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ESTIMATING THE FINITE POPULATION VERSIONS OF MEAN RESIDUAL LIFE-TIME FUNCTION AND HAZARD FUNCTION USING BAYES METHOD

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Abstract. In this paper we introduce a finite population version of the mean residual life-time (MRL) function and the hazard function, and study Bayesian estimation of these functions. The unknown parameter is the complete set $(y_1, \dots, y_N)$ of lifetimes of the N units which constitute the complete population. A hierarchical type prior is used, where the y_i's are assumed conditionally independent given a random parameter θ. The data consists of a random sample of n values of y_i. The Bayes estimators of MRL and hazard functions, respectively, are then obtained as the posterior expectations of the unknown functions.

Key words and phrases: Finite population, mean residual life-time function, hazard function, exponential distribution, prior distribution, posterior distribution, exchangeability, type I censoring, type II censoring.

1. Introduction

Consider a finite population of N units $\{1, 2, \dots, N\}$, with y_i being the lifetime of unit i, $i = 1, \dots, N$. Under Bayesian framework, we regard the finite population of interest as a sample of size N from an infinite or superpopulation, and regard the stochastic procedure generating the surveyor's sample of n units as the following two-step procedure: Step 1: Draw a sample of size N from an infinite super-population. Step 2: Draw a sample of size $n < N$ from the large sample of size N obtained in Step 1. Actually, Step 1 is an imaginary step and Step 2 is the real sample survey conducted by the surveyor.

There are two kinds of reliability evaluation; one where we call for inference on infinite population parameters and another where we are concerned with estimates of the parameters of definite finite population. The conceptual or infinite population is, by definition, in the analyst mind. The real population or finite population, e.g., a collection of parts from a machine, is what it is.

The focus of our attention in this paper are two important parameters of the finite population namely the population mean residual life-time (MRL) function,

$$\gamma_N(t) = \frac{\sum_{k=1}^{N}(y_k - t)I(y_k > t)}{\sum_{k=1}^{N} I(y_k > t)}, \tag{1.1}$$

and hazard function for specified interval $(t, t+\Delta)$,

$$\lambda_N(t, t+\Delta) = \frac{\sum_{k=1}^{N} I(t < y_k \leq t+\Delta)}{\sum_{k=1}^{N} I(y_k > t)}. \tag{1.2}$$

Here $I(A)$ denotes the indicator function of the set A. Physical interpretations of (1.1) and (1.2) are as follows. The $\gamma_N(t)$ in Equation (1.1) is the average remaining life among those population members who have survived until time t. The $\lambda_N(t, t+\Delta)$ in Equation (1.2) is the ratio of the number of deaths (or failures) in the interval $(t, t+\Delta)$ to the number of surviving at the beginning of the interval. From Equations (1.1) and (1.2) we observe that $\gamma_N(t)$ and $\lambda_N(t, t+\Delta)$ are well defined only when $0 \leq t < \max_{1 \leq i \leq N} y_i$. When t is larger, the numerators and denominators are zero. (As usual we define $\frac{0}{0} = 0$.)

The problem of calculating both functions arises quite often in public health, demography, actuarial science and industrial reliability application. Because of limited time and limited budget, life-times of units in the entire population are not observed, $\gamma_N(t)$ and $\lambda_N(t, t+\Delta)$ can not be calculated precisely. Therefore, one has to estimate both of them from a given sample. This paper considers Bayes estimation of both functions which are nonlinear functions of the population units. The prediction approach for linear functions of the population units are developed in the literature under general assumptions; see Bolfarine (1989, 1990) and references therein.

There are many situations of the "analytical studies" type which call for estimating MRL function and hazard function of the infinite population. The MRL function and the hazard function versions for an infinite population are known in the literature and also they are described in the next section. (For more details see Barlow and Proschan (1981).) In this manuscript we will also focus on these two parameters.

The outline of the paper is as follows. In Section 2 we drive Bayesian estimators of both measures under finite population and infinite population. We give general ways of deriving estimators and for illustration of our method we consider a situation where conditioned on a parameters λ the lifetimes are exponential and where λ is Gamma with parameters α and β. Section 3 studies properties of our proposed estimators. Section 4, applies methodology of Section 2 to one example.

2. Bayesian approach

In this section we present a Bayesian approach to estimation of the MRL and the hazard functions. First we estimate MRL and hazard functions of infinite population.

2.1 *The MRL and hazard functions of an infinite population and their estimates*

In this case we think of a population consists of infinitely many independent and identical units. From a practical point of view, it is of course hard to see how one can create infinitely many such units. Still conceptually it might make sense to work sometimes with such infinite populations at least as limiting case.

Let Y be the lifetime of any unit from this population. The MRL and hazard functions at time t are defined as

$$\gamma(t) = E(Y - t \mid Y > t), \tag{2.1}$$

and

$$\lambda(t) = \lim_{\Delta \to 0} P(t < Y \leq t + \Delta \mid Y > t), \tag{2.2}$$

respectively.

Assuming that Y is an exponential with the hazard function λ and that λ is distributed as Gamma distribution with parameters α and β, we get from (2.1) that

$$\gamma(t) = \int_0^\infty E(Y - t \mid Y > t, \lambda) g(\lambda \mid Y > t) d\lambda, \tag{2.3}$$

where $g(\lambda \mid Y > t)$ denotes the conditional density of λ given that $Y > t$. It should be noted that since we are conditioning on the event that $Y > t$, we must use the conditional density of λ given $Y > t$. It is easy to verify that $E(Y - t \mid Y > t, \lambda) = \lambda^{-1}$ and $g(\lambda \mid Y > t)$ is Gamma with parameters α and $\beta + t$. Inserting this to (2.3) we get that

$$\gamma(t) = \int_0^\infty (\lambda)^{-1} \frac{(\beta + t)^\alpha}{\Gamma(\alpha)} \lambda^{\alpha - 1} \exp(-(\beta + t)\lambda) d\lambda. \tag{2.4}$$

Assuming that $\alpha > 1$, the integral in (2.4) exists and the solution is given by

$$\gamma(t) = \frac{\beta + t}{\alpha - 1}. \tag{2.5}$$

By similar arguments the hazard function $\lambda(t)$ in (2.2) is given by

$$\begin{aligned} \lambda(t) &= \int_0^\infty \lambda \frac{(\beta + t)^\alpha}{\Gamma(\alpha)} \lambda^{\alpha - 1} \exp(-(\beta + t)\lambda) d\lambda \\ &= \frac{\alpha}{\beta + t}. \end{aligned} \tag{2.6}$$

Given n observations $y_1, \ldots, y_n$ from this population we can update both (2.5) and (2.6) in the usual Bayesian fashion. Hence the Bayesian estimators of $\gamma(t)$ and $\lambda(t)$ become as follows

$$\gamma_B(t \mid y_1, \ldots, y_n) = \frac{\beta + \sum_{i=1}^n y_i + t}{\alpha + n - 1}, \tag{2.7}$$

and

$$\lambda_B(t \mid y_1, \ldots, y_n) = \frac{\alpha + n}{\beta + \sum_{i=1}^{n} y_i + t}, \tag{2.8}$$

respectively.

We observe that (2.5) and (2.7) are increasing functions of t while (2.6) and (2.8) are decreasing functions of t. When $n \to \infty$, however, (2.7) and (2.8) converges almost surely to the constants λ^{-1} and λ respectively.

Suppose now that the experiment is terminated after the r smallest observations from a sample of size n have been observed. This is referred to as type II censoring. In this case the Bayesian estimators of $\gamma(t)$ and $\lambda(t)$ become

$$\gamma_B^{(2)}(t \mid y(1), \ldots, y(r)) = \frac{\beta + \sum_{i=1}^{r} y(i) + (n-r)y(r) + t}{\alpha + r - 1}, \tag{2.9}$$

and

$$\lambda_B^{(2)}(t \mid y(1), \ldots, y(r)) = \frac{\alpha + r}{\beta + \sum_{i=1}^{r} y(i) + (n-r)y(r) + t}, \tag{2.10}$$

respectively, where $y(i)$, $i = 1, \ldots, n$, denotes the i-th order statistic.

If the experiment is terminated after a fixed time T, this is referred to as type I censoring, and if r smallest observations are observed prior to termination. Then the Bayesian estimators of $\gamma(t)$ and $\lambda(t)$ may be expressed as

$$\gamma_B^{(1)}(t \mid y(1), \ldots, y(r)) = \frac{\beta + \sum_{i=1}^{r} y(i) + (n-r)T + t}{\alpha + r - 1}, \tag{2.11}$$

and

$$\lambda_B^{(1)}(t \mid y(1), \ldots, y(r)) = \frac{\alpha + r}{\beta + \sum_{i=1}^{r} y(i) + (n-r)T + t}. \tag{2.12}$$

2.2 *Bayesian estimators of $\gamma_N(t)$ and $\lambda_N(t, t+\Delta)$*

In this case we are concerned with Bayes estimates of $\gamma_N(t)$ and $\lambda_N(t, t+\Delta)$ of the finite population. Let s be a subset of $\{1, 2, \ldots, N\}$ and let $n(s)$ be the number of elements in s. For simplicity we shall consider only samples of fixed size n. Let $\pi(y_1, \ldots, y_N)$ denote the prior density or probability function of the Bayesian statistician. The prior density $\pi(y_1, \ldots, y_N)$ would be chosen to represent and summarize the prior beliefs and information of statistician about $(y_1, \ldots, y_N)$. Given the sample $(s, y(s))$ one then can find the conditional joint density of $(y_1, \ldots, y_N)$ given $(s, y(s))$. Here,

$$y(s) = \{y_i; i \epsilon s\}, \tag{2.13}$$

and

$$y(\bar{s}) = \{y_i; i \epsilon \bar{s}\}. \tag{2.14}$$

Estimation in finite population sampling can now be thought of as a prediction problem. That is, predicting the unseen, $y(\bar{s})$, from the seen $y(s)$.

For estimating $\gamma_N(t)$ under squared error loss, the Bayes estimator against the prior π is

$$\hat{\gamma}_N(t) = E_\pi \left[\frac{a_1(t) + W(t)}{b(t) + V(t)} \,\middle|\, (s, y(s)) \right] - t, \tag{2.15}$$

where

$$a_1(t) = \sum_{i \epsilon s} y_i I(y_i > t), \tag{2.16}$$

$$b(t) = \sum_{i \epsilon s} I(y_i > t), \tag{2.17}$$

$$W(t) = \sum_{i \epsilon \bar{s}} y_i I(y_i > t), \tag{2.18}$$

and

$$V(t) = \sum_{i \epsilon \bar{s}} I(y_i > t). \tag{2.19}$$

In Equation (2.15), given $(s, y(s))$, both $a_1(t)$ and $b(t)$ are treated as constants.

In many situations it can be quite difficult to specify a sensible prior distribution π and then carry out a Bayesian analysis. This is particularly true for problems with a large dimensional parameter space, of which finite population sampling forms an important subset. For such large scaled problems a Bayesian analysis seems impossible without some simplifying assumptions that allow us to model our prior information. A naive first guess might be to assume that $y = (y_1, \ldots, y_N)$ are independent and identically distributed. Under this prior, we see from Equation (2.15), the Bayes estimator of $\gamma_N(t)$ reduces to $E_\pi(\frac{a_1(t)+W(t)}{b(t)+V(t)})$. Because of the independence we see that the seen, $y(s)$, gives us no information about the unseen $y(s')$. In order to relate the unseen to the seen, we need a prior which makes the $y_1, \ldots, y_N$ dependent. Since in many cases, for example in reliability all the units in the finite population are manufactured by the same process, $y_1, \ldots, y_N$ enjoy some similarity, therefore a judgment of exchangeability among units is reasonable. A technique for defining exchangeable distributions which is quite useful in finite population sampling, was proposed by Ericson (1969), see also Ghosh and Meeden (1996). We use his technique to construct $\pi(y_1, \ldots, y_N)$.

Let $\theta = (\theta_1, \ldots, \theta_m)$ be a vector valued parameter. We assume that θ has a probability distribution given by probability density function g. Moreover, we assume that given θ, $y_1, \ldots, y_N$ are independent and have a common probability density function $f(\cdot \mid \theta)$. Unconditionally, i.e. integrating out θ, this defines a probability density for $y_1, \ldots, y_N$ given by

$$\pi(y_1, \ldots, y_N) = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{N} f(y_i \mid \theta) \right) g(\theta) d\theta. \tag{2.20}$$

In the Bayesian framework θ is sometimes called a hyperparameter and is introduced as a mixing parameter to generate exchangeable distributions from independent and identically distributed distributions.

For our purpose as we did in Subsection 2.1, we assume that in (2.20)

$$f(y \mid \lambda) = \lambda \exp(-\lambda y). \tag{2.21}$$

and

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\beta\lambda), \tag{2.22}$$

where $\alpha, \beta > 0$ and are all known. It should be mentioned that $\frac{\alpha}{\beta}$ is our prior guess about the mean of the population and $\frac{\alpha}{\beta^2}$ is a measure of how certain we are about the choice of the mean.

From (2.15), it is clear that

$$\hat{\gamma}_N(t) = \sum_{k=0}^{N-n} \frac{1}{b(t)+k} [a_1(t) + E(W(t) \mid (s, y(s)), V(t) = k)] \\ \times P(V(t) = k \mid (s, y(s))) - t. \tag{2.23}$$

The main issue now is to compute (2.23) for an exponential model. Given the model assumptions in (2.21) and (2.22),

$$\begin{aligned} &P(V(t) = k \mid (s, y(s))) \\ &= \int_0^\infty \binom{N-n}{k} (\exp(-k\lambda t))(1 - \exp(-\lambda t))^{N-n-k} \\ &\times \frac{(\beta + a_1(0))^{n+\alpha}}{\Gamma(n+\alpha)} \lambda^{n+\alpha-1} \exp(-(\beta + a_1(0))\lambda) d\lambda \\ &= \sum_{j=0}^{N-n-k} (-1)^j \frac{(N-n)!}{k! j! (N-n-(j+k))!} \left(\frac{\beta + a_1(0)}{\beta + a_1(0) + (k+j)t} \right)^{n+\alpha}. \end{aligned} \tag{2.24}$$

Note that the first term in the integral is $P(V(t) = k \mid \lambda, s, y(s))$ and the second term is the conditional density of λ given $(s, y(s))$. Similarly,

$$E(W(t) \mid (s, y(s)), V(t) = k) = kt + kE\left(\frac{1}{\lambda} \,\middle|\, (s, y(s)), V(t) = k \right). \tag{2.25}$$

To get the expression (2.25), we use the fact that given $V(t) = k$ and λ there is a sample of size k from $(\frac{f(y|\lambda)}{\bar{F}(t|\lambda)}) I(y > t)$. Now to compute the second term in Equation (2.25) we need the posterior distribution of λ given $(s, y(s))$ and $V(t) = k$. One can easily verify that the posterior distribution of $(\lambda \mid (s, y(s)), V(t) = k)$ is

$$\left(\sum_{j=0}^{N-n-k} (-1)^j \frac{\lambda^{n+\alpha-1}}{j!(N-n-(k+j))!} \exp(-(\beta + a_1(0) + (k+j)t)\lambda) \right) \\ \times \left(\sum_{j=0}^{N-n-k} (-1)^j \frac{(\beta + a_1(0) + (k+j)t)^{-n-\alpha} \Gamma(n+\alpha)}{j!(N-n-(k+j))!} \right)^{-1}. \tag{2.26}$$

Using (2.26), Equation (2.25) reduces to

$$(2.27)\qquad \begin{aligned} &E(W(t) \mid (s, y(s)), V(t) = k) \\ &= kt + \Bigg[k(n+\alpha-1)^{-1} \\ &\qquad \times \sum_{j=0}^{N-n-k} (-1)^j \frac{(\beta + a_1(0) + (k+j)t)^{-n-\alpha+1}}{j!(N-n-(k+j))!} \\ &\qquad \times \left(\sum_{j=0}^{N-n-k} \frac{(-1)^j(\beta + a_1(0) + (k+j)t)^{-n-\alpha}}{(N-n-(k+j))!j!} \right)^{1} \Bigg]. \end{aligned}$$

Substituting from (2.24) and (2.27), into Equation (2.23) one finds that the Bayes estimator of $\gamma_N(t)$ is

$$(2.28)\qquad \begin{aligned} \hat{\gamma}_N(t) = &\sum_{k=0}^{N-n} \sum_{j=0}^{N-n-k} (-1)^j \\ &\times \frac{(N-n)!(a_1(t)+kt)\left(\dfrac{\beta + a_1(0)}{\beta + a_1(0) + (k+j)t}\right)^{n+\alpha}}{(k!j!(N-n-(j+k))!)(b(t)+k)} \\ &+ \sum_{k=1}^{N-n} \sum_{j=0}^{N-n-k} \sum_{i=0}^{N-n-k} \left[\frac{(n+\alpha-1)^{-1}(-1)^{j+i}}{j!(N-n-(k+j))!i!} \right. \\ &\qquad \left. \times \frac{(N-n)!}{(N-n-(i+k))!(k-1)!} \right] \\ &\times \frac{1}{b(t)+k} \left(\frac{\beta + a_1(0)}{\beta + a_1(0) + (k+j)t} \right)^{n+\alpha} \\ &\times (\beta + a_1(0) + (k+i)t)^{-n-\alpha+1} \\ &\times \left(\sum_{\ell=0}^{N-n-k} (-1)^\ell \frac{(\beta + a_1(0) + (k+\ell)t)^{-n-\alpha}}{(N-n-(k+\ell))!\ell!} \right)^{-1} - t. \end{aligned}$$

For estimating $\lambda_N(t, t+\Delta)$ under squared error loss, the Bayes estimator against the prior π is

$$(2.29)\qquad \hat{\lambda}_N(t, t+\Delta) = E\left[\frac{a_2(t) + \sum_{i \epsilon \bar{s}} I(t < Y_i \le t + \Delta)}{b(t) + V(t)} \,\middle|\, s, y(s) \right],$$

where

$$(2.30)\qquad a_2(t) = \sum_{i \epsilon s} I(t < y_i \le t + \Delta).$$

Given the model assumptions in (2.21) and (2.22), by reasoning similar to that used in obtaining Equation (2.28) and after some algebraic manipulation it may be verified that the final expression for the Bayes estimator of $\lambda_N(t, t+\Delta)$ is

$$
\begin{aligned}
(2.31)\quad \hat{\lambda}_N(t,t+\Delta) = \sum_{k=0}^{N-n}\sum_{j=0}^{N-n-k} & \frac{(N-n)!a_2(t)}{(b(t)+k)} \\
& \times \frac{(-1)^j\left(\frac{\beta+a_1(0)}{\beta+a_1(0)+(k+j)t}\right)^{n+\alpha}}{k!j!(N-n-(j+k))!} \\
& + \sum_{k=1}^{N-n}\sum_{j=0}^{N-n-k}\sum_{i=0}^{N-n-k}\frac{(N-n)!}{b(t)+k} \\
& \times \frac{(-1)^{i+j}\left(\frac{\beta+a_1(0)}{\beta+a_1(0)+(k+j)t}\right)^{n+\alpha}}{(k-1)!j!(N-n-(k+j))!} \\
& \times [(\beta+a_1(0)+(k+i)t)^{-n-\alpha} \\
& \quad - (\beta+a_1(0)+(k+i)t+\Delta)^{-n-\alpha}] \\
& \times \frac{1}{i!(N-n-(i+k))!} \\
& \times \left(\sum_{\ell=0}^{N-n-k}(-1)^\ell\frac{(\beta+a_1(0)+(k+\ell)t)^{-n-\alpha}}{\ell!(N-n-(\ell+k))!}\right)^{-1}.
\end{aligned}
$$

Now, supposing $\Delta \to 0$, from (2.31), one can show that

$$
\begin{aligned}
(2.32)\quad & \lim_{\Delta\to 0}\frac{\hat{\lambda}_N(t,t+\Delta)}{\Delta} \\
& = \sum_{k=0}^{N-n}\sum_{j=0}^{N-n-k}\frac{a_3(t)(-1)^j\left(\frac{\beta+a_1(0)}{\beta+a_1(0)+(k+j)t}\right)^{n+\alpha}(N-n)!}{(b(t)+k)k!j!(N-n-(j+k))!} \\
& + \sum_{k=1}^{N-n}\sum_{j=0}^{N-n-k}\sum_{i=0}^{N-n-k}\left[\frac{(-1)^{i+j}\left(\frac{\beta+a_1(0)}{\beta+a_1(0)+(k+j)t}\right)^{n+\alpha}}{(b(t)+k)(k-1)!j!i!}\right. \\
& \qquad \left.\times\frac{(N-n)!}{(N-n-(k+j))!(N-n-(i+k))!}\right] \\
& \times (n+\alpha)(\beta+a_1(0)+(k+i)t)^{-n-\alpha-1} \\
& \times \left(\sum_{\ell=0}^{N-n-k}(-1)^\ell\frac{(\beta+a_1(0)+(k+\ell)t)^{-n-\alpha}}{\ell!(N-n-(k+\ell))!}\right)^{-1},
\end{aligned}
$$

where $a_3(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \times \sum_{i \epsilon s} I(t < y_i \leq t + \Delta)$.

Under type II censoring one can consider the following two cases:

Case I. $\sum_{i=1}^{r} I(y(i) > t) \geq 1$. Under this case by reasoning similar to that used in obtaining Equation (2.28) it may be verified that the final expression for the Bayes estimator of $\gamma_N(t)$ is similar to the expression (2.28) where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)y(r)$ and $a_1(t)$ is replaced by

$$\sum_{i=1}^{r} y(i) I(y(i) > t) + (n-r)y(r) + (n-r)\frac{\beta + \sum_{i=1}^{r} y(i) + (n-r)y(r)}{\alpha + r - 1}.$$

Also, the Bayes estimator of $\lambda_N(t, t+\Delta)$ is similar to the expression (2.31) where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)y(r)$ and $a_2(t)$ is replaced by

$$\sum_{i=1}^{r} I(t < y(i) \leq t + \Delta) + (n-r)\frac{\alpha + r}{\beta + \sum_{i=1}^{r} y(i) + (n - r - 1)y(r) + t + \Delta}.$$

Case II. $\sum_{i=1}^{r} I(y(i) > t) = 0$. Under this case the Bayes estimator of $\gamma_N(t)$ is similar to the expression (2.28), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n - r)y(r)$, n is replaced by r, and $b(t) = a_1(t) = 0$. Similarly the Bayes estimator of $\lambda_N(t, t+\Delta)$ is similar to the expression in (2.31), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)y(r)$, n is replaced by r, and $b(t) = a_2(t) = 0$. In this case $\frac{0}{0} = 0$.

Under type I censoring again one can consider two cases:

Case I. $\sum_{i=1}^{r} I(y(i) > t) \geq 1$. Under this case, the Bayes estimator of $\gamma_N(t)$ is similar to the expression in (2.28), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)T$ and $a_1(t)$ is replaced by

$$\sum_{i=1}^{r} y(i) I(y(i) > t) + (n-r)T + (n-r)\frac{\beta + \sum_{i=1}^{r} y(i) + (n-r)T}{\alpha + r - 1}.$$

Similarly, the Bayes estimator of $\lambda_N(t, t+\Delta)$ is similar to the expression in (2.31), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)T$ and $a_1(t)$ is replaced by

$$\sum_{i=1}^{r} I(t < y(i) \leq t + \Delta) + (n-r)\frac{\alpha + r}{\beta + \sum_{i=1}^{r} y(i) + (n - r - 1)T + t + \Delta}.$$

Case II. $\sum_{i=1}^{r} I(y(i) > t) = 0$. Under this case the Bayes estimator of $\gamma_N(t)$ is similar to the expression in (2.28), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)T$, n is replaced by r and $a_1(t) = b(t) = 0$. Similarly, the Bayes estimator of $\lambda_N(t, t+\Delta)$ is similar to the expression in (2.31), where $a_1(0)$ is replaced by $\sum_{i=1}^{r} y(i) + (n-r)T$, n is replaced by r and $b(t) = a_2(t) = 0$. In this case $\frac{0}{0} = 0$.

An important difference between estimators derived in Subsections 2.1 and 2.2 is the following. The classical MRL-concept in (2.1) is defined as the expected remaining lifetime at time t given that the unit has survived time t. The $\gamma_N(t)$ in Equation (1.1) is the average remaining life among those population members who have survived until time t. When estimating $\gamma_N(t)$, it is done based on complete information about the lifetimes of the observed units and no information at all about the unobserved units. Thus, the estimates provided in Subsection 2.2 addresses a different problem compared to the classical MRL estimates in Subsection 2.1. Similar things can be said about hazard function estimates. In the next section we discuss in more details regarding these estimators.

Another function which certainly is of great interest is the finite population survivor function $\bar{F}_N(t)$,

$$\bar{F}_N(t) = \frac{1}{N}\sum_{i=1}^{N} I(y_i > t).$$

For estimating $\bar{F}_N$ under squared error loss, the Bayes estimator against the prior π is

$$\hat{\bar{F}}_N(t) = \frac{1}{N}E(b(t) + V(t) \mid (s, y(s))), \tag{2.33}$$

where $b(t)$ and $V(t)$ are given by (2.17) and (2.19) respectively. By reasoning similar to that used in obtaining Equation (2.28) and after some algebraic manipulation, it may be verified that under Model (2.21) and (2.22) the expression (2.33) reduces to

$$\hat{\bar{F}}_N(t) = \frac{b(t)}{N} + \frac{1}{N}\sum_{k=1}^{N-n}\sum_{j=0}^{N-n-k}(-1)^j\frac{(N-n)!}{(k-1)!j!(N-n-(j+k))!} \times \left(\frac{(\beta + a_1(0))}{\beta + a_1(0) + (k+j)t}\right)^{n+\alpha}. \tag{2.34}$$

Under type II censoring the Bayes estimator of $\bar{F}_N(t)$ is similar to Expression (2.34) where $a_1(0) = \sum_{i=1}^{r} y(i) + (n-r)y(r)$, $b(t) = \sum_{i=1}^{r} I(y(i) > t) + (n-r)$ if $\sum_{i=1}^{r}(y(i) > t) > 1$, $b(t) = 0$ if $\sum_{i=1}^{r} I(y(i) > t) = 0$ and $n = r$ if $\sum_{i=1}^{r} I(y(i) > t) = 0$. Similarly under type I censoring, the Bayes estimator of $\bar{F}_N(t)$ is similar to Expression (2.34) where $a_1(0) = \sum_{i=1}^{r} y(i) + (n-r)T$, $b(t) = \sum_{i=1}^{r} I(y(i) > t) + (n-r)$ if $\sum_{i=1}^{r} I(y(i) > t) > 1$ and $b(t) = 0$ if $\sum_{i=1}^{r} I(y(i) > t) = 0$, and $n = r$ if $\sum_{i=1}^{r} I(y(i) > t) = 0$.

Optimal prediction of the finite population distribution function $F_N(t) = \frac{1}{N}\sum_{i=1}^{N} I(y_i \leq t)$ is considered in Bolfarine and Sandoval (1993). They assumed that $y_1, \ldots, y_N$ are independent and their approach was non-Bayesian.

Remark 2.1. We can obtain the posterior distribution functions of $\gamma_N(t)$, $\lambda_N(t)$ and $\bar{F}_N(t)$ given $(s, y(s))$ using arguments similar to that used in obtaining Equations (2.28), (2.31) and (2.34). However, the forms of these distributions are very complicated. It should be mentioned that we only used the means of these distributions in constructing our Bayesian estimators.

3. Properties of the estimators

In this section we study properties of our estimators proposed in Subsections 2.1 and 2.2.

The results of this section are summarized as follows.

THEOREM 3.1.
(a) *As* $N \to \infty$, $\hat{\gamma}_N(t)$ *converges to* $\gamma_B(t)$.
(b) *As* $N \to \infty$, $\hat{\lambda}_N(t)$ *converges to* $\lambda_B(t)$.

PROOF. (a) From (2.15), for large N since the leading terms are $W(t)$ and $V(t)$ we get that $\hat{\gamma}_N(t)$ and $E_\pi[\frac{W(t)}{V(t)} \mid (s, y(s))] - t$ are equivalent. From (2.18) and (2.19) it suffices to notice that as $N \to \infty$,

$$E_\pi\left[\frac{W(t)}{V(t)} \,\middle|\, (s, y(s))\right] - t$$

goes to

$$E_\pi[Y - t \mid Y > t, (s, y(s))] = \gamma_B(t).$$

This completes the proof.
(b) Similar arguments can be said about $\hat{\lambda}_N(t)$.

It should be pointed out that the results of Theorem 3.1 always holds irrespective of type of distribution. Theorem 3.1 states that for large N estimators proposed in Subsections 2.1 and 2.2 are equivalent. Similar results can be obtained for both type I censoring and type II censoring.

4. Illustrative example

The following example illustrates the procedure proposed in Section 2.

Example. The Lifetime Light Bulb Company makes an incandescent filament that they believe does not wear out during an extended period of normal use. The company takes a random sample of 100 bulbs and test them all until failure. The sample data is given in Table 1. It should be mentioned that the company came up with a test plan that can simulate a month of typical use by a buyer in 1 hour of laboratory testing (using higher than normal voltages).

Tobias and Trindade (1995) state that the exponential does a fair job of describing the survival function of a bulb.

From Equation (2.28) we find $\hat{\gamma}_N(t)$. In (2.28), we choose α and β to provide a rather vague specification. In particular $\beta = .002$ and $\alpha = (\lambda(0))\beta$, where $\lambda(0) = \frac{100}{\sum_{i=1}^{100} y_i}$. That is, we employed data motivated guess for the prior mean for λ. The analysis is not sensitive to these choices. Figure 1 gives $\hat{\gamma}_N(t)$ for various values of N. Similarly, from Equation (2.32) we find $\hat{\lambda}_N(t) = \lim_{\Delta\to 0} \frac{\hat{\lambda}_N(t,t+\Delta)}{\Delta}$. Figure 2 gives $\hat{\lambda}_N(t)$ for various values of N. Figures 1 and 2 also give the classical Bayes

Table 1. Sample data of equivalent month of bulb failure.

1	2	2	3	4	5	7	8	9	10
11	13	15	16	17	17	18	18	18	20
20	21	21	24	27	29	30	37	40	40
40	41	46	47	48	52	54	54	55	55
64	65	65	65	67	76	76	79	80	80
82	86	87	89	94	96	100	101	102	104
105	109	109	120	123	141	150	156	156	161
164	167	170	178	181	191	193	206	211	212
214	236	238	240	265	304	317	328	355	363
365	369	389	404	427	435	500	522	547	889

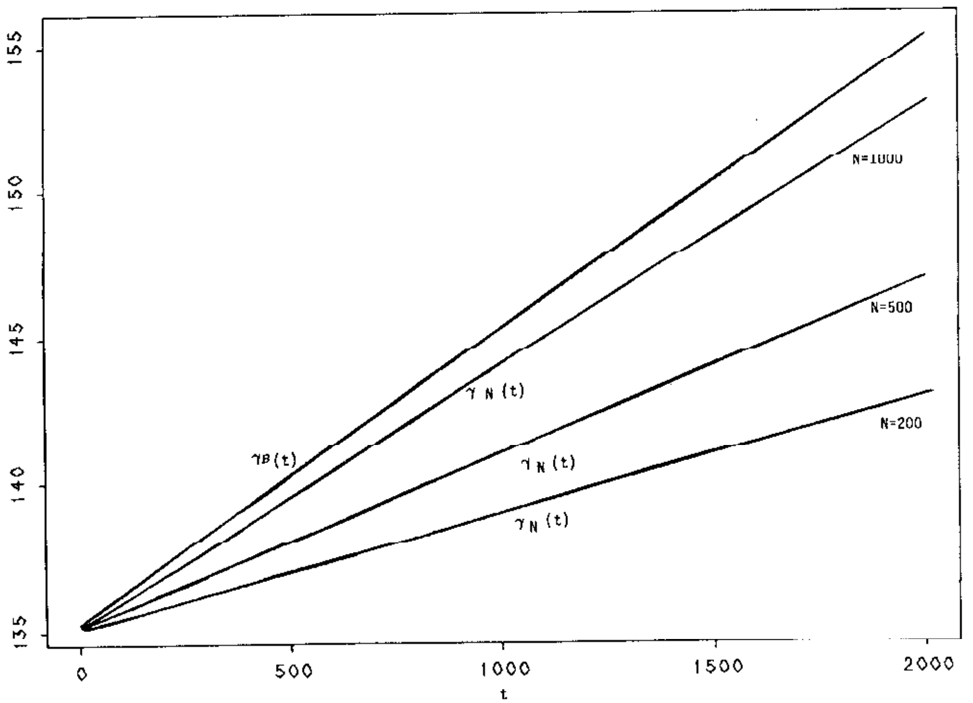

Fig. 1.

estimators of MRL function $\gamma_B(t)$ and hazard function $\lambda_B(t)$ given by Equations (2.7) and (2.8) respectively for the same values of α and β. It should be pointed out that $\gamma_B(t) = \frac{13563.002+t}{99.0014}$ and $\lambda_B(t) = \frac{100.0014}{13563.002+t}$.

From Fig. 1 we see that our estimator of $\gamma_N(t)$, $\hat{\gamma}_N(t)$ is consistently smaller than $\gamma_B(t)$. However, as we also proved in Section 3, when N gets larger our estimator approaches the classical Bayes estimator. Similar things can be said

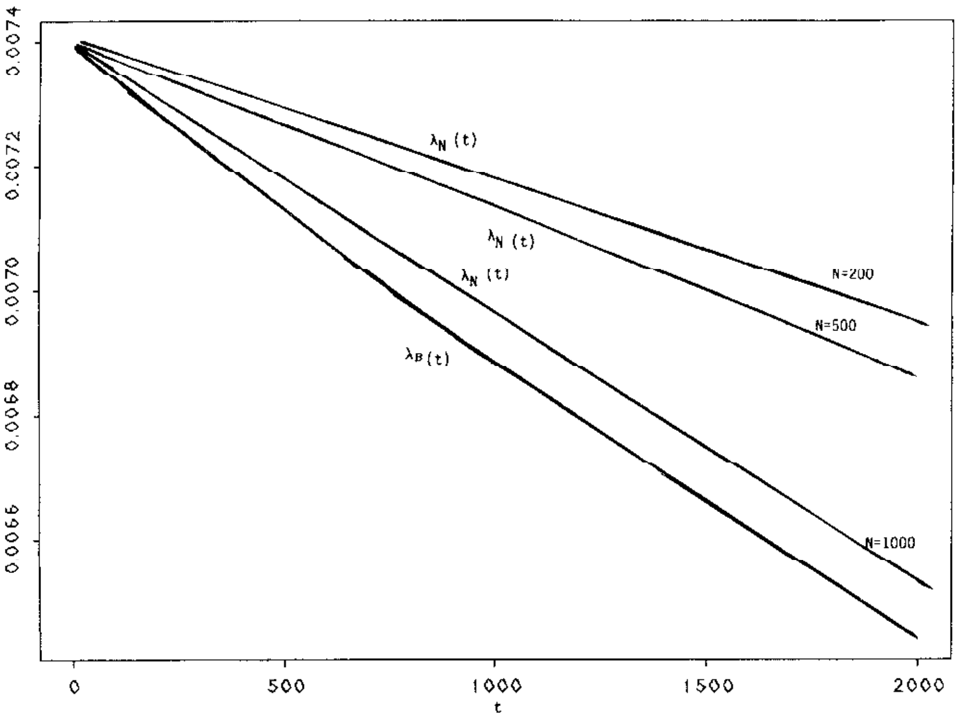

Fig. 2.

about $\lambda_B(t)$ and $\hat{\lambda}_N(t)$. However in this case our estimator is consistently larger than $\lambda_B(t)$.

References

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing*, To Begin with, Silver Spring, MD.

Bolfarine, Heleno (1989). Bayesian modeling in finite populations, *South African Statist. J.*, **23**, 157–166.

Bolfarine, Heleno (1990). Bayesian linear prediction in finite populations, *Ann. Inst. Statist. Math.*, **42**, 435–444.

Bolfarine, Heleno and Sandoval, M. C. (1993). Prediction of the finite population distribution function under Gaussian Supper Population models, *Austral. J. Statist.*, **35**, 195–204.

Ericson, W. A. (1969). Subjective Bayesian models in sampling finite populations, *J. Roy. Statist. Soc. Ser. B*, **31**, 195–234.

Ghosh, M. and Meeden, G. (1996). Bayesian inference in-finite population, Forthcoming monograph by Chapman and Hall.

Tobias, P. A. and Trindade, D. C. (1995). *Applied Reliability*, van Nostrand, New York.