \in N_m$, we denote by $\phi_1^{(x)}(t,t_l,\dots,t_k)$, $\phi_2^{(x)}(t,t_l,\dots,t_k)$, $\phi_3^{(x)}(t,t_l,\dots,t_k)$ and $\phi_4^{(x)}(t,t_l,\dots,t_k)$ the joint p.g.f.s of the conditional distributions of $(\tau,\mu_l,\dots,\mu_k)$, $(\tau,\nu_l,\dots,\nu_k)$, $(\tau,\xi_l,\dots,\xi_k)$ and $(\tau,\eta_l,\dots,\eta_k)$ given that $X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_m$, respectively.

Let A_i, C and I_{xy} be as in the proof of Proposition 3.1 and for each $x \in N_m$ and $j = 1,2,3,4$, we denote by $\phi_j^{(x)}(t,t_l,\dots,t_k \mid A_i)$, $\phi_j^{(x)}(t,t_l,\dots,t_k \mid C)$ the respective conditional p.g.f.'s. Since A_i, $i = 0,1,\dots,k-1$ and C construct a partition of the sample space, we have the following system of 2^m equations of conditional p.g.f.'s.

For each $x \in N_m$,

$$\phi_j^{(x)}(t,t_l,\dots,t_k)$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} P(A_i)\phi_j^{(x)}(t, t_l, \ldots, t_k \mid A_i) + P(C)\phi_j^{(x)}(t, t_l, \ldots, t_k \mid C) \\
&= q_x t \phi_j^{(f_0(x))}(t, t_l, \ldots, t_k) \\
&\quad + p_x t \sum_{i=1}^{m-2} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t) \right] q_{f_1^i(x)} t \phi_j^{(f_0 \circ f_1^i(x))}(t, t_l, \ldots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} t) \right] q_{f_1^{m-1}(x)} t \phi_j^{(2^m-2)}(t, t_l, \ldots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] \sum_{i=m}^{l-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \phi_j^{(2^m-2)}(t, t_l, \ldots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] \sum_{i=l}^{k-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \varphi_{ij} \phi_j^{(2^m-2)}(t, t_l, \ldots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] (p_{2^m-1} t)^{k-m} \varphi_{kj},
\end{aligned}
$$

where

$$
\varphi_{ij} = \begin{cases} t_l^{[i/l]} \cdots t_k^{[i/k]}, & \text{if } j = 1, \\ t_l \cdots t_i, & \text{if } j = 2, \\ t_l^{i-l+1} t_{l+1}^{i-l} \cdots t_i, & \text{if } j = 3, \\ t_i, & \text{if } j = 4. \end{cases}
$$

For each $x, y \in N_m$ and $i \in L_m$, we set that

$$
\begin{aligned}
\alpha'_{x,i}(t) &= p_x t \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t) \right] q_{f_1^i(x)} t, \\
\beta'_x(t) &= p_x t \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} t) \right] q_{f_1^{m-1}(x)} t \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] \sum_{i=m}^{l-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] \sum_{i=l}^{k-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \varphi_{ij}, \\
\gamma'_x(t) &= p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] (p_{2^m-1} t)^{k-m} \varphi_{kj}.
\end{aligned}
$$

For each $x, y \in N_m$,

$$a'_{x,y}(t) = q_x t 1\{f_0(x) = y\} + \sum_{i \in I_{x,y}} \alpha'_{x,i}(t) + \beta'_x(t) 1\{y = 2^m - 2\},$$
$$b'_{x,y}(t) = 1\{x = y\} - a'_{x,y}(t).$$

Define

$$B' = (b'_{x,y}(t))_{x,y \in N_m},$$
$$(B')^{-1} = (c'_{x,y})_{x,y \in N_m},$$
$$c'_{x,y} = \frac{\Delta'_{y,x}}{|B'|},$$

where $\Delta'_{x,y}$ is the (x, y)-cofactor of the matrix B'.

From the above system of equations, we have

$$\phi_j^{(x)}(t, t_l, \ldots, t_k) = \sum_{i=0}^{2^m-1} \frac{\Delta'_{i,x}}{|B'|} \gamma'_i(t) = \frac{1}{|B'|} \sum_{i=0}^{2^m-1} \Delta'_{i,x} \gamma'_i(t).$$

Here, we set that

$$\psi_j(t, t_l, \ldots, t_k) = |B'|,$$
$$\psi_j^{(x)}(t, t_l, \ldots, t_k) = \sum_{i=0}^{2^m-1} \Delta'_{i,x} \gamma'_i(t) \qquad (\equiv \psi_j^{(x)})$$

for each $x \in N_m$.

We note that

$$\psi_j^{(x)} = \begin{vmatrix} b'_{00}(t) & \cdots & b'_{0,x-1}(t) & \gamma'_0(t) & b'_{0,x+1}(t) & \cdots & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b'_{2^m-1,0}(t) & \cdots & b'_{2^m-1,x-1}(t) & \gamma'_{2^m-1}(t) & b'_{2^m-1,x-1}(t) & \cdots & b'_{2^m-1,2^m-1}(t) \end{vmatrix}$$

and

$$b'_{x,2^m-2}(t) = \Xi'_x(t) - \Gamma'_x(t) \sum_{i=m}^{l-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t - \Gamma'_x(t)$$
$$\cdot \sum_{i=l}^{k-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \varphi_{ij},$$

where

$$\Gamma'_x(t) = p_x t \left[\prod_{j-1}^{m-1} (p_{f_1^j(x)} t) \right],$$
$$\Xi'_x(t) = 1\{x = 2^m - 2\} - q_x t 1\{f_0(x) = 2^m - 2\} - \sum_{i \in I_{x,2^m-2}} \alpha'_{x,i}(t)$$
$$- p_x t \left[\prod_{j-1}^{m-2} (p_{f_1^j(x)} t) \right] q_{f_1^{m-1}(x)} t.$$

Here, we see that

$$\gamma'_x(t) = \gamma_x(t,1)\varphi_{kj}$$

and when $y \neq 2^m - 2$,

$$b'_{x,y}(t) = b_{x,y}(t,1,1).$$

When $x = 2^m - 2$, we have

$$\begin{aligned}
\psi_j^{(2^m-2)} &= \begin{vmatrix} b'_{00}(t) & \cdots & b'_{0,2^m-3}(t) & \gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & b'_{2^m-1,2^m-3}(t) & \gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix} \\
&= \varphi_{kj} \begin{vmatrix} b'_{00}(t) & \cdots & b'_{0,2^m-3}(t) & \gamma_0(t,1) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & b'_{2^m-1,2^m-3}(t) & \gamma_{2^m-1}(t,1) & b'_{2^m-1,2^m-1}(t) \end{vmatrix} \\
&= \psi^{(2^m-2)}(t,1,1)\varphi_{kj}.
\end{aligned}$$

On the other hand, when $x \neq 2^m - 2$,

$$\begin{aligned}
\psi_j^{(x)} &= \begin{vmatrix} b'_{00}(t) & \cdots & \gamma'_0(t) & \cdots & \Xi'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma'_{2^m-1}(t) & \cdots & \Xi'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix} \\
&\quad - \sum_{i=m}^{l-1} (p_{2^m-1}t)^{i-m} q_{2^m-1} t \\
&\qquad \cdot \begin{vmatrix} b'_{00}(t) & \cdots & \gamma'_0(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma'_{2^m-1}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix} \\
&\quad - \sum_{i=l}^{k-1} (p_{2^m-1}t)^{i-m} q_{2^m-1} t \varphi_{ij} \\
&\qquad \cdot \begin{vmatrix} b'_{00}(t) & \cdots & \gamma'_0(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma'_{2^m-1}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}.
\end{aligned}$$

We note that

$$\begin{aligned}
\gamma'_x(t) &= \Gamma_x(t)'(p_{2^m-1}t)^{k-m}\varphi_{kj}, \\
\Xi'_x(t) &= \Xi_x(t,1,1).
\end{aligned}$$

Then, we obtain

$$\begin{vmatrix} b'_{00}(t) & \cdots & \gamma'_0(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma'_{2^m-1}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix} = 0.$$

Here, we have

$$\psi_j^{(x)} = \begin{vmatrix} b'_{00}(t) & \cdots & \gamma'_0(t) & \cdots & \Xi'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma'_{2^m-1}(t) & \cdots & \Xi'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}$$

$$= \varphi_{kj} \begin{vmatrix} b'_{00}(t) & \cdots & \gamma_0(t,1) & \cdots & \Xi_0(t,1,1) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \gamma_{2^m-1}(t,1) & \cdots & \Xi_{2^m-1}(t,1,1) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}$$

$$= \psi^{(x)}(t,1,1)\varphi_{kj}.$$

Then, we have

THEOREM 4.1.

$$\phi_j^{(x)}(t, t_l, \ldots, t_k) = \frac{\psi^{(x)}(t,1,1)\varphi_{kj}}{\psi_j(t, t_l, \ldots, t_k)}, \qquad for \quad x \in N_m.$$

COROLLARY 4.2. *For $m \le l < k$, the conditional joint distribution of $(\mu_l, \ldots, \mu_k)$ does not depend on the values $X_{-m+1}, X_{-m+2}, \ldots, X_0$. In particular, if $k = l(n+1)$ holds for a positive integer n, then for every $l = m, \ldots, k-1$, the marginal distribution of μ_l is the geometric distribution of order n, i.e., $\mu_l \sim G_n(p^l_{2^m-1}, n+1)$.*

PROOF. By substituting $t = u = s = 1$ in the formula of Proposition 3.1, we obtain that for each $x \in N_m$,

$$\psi^{(x)}(1,1,1) = \psi(1,1,1).$$

By setting $j = 1$, $t = 1$ in the formula of Theorem 4.1, we see that

$$\phi_1^{(0)} = \phi_1^{(1)} = \cdots = \phi_1^{(2^m-1)}.$$

The second statement is easy to prove by substituting 1 for all arguments except for t_l in the formula of Theorem 4.1.

Here, we note that

$$\begin{aligned} b'_{x,2^m-2}(1) &= 1\{x = 2^m - 2\} - q_x 1\{f_0(x) = 2^m - 2\} - \sum_{i \in I_{x,2^m-2}} \alpha'_{x,i}(1) \\ &\quad - p_x \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)})\right] q_{f_1^{m-1}(x)} - \Gamma'_x(1) \sum_{i=m}^{l-1} (p_{2^m-1})^{i-m} q_{2^m-1} \\ &\quad - \Gamma'_x(1) \sum_{i=l}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} t_l^{[i/l]} \\ &= \Xi''_x + \Gamma'_x(1)(p_{2^m-1})^{l-m} - \Gamma'_x(1) \sum_{i=l}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} t_l^{[i/l]} \\ &= \Xi''_x + \Gamma'_x(1)\Gamma''(t_l), \end{aligned}$$

where

$$\Xi''_x = 1\{x = 2^m - 2\} - q_x 1\{f_0(x) = 2^m - 2\} - \sum_{i \in I_{x,2^m-2}} \alpha'_{x,i}(1) - p_x \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)}) \right],$$

$$\Gamma''(t_l) = p_{2^m-1}^{l-m} - \sum_{i=l}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} t_l^{[i/l]}.$$

Here, we see that

$$\begin{aligned} \sum_{x=0}^{2^m-1} b'_{x,y}(1) &= 1 - q_x - p_x \sum_{i=1}^{m-1} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)}) \right] q_{f_1^i(x)} \\ &\quad - p_x \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)}) \right] \sum_{i=m}^{l-1} (p_{2^m-1})^{i-m} q_{2^m-1} \\ &\quad - p_x \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)}) \right] \sum_{i=l}^{k-1} (p_{2^m-1})^{i-m} q_{2^m-1} t_l^{[i/l]} \\ &= \Gamma'_x(1)\Gamma''(t_l). \end{aligned}$$

We also give

$$\begin{aligned} \sum_{x=0}^{2^m-1} b'_{x,y}(1) &= \sum_{\substack{x=0 \\ x \neq 2^m-2}}^{2^m-1} b'_{x,y}(1) + b'_{x,2^m-2}(1) \\ &= \sum_{\substack{x=0 \\ x \neq 2^m-2}}^{2^m-1} b'_{x,y}(1) + \Xi''_x + \Gamma'_x(1)\Gamma''(t_l). \end{aligned}$$

Here, we obtain

$$\sum_{\substack{x=0 \\ x \neq 2^m-2}}^{2^m-1} b'_{x,y}(1) + \Xi''_x = 0.$$

Then, we obtain

$$\left| \begin{matrix} b'_{00}(t) & \cdots & \Xi''_0 & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Xi''_{2^m-1} & b'_{2^m-1,2^m-1}(t) \end{matrix} \right|_{t=1} = 0.$$

Consequently, we have

$$\psi_j(1, t_l, 1, \ldots, 1) = \left| \begin{matrix} b'_{00}(t) & \cdots & b'_{0,2^m-2}(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & b'_{2^m-1,2^m-2}(t) & b'_{2^m-1,2^m-1}(t) \end{matrix} \right|_{t=1}$$

$$
= \begin{vmatrix} b'_{00}(t) & \cdots & \Xi''_0 & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Xi''_{2^m-1} & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}
+ \Gamma''(t_l) \begin{vmatrix} b'_{00}(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}
$$

$$
= \Gamma''(t_l) \begin{vmatrix} b'_{00}(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}.
$$

Similarly, we have

$$
\psi^{(x)}(1,1,1) = \begin{vmatrix} b'_{00}(t) & \cdots & \Xi''_0 & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Xi''_{2^m-1} & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}
+ \Gamma''(1) \begin{vmatrix} b'_{00}(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}
$$

$$
= \Gamma''(1) \begin{vmatrix} b'_{00}(t) & \cdots & \Gamma'_0(t) & b'_{0,2^m-1}(t) \\ \vdots & & \vdots & \vdots \\ b'_{2^m-1,0}(t) & \cdots & \Gamma'_{2^m-1}(t) & b'_{2^m-1,2^m-1}(t) \end{vmatrix}_{t=1}.
$$

Consequently, we have

$$
\begin{aligned}
\phi_1^{(x)} &= \frac{[p_{2^m-1}^{l-m} - \sum_{i=l}^{k-1} p_{2^m-1}^{i-m} q_{2^m-1}] t_l^{[k/l]}}{p_{2^m-1}^{l-m} - \sum_{i=l}^{k-1} p_{2^m-1}^{i-m} q_{2^m-1} t_l^{[i/l]}} \\
&= \frac{p_{2^m-1}^{k-l} t_l^{[k/l]}}{1 - \sum_{i=l}^{k-1} p_{2^m-1}^{i-l} q_{2^m-1} t_l^{[i/l]}} \\
&= \frac{(p_{2^m-1}^l)^n t_l^{n+1}}{1 - \sum_{j=0}^{n-1} (p_{2^m-1}^l)^j (1 - p_{2^m-1}^l) t_l^{j+1}}.
\end{aligned}
$$

This completes the proof.

Remark 3. The second statement in Corollary 4.2 coincides with Theorem 23.2.2 of Hirano *et al.* (1997).

COROLLARY 4.3. *For $m \le l < k$, the conditional joint distribution of $(\nu_l, \dots, \nu_k)$ does not depend on the values $X_{-m+1}, X_{-m+2}, \dots, X_0$. Especially, for every $l = m, \dots, k-1$, the marginal distribution of ν_l is the geometric distribution of order 1, i.e., $\nu_l \sim G_1(p_{2^m-1}^{k-l})$.*

Remark 4. The second statement in Corollary 4.3 agrees with Theorem 23.2.4 of Hirano *et al.* (1997).

COROLLARY 4.4. *For $m \le l < k$, the conditional joint distribution of $(\xi_l, \dots, \xi_k)$ does not depend on the values $X_{-m+1}, X_{-m+2}, \dots, X_0$. Especially, for every $l = m, \dots, k-1$, the marginal distribution of ξ_l is the shifted geometric distribution of order $k-l$, i.e., $\xi_l \sim G_{k-l}(p_{2^m-1}, k-l+1)$.*

Remark 5. The second statement in Corollary 4.3 is in keeping with Theorem 23.2.1 of Hirano *et al.* (1997).

COROLLARY 4.5. *For $m \le l < k$, the conditional joint distribution of $(\eta_l, \dots, \eta_k)$ does not depend on the values $X_{-m+1}, X_{-m+2}, \dots, X_0$. Especially, for every $l = m, \dots, k-1$, the marginal distribution of η_l is the geometric distribution, i.e., $\eta_l \sim G\left(\frac{p_{2^m-1}^{k-l}}{p_{2^m-1}^{k-l} + q_{2^m-1}}\right)$.*

Remark 6. The second statement in Corollary 4.5 tallies with Theorem 23.2.3 of Hirano *et al.* (1997).

4.2 *Case $l < m$*

In this subsection, we consider the p.g.f. of the joint distribution of the numbers of trials and success-runs of length l ($l < m \le k$) until τ.

Let $\phi_j^{(x)}(t, t_l, \dots, t_k \mid A_i)$, $\phi_j^{(x)}(t, t_l, \dots, t_k \mid C)$ be as in the proof of Theorem 4.1. Then, we have the following system of 2^m equations of conditional p.g.f.'s by considering all possibilities of the first occurrence of 0.

For each $x \in N_m$,

$$
\begin{aligned}
\phi_j^{(x)}(t, t_l, \dots, t_k) &= \sum_{i=0}^{k-1} P(A_i)\phi_j^{(x)}(t, t_l, \dots, t_k \mid A_i) + P(C)\phi_j^{(x)}(t, t_l, \dots, t_k \mid C) \\
&= q_x t \phi_j^{(f_0(x))}(t, t_l, \dots, t_k) \\
&\quad + p_x t \sum_{i=1}^{l-1} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t)\right] q_{f_1^i(x)} t \phi_j^{(f_0 \circ f_1^i(x))}(t, t_l, \dots, t_k) \\
&\quad + p_x t \sum_{i=l}^{m-2} \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t)\right] q_{f_1^i(x)} t \varphi_{ij} \phi_j^{(f_0 \circ f_1^i(x))}(t, t_l, \dots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} t)\right] q_{f_1^{m-1}(x)} t \varphi_{m-1,j} \phi_j^{(2^m-2)}(t, t_l, \dots, t_k) \\
&\quad + p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t)\right] \sum_{i=m}^{k-1} (p_{2^m-1} t)^{i-m} \\
&\qquad \cdot q_{2^m-1} t \varphi_{ij} \phi_j^{(2^m-2)}(t, t_l, \dots, t_k)
\end{aligned}
$$

$$+ p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] (p_{2^m-1} t)^{k-m} \varphi_{kj},$$

where

$$\varphi_{ij} = \begin{cases} t_l^{[i/l]} \cdots t_i^{[i/k]}, & \text{if } j = 1, \\ t_l \cdots t_i, & \text{if } j = 2, \\ t_l^{i-l+1} t_{l+1}^{i-l} \cdots t_i, & \text{if } j = 3, \\ t_i, & \text{if } j = 4. \end{cases}$$

For each $x, y \in N_m$ and $i \in L_m$,

$$\alpha''_{x,i}(t) = \begin{cases} p_x t \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t) \right] q_{f_1^i(x)} t, & \text{if } 1 \le i \le l-1, \\ p_x t \left[\prod_{j=1}^{i-1} (p_{f_1^j(x)} t) \right] q_{f_1^i(x)} t \varphi_{ij}, & \text{if } l \le i \le m-2, \end{cases}$$

$$\beta''_x(t) = p_x t \left[\prod_{j=1}^{m-2} (p_{f_1^j(x)} t) \right] q_{f_1^{m-1}(x)} t \varphi_{m-1,j}$$

$$+ p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] \sum_{i=m}^{k-1} (p_{2^m-1} t)^{i-m} q_{2^m-1} t \varphi_{ij},$$

$$\gamma''_x(t) = p_x t \left[\prod_{j=1}^{m-1} (p_{f_1^j(x)} t) \right] (p_{2^m-1} t)^{k-m} \varphi_{kj}.$$

For each $x, y \in N_m$,

$$a''_{x,y}(t) = q_x t 1\{y = 2x\} + \sum_{i \in I_{x,y}} \alpha''_{x,i}(t) + \beta''_x(t) 1\{y = 2^m - 2\},$$

$$b''_{x,y}(t) = 1\{x = y\} - a''_{x,y}(t).$$

Define

$$B'' = (b''_{x,y}(t))_{x,y \in N_m},$$
$$(B'')^{-1} = (c''_{x,y})_{x,y \in N_m},$$
$$c''_{x,y} = \frac{\Delta''_{y,x}}{|B''|},$$

where $\Delta''_{x,y}$ is the (x, y)-cofactor of the matrix B''.

From the above system of linear equations, we have

$$\phi_j^{(x)}(t, t_l, \ldots, t_k) = \sum_{i=0}^{2^m-1} \frac{\Delta''_{i,x}}{|B''|} \gamma''_i(t) = \frac{1}{|B''|} \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i(t).$$

Here, we set that for each $x \in N_m$,

$$\psi_j(t, t_l, \dots, t_k) = |B''|,$$
$$\psi_j^{(x)}(t, t_l, \dots, t_k) = \sum_{i=0}^{2^m-1} \Delta''_{i,x} \gamma''_i(t).$$

Then, we have

THEOREM 4.6.

$$\phi_j^{(x)}(t, t_l, \dots, t_k) = \frac{\psi_j^{(x)}(t, t_l, \dots, t_k)}{\psi_j(t, t_l, \dots, t_k)}, \qquad for \quad x \in N_m.$$

Remark 7. In general, when $l < m$, the corresponding marginal distributions depend on the initial condition of the m-th order Markov chain and are not necessarily as simple as in the case $m \leq l$.

Remark 8. By using these results, we can easily obtain the p.g.f. of the joint distribution of the numbers of success-runs of length l and trials until τ by means of a computer algebra. As Remark 2, we use a computer algebra, REDUCE ver. 3.5 for generating the system of equations of conditional p.g.f.'s.

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