Ann. Inst. Statist. Math.
Vol. 48, No. 4, 757–771 (1996)

RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF PROGRESSIVE TYPE-II RIGHT CENSORED ORDER STATISTICS FROM EXPONENTIAL AND TRUNCATED EXPONENTIAL DISTRIBUTIONS

RITA AGGARWALA AND N. BALAKRISHNAN

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

(Received June 26, 1995; revised December 28, 1995)

Abstract. In this paper, we establish several recurrence relations satisfied by the single and product moments of progressive Type-II right censored order statistics from an exponential distribution. These relations may then be used, for example, to compute all the means, variances and covariances of exponential progressive Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \dots, R_m)$, $m \leq n$. The results presented in the paper generalize the results given by Joshi (1978, *Sankhyā Ser. B*, **39**, 362–371; 1982, *J. Statist. Plann. Inference*, **6**, 13–16) for the single moments and product moments of order statistics from the exponential distribution.

To further generalize these results, we consider also the right truncated exponential distribution. Recurrence relations for the single and product moments are established for progressive Type-II right censored order statistics from the right truncated exponential distribution.

Key words and phrases: Progressive Type-II right censored order statistics, single moments, product moments, recurrence relations, exponential distribution, right truncated exponential distribution.

1. Introduction

The scheme of progressive censoring is a useful one in the realm of reliability and life time studies. Its allowance for removal of live units from the test at various stages during the experiment will potentially save the experimenter cost while still allowing for the observation of some extreme data. A number of authors have discussed inference problems for various distributions when data are obtained by progressive censoring, including Cohen (1963, 1966, 1975, 1976), Mann (1969, 1971), Gibbons and Vance (1983), Cohen and Whitten (1988), Cohen (1991) and, more recently, Viveros and Balakrishnan (1994). Balakrishnan and Sandhu (1995a, 1995b) have also discussed some mathematical properties of the special set of order statistics which are obtained through this progressive censoring in the special cases when failure times are uniformly or exponentially distributed.

We begin with the following assumptions and notation: Suppose n independent items are put on test with continuous, identically distributed failure times $X_1, X_2, \ldots, X_n$. Suppose further that a censoring scheme $(R_1, R_2, \ldots, R_m)$ is fixed such that immediately following the first failure, R_1 surviving items are removed from the experiment at random; immediately following the first failure after that point, i.e. the second observed failure, R_2 surviving items are removed from the experiment at random; this process continues until, at the time of the m-th observed failure, R_m items are removed from the test at random. We will denote the m ordered observed failure times by $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$, and call them the progressive Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$. It is clear that $n = m + R_1 + R_2 + \cdots + R_m$. If the failure times of the n items originally on test are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, then the joint probability density function of $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is given by (Balakrishnan and Sandhu (1995a))

$$f_{1,2,\ldots,m:m:n}(x_1, \ldots, x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i}, \quad 0 < x_1 < \cdots < x_m < \infty \tag{1.1}$$

where $A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$. Here, note that all the factors in $A(n, m-1)$ are positive integers. Similarly, for convenience in notation, let us define

$$\begin{aligned} A(p,q) = p(p - R_1 - 1)(p - R_1 - R_2 - 2) \cdots \\ (p - R_1 - R_2 - \cdots - R_q - q) \end{aligned} \qquad \text{for} \quad q = 0, 1, \ldots, p-1, \tag{1.2}$$

with all the factors being positive integers.

In this paper, by assuming the underlying distribution of failure times is exponential, we derive recurrence relations for the single and product moments of the corresponding progressive Type-II right censored order statistics which will allow for the recursive computation of these moments for all sample sizes and all possible censoring schemes. We further generalize this result to right truncated exponential failure time distributions. The results which are obtained generalize the results given by Joshi (1978, 1982) for the usual order statistics from the exponential and right truncated exponential distributions.

2. Recurrence relations for single moments—exponential distribution

Let $X_1, X_2, \ldots, X_n$ denote a random sample from the standard exponential distribution, that is, with probability density function $f(x) = e^{-x}$ and cumulative distribution function $F(x) = 1 - e^{-x}$. The characterizing differential equation for the standard exponential distribution is $f(x) = 1 - F(x)$. This relationship will be exploited in this section to derive recurrence relations for the single moments of exponential progressive Type-II right censored order statistics, which can be written from (1.1) as:

$$
\begin{aligned}
(2.1)\quad & \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} \\
&= E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}]^k \\
&= A(n, m-1) \\
&\cdot \iint\limits_{0<x_1<\cdots<x_m<\infty} \cdots \int x_i^k f(x_1)[1-F(x_1)]^{R_1} f(x_2)[1-F(x_2)]^{R_2} \cdots \\
&\qquad f(x_m)[1-F(x_m)]^{R_m} dx_1 \cdots dx_m.
\end{aligned}
$$

When $k = 1$, the superscript in the notation of the mean of the progressive Type-II right censored order statistic may be omitted without any confusion.

THEOREM 2.1. *For $2 \le m \le n$ and $k \ge 0$,*

$$
\begin{aligned}
(2.2)\quad \mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} &= \frac{1}{R_1+1}[(k+1)\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k)}} \\
&\qquad - (n - R_1 - 1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}}],
\end{aligned}
$$

and for $m = 1$, $n = 1, 2, \ldots$ and $k \ge 0$,

$$
(2.3)\quad \mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{n}\mu_{1:1:n}^{(n-1)^{(k)}}.
$$

PROOF. Relations in (2.2) and (2.3) may be proved by following exactly the same steps as those used in proving Theorem 2.2, which is presented next.

Remark 1. Although it has not been stated, it is true that the first progressive Type-II right censored order statistic is the same as the first usual order statistic from a sample of size n, regardless of the censoring scheme employed. This is because no censoring has taken place before this time. However, these recurrence relations have been included in order to establish completeness even without this knowledge.

THEOREM 2.2. *For $2 \le i \le m-1$, $m \le n$ and $k \ge 0$,*

$$
\begin{aligned}
(2.4)\quad \mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} &= \frac{1}{R_i+1}[(k+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} \\
&\qquad - (n - R_1 - R_2 - \cdots - R_i - i) \\
&\qquad \cdot \mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} \\
&\qquad + (n - R_1 - R_2 - \cdots - R_{i-1} - i + 1) \\
&\qquad \cdot \mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}].
\end{aligned}
$$

PROOF. Let us consider, substituting $f(x_i) = 1 - F(x_i)$ into (2.1),

$$\begin{aligned}
(2.5)\quad & \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} \\
&= A(n,m-1) \underset{0<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} \\
&\cdot \left\{ \int_{x_{i-1}}^{x_{i+1}} x_i^k [1-F(x_i)]^{R_i+1} dx_i \right\} f(x_1)[1-F(x_1)]^{R_1} \cdots \\
& f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}} \cdots \\
& f(x_m)[1-F(x_m)]^{R_m} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m.
\end{aligned}$$

Integrating the innermost integral by parts, we obtain upon simplification,

$$\begin{aligned}
& \int_{x_{i-1}}^{x_{i+1}} x_i^k [1-F(x_i)]^{R_i+1} dx_i \\
&= \frac{1}{k+1} \Big\{ x_{i+1}^{k+1} [1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+1} [1-F(x_{i-1})]^{R_i+1} \\
&\qquad + (R_i+1) \int_{x_{i-1}}^{x_{i+1}} x_i^{k+1} [1-F(x_i)]^{R_i} f(x_i) dx_i \Big\}.
\end{aligned}$$

Substituting this into equation (2.5), we obtain

$$\begin{aligned}
& \mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} \\
&= \frac{n-R_1-\cdots-R_i-i}{k+1} \mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} \\
&\quad - \frac{n-R_1-\cdots-R_{i-1}-i+1}{k+1} \mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}} \\
&\quad + \frac{R_i+1}{k+1} \mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}}.
\end{aligned}$$

Rewriting the above equation, we obtain the relation in (2.4).

THEOREM 2.3. *For $2 \le m \le n$ and $k \ge 0$,*

$$(2.6)\quad \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{k+1}{R_m+1} \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} + \mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}}.$$

PROOF. The relation in (2.6) may be proved by following exactly the same steps as those used earlier in proving Theorem 2.2.

Remark 2. Using these recurrence relations, we can obtain all the single moments of all progressive Type-II right censored order statistics for all sample sizes and censoring schemes $(R_1, \ldots, R_m)$ in a simple recursive manner. The recursive algorithm will be described in detail in Section 4.

3. Recurrence relations for product moments—exponential distribution

Using (1.1) we can write the (i,j)-th product moment of the progressive Type-II right censored order statistics as follows:

$$(3.1)\quad \begin{aligned} &\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)} \\ &= E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)} X_{j:m:n}^{(R_1,R_2,\ldots,R_m)}] \\ &= A(n,m-1) \\ &\cdot \iint\limits_{0<x_1<\cdots<x_m<\infty}\cdots\int x_i x_j f(x_1)[1-F(x_1)]^{R_1} f(x_2)[1-F(x_2)]^{R_2}\cdots \\ &\qquad f(x_m)[1-F(x_m)]^{R_m}\,dx_1\cdots dx_m. \end{aligned}$$

In this section, assuming a standard exponential distribution for the failure times, we will again exploit the characterizing differential equation $f(x) = 1-F(x)$ to obtain recurrence relations for these product moments which will enable us to compute all the product moments of progressive Type-II right censored order statistics for all sample sizes and all possible censoring schemes.

THEOREM 3.1. *For $1 \le i < j \le m-1$ and $m \le n$,*

$$(3.2)\quad \begin{aligned} \mu_{i,j:m:n}^{(R_1,\ldots,R_m)} = \frac{1}{R_j+1}[\mu_{i:m:n}^{(R_1,\ldots,R_m)} &- (n-R_1-R_2-\cdots-R_j-j) \\ &\cdot \mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)} \\ &+ (n-R_1-R_2-\cdots-R_{j-1}-j+1) \\ &\cdot \mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)}]. \end{aligned}$$

PROOF. For $1 \le i < j \le m-1$, from (2.1) and the fact that $f(x_j) = 1-F(x_j)$,

$$(3.3)\quad \begin{aligned} &\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} \\ &= A(n,m-1) \int\limits_{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}\cdots\int\cdots\int \\ &\cdot \left\{\int_{x_{j-1}}^{x_{j+1}}[1-F(x_j)]^{R_j+1}dx_j\right\} x_i f(x_1)[1-F(x_1)]^{R_1}\cdots \\ &f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}}\cdots \\ &f(x_m)[1-F(x_m)]^{R_m}dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_m. \end{aligned}$$

Integrating the innermost integral by parts, we obtain

$$\begin{aligned} \int_{x_{j-1}}^{x_{j+1}}[1-F(x_j)]^{R_j+1}dx_j &= x_{j+1}[1-F(x_{j+1})]^{R_j+1} - x_{j-1}[1-F(x_{j-1})]^{R_j+1} \\ &\quad + (R_j+1)\int_{x_{j-1}}^{x_{j+1}} x_j[1-F(x_j)]^{R_j} f(x_j)dx_j, \end{aligned}$$

which, when substituted into equation (3.3), results in the following:

$$\begin{aligned}\mu_{i:m:n}^{(R_1,\dots,R_m)} &= (n - R_1 - \cdots R_j - j)\mu_{i,j:m-1:n}^{(R_1,\dots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\dots,R_m)} \\ &\quad - (n - R_1 - \cdots - R_{j-1} - j + 1)\mu_{i,j-1:m-1:n}^{(R_1,\dots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\dots,R_m)} \\ &\quad + (R_j + 1)\mu_{i,j:m:n}^{(R_1,\dots,R_m)}.\end{aligned}$$

Upon rearrangement of this equation, we obtain the relation in (3.2).

Remark 3. Notice that Theorem 3.1 holds even for $j = i+1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,\dots,R_m)} = \mu_{i:m:n}^{(R_1,\dots,R_m)^{(2)}}$.

THEOREM 3.2. *For* $1 \le i \le m-1$ *and* $m \le n$,

$$\tag{3.4} \begin{aligned}\mu_{i,m:m:n}^{(R_1,\dots,R_m)} = \frac{1}{R_m + 1}\Big[\mu_{i:m:n}^{(R_1,\dots,R_m)} &+ (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1) \\ &\cdot \mu_{i,m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},R_{m-1}+R_m+1)}\Big].\end{aligned}$$

PROOF. The relation in (3.4) may be proved by following exactly the same steps as those used earlier in proving Theorem 3.1.

Remark 4. Using these recurrence relations, we can obtain all the product moments of progressive Type-II right censored order statistics for all sample sizes and censoring schemes $(R_1, \dots, R_m)$.

Remark 5. For the special case $R_1 = \cdots = R_m = 0$, so that $m = n$ and all n usual order statistics $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, whose k-th moments are denoted $\mu_{i:n}^{(k)}$ for $1 \le i \le n$ and whose product moments are denoted $\mu_{i,j:n}$ for $1 \le i < j \le n$, are obtained, the relations in Sections 2 and 3 reduce to the following:

From equation (2.2): For $k \ge 0$,

$$\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n-1:n}^{(1,0,\dots,0)^{(k+1)}}.$$

From equation (2.4): For $2 \le i \le n-1$ and $k \ge 0$,

$$\begin{aligned}\mu_{i:n}^{(k+1)} &= (k+1)\mu_{i:n}^{(k)} - (n-i)\mu_{i:n-1:n}^{(0,\dots,0,1,0,\dots,0)^{(k+1)}} \\ &\quad + (n-i+1)\mu_{i-1:n-1:n}^{(0,\dots,0,1,0,\dots,0)^{(k+1)}},\end{aligned}$$

where, in the superscript of the second term on the right hand side, the 1 is in the i-th position, and in the superscript of the third term on the right hand side, the 1 is in the $(i-1)$-th position.

From equation (2.6): For $k \ge 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n-1:n}^{(0,\dots,0,1)^{(k+1)}}.$$

Now, upon realizing that if $R_1 = R_2 = \cdots = R_{j-1} = 0$, so that there is no censoring before the time of the j-th failure, then the first j progressive Type-II right censored order statistics are simply the first j usual order statistics, the above three relations simply reduce to the following results in the case of $R_1 = R_2 = \cdots = R_m = 0$:

$$\text{(i)}\quad \mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n}^{(k+1)} \quad \text{or}$$
$$\mu_{1:n}^{(k+1)} = \frac{k+1}{n}\mu_{1:n}^{(k)}, \qquad n \geq 1;$$
$$\text{(ii)}\quad \mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} - (n-i)\mu_{i:n}^{(k+1)} + (n-i+1)\mu_{i-1:n}^{(k+1)} \quad \text{or}$$
$$\mu_{i:n}^{(k+1)} = \mu_{i-1:n}^{(k+1)} + \frac{k+1}{n-i+1}\mu_{i:n}^{(k)}, \qquad 2 \leq i \leq n-1;$$

and

$$\text{(iii)}\quad \mu_{n:n}^{(k+1)} = \mu_{n-1:n}^{(k+1)} + (k+1)\mu_{n:n}^{(k)}, \qquad n \geq 2.$$

These recurrence relations are the ones given by Joshi (1978).

Following similar arguments, substituting $R_1 = R_2 = \cdots = R_m = 0$ into equations (3.2) and (3.4) gives, for $1 \leq i < j \leq n$,

$$\mu_{i,j:n} = \frac{1}{n-j+1}\mu_{i:n} + \mu_{i,j-1:n},$$

which is the relation given in Joshi (1982).

4. Recursive algorithm—exponential distribution

Using the recurrence relations established in Sections 2 and 3, the means, variances and covariances of all progressive Type-II right censored order statistics from the standard exponential distribution can be readily computed as follows:

Setting $k = 0$, equation (2.3) will give us the values $\mu_{1:1:n}^{(n-1)} = 1/n$, $n = 1, 2, \ldots$ which in turn, again using (2.3) with $k = 1$, will give us the values $\mu_{1:1:n}^{(n-1)^{(2)}} = 2/n^2$, $n = 1, 2, \ldots$. Thus, all first and second moments with $m = 1$ for all sample sizes n will be obtained. Next, using equation (2.2), we can determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)}$, $n = 2, 3, \ldots$, which can in turn be used, with (2.2), to determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$, $n = 2, 3, \ldots$. Equation (2.6) can then be used to obtain $\mu_{2:2:n}^{(R_1,R_2)}$ for all R_1, R_2, and $n \geq 2$, and these values can be used to obtain all moments of the form $\mu_{2:2:n}^{(R_1,R_2)^{(2)}}$ using (2.6) again. Equation (2.2) can now be used again to obtain $\mu_{1:3:n}^{(R_1,R_2,R_3)}$, $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(2)}}$ for all n, R_1, R_2, and R_3 and (2.4) can be used next to obtain all moments of the form $\mu_{2:3:n}^{(R_1,R_2,R_3)}$, $\mu_{2:3:n}^{(R_1,R_2,R_3)^{(2)}}$. Finally, (2.6) can be used to obtain all moments of the form $\mu_{3:3:n}^{(R_1,R_2,R_3)}$, $\mu_{3:3:n}^{(R_1,R_2,R_3)^{(2)}}$. This process can be continued until all desired first and second order moments (and therefore all variances) are obtained.

From equation (3.4), all moments of the form $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_m)}$, $m = 2, 3, \ldots, n$, can be determined, since only single moments, which have already been obtained, are needed to calculate them. Then, using (3.2), all moments of the form $\mu_{i-1,i:m:n}^{(R_1,\ldots,R_m)}$, $2 \leq i < m$, can be obtained. From this point, using (3.4), we can obtain all moments of the form $\mu_{m-2,m:m:n}^{(R_1,\ldots,R_m)}$, $m = 3, 4, \ldots, n$, and, subsequently, using (3.2), all moments of the form $\mu_{i-2,i:m:n}^{(R_1,\ldots,R_m)}$, $3 \leq i < m$. Continuing this way, all the desired product moments (and therefore all covariances) can be obtained.

Remark 6. Means, variances and covariances of progressive Type-II right censored order statistics from the standard exponential distribution can be obtained explicitly, as described, for example, in Thomas and Wilson (1972) and Viveros and Balakrishnan (1994). Thus, this method of recursion is an alternative method, and can be used for any order moments. However, for many distributions, explicit expressions for single and product moments of progressive Type-II right censored order statistics are not easily obtained, and this recursive method of computation will be very useful in such cases. This is the primary reason why such recursive methods have been developed for a variety of distributions for the usual order statistics; for example, see Arnold and Balakrishnan (1989).

5. Recurrence relations for single moments—right truncated exponential distribution

In this and the following section, we will present recurrence relations for the single and product moments of progressive Type-II right censored order statistics from a right truncated exponential distribution which generalize the results presented in Sections 2 and 3 of this paper. The results developed will, as mentioned earlier, generalize the results given by Joshi (1978, 1982) for usual order statistics. Proofs of the relations are similar to those presented earlier for the exponential distribution and will therefore be omitted.

The probability density function for the right truncated exponential distribution is given by:

$$f(x) = \frac{e^{-x}}{P}, \qquad 0 \leq x \leq P_1, \qquad \text{where} \qquad P_1 = -\ln(1-P).$$

Here, $1 - P$ is the proportion of right truncation of the standard exponential distribution. Thus, the characterizing differential equation for this distribution is:

$$f(x) = \frac{1}{P} - F(x) = \left(\frac{1}{P} - 1\right) + 1 - F(x).$$

This equation will be exploited in order to derive the complete recurrence relations for the single and product moments of progressive Type-II right censored order statistics from a right truncated exponential distribution.

THEOREM 5.1. *For* $k \geq 0$,

$$\mu_{1:1:1}^{(0)^{(k+1)}} = (k+1)\mu_{1:1:1}^{(0)^{(k)}} - \left(\frac{1}{P} - 1\right) P_1^{k+1}. \tag{5.1}$$

THEOREM 5.2. *For $n \geq 2$ and $k \geq 0$,*

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{n}\mu_{1:1:n}^{(n-1)^{(k)}} - \left(\frac{1}{P} - 1\right)\mu_{1:1:n-1}^{(n-2)^{(k+1)}}. \tag{5.2}$$

THEOREM 5.3. *For $2 \leq m \leq n-1$, $R_1 \geq 1$ and $k \geq 0$,*

$$\begin{aligned} \mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} &= \frac{1}{R_1+1}\Bigg\{(k+1)\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k)}} - \left(\frac{1}{P}-1\right) \\ &\quad \cdot \left[\frac{n(n-R_1-1)}{n-1}\mu_{1:m-1:n-1}^{(R_1+R_2,R_3,\ldots,R_m)^{(k+1)}}\right. \\ &\quad \left. + \frac{n}{n-1}R_1\mu_{1:m:n-1}^{(R_1-1,R_2,\ldots,R_m)^{(k+1)}}\right] \\ &\quad - (n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}}\Bigg\}. \end{aligned} \tag{5.3}$$

THEOREM 5.4. *For $2 \leq m \leq n$, $R_1 = 0$ and $k \geq 0$,*

$$\begin{aligned} \mu_{1:m:n}^{(0,R_2,\ldots,R_m)^{(k+1)}} &= (k+1)\mu_{1:m:n}^{(0,R_2,\ldots,R_m)^{(k)}} \\ &\quad - n\left(\frac{1}{P}-1\right)\mu_{1:m-1:n-1}^{(R_2,\ldots,R_m)^{(k+1)}} \\ &\quad - (n-1)\mu_{1:m-1:n}^{(R_2+1,R_3,\ldots,R_m)^{(k+1)}}. \end{aligned} \tag{5.4}$$

THEOREM 5.5. *For $2 \leq i \leq m-1$, $m \leq n-1$, $R_i \geq 1$ and $k \geq 0$,*

$$\begin{aligned} \mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} &= \frac{1}{R_i+1}\Bigg\{(k+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} - \left(\frac{1}{P}-1\right) \\ &\quad \cdot \left[\frac{A(n,i)}{A(n-1,i-1)}\mu_{i:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1},R_{i+2},\ldots,R_m)^{(k+1)}}\right. \\ &\quad - \frac{A(n,i-1)}{A(n-1,i-2)}\mu_{i-1:m-1:n-1}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i,R_{i+1},\ldots,R_m)^{(k+1)}} \\ &\quad \left. + \frac{A(n,i-1)}{A(n-1,i-1)}R_i\mu_{i:m:n-1}^{(R_1,\ldots,R_{i-1},R_i-1,R_{i+1},\ldots,R_m)^{(k+1)}}\right] \\ &\quad - (n-R_1-\cdots-R_i-i) \\ &\quad \cdot \mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} \\ &\quad + (n-R_1-\cdots-R_{i-1}-i+1) \\ &\quad \cdot \mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}\Bigg\}. \end{aligned} \tag{5.5}$$

THEOREM 5.6. *For $2 \le i \le m-1$, $m \le n$, $R_i = 0$ and $k \ge 0$,*

$$(5.6)\quad \mu_{i:m:n}^{(R_1,\ldots,R_{i-1},0,R_{i+1},\ldots,R_m)^{(k+1)}} = (k+1)\mu_{i:m:n}^{(R_1,\ldots,R_{i-1},0,R_{i+1},\ldots,R_m)^{(k)}} - \left(\frac{1}{P}-1\right)\frac{A(n,i-1)}{A(n-1,i-2)} \cdot [\mu_{i:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_{i+1},\ldots,R_m)^{(k+1)}} - \mu_{i-1:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_{i+1},\ldots,R_m)^{(k+1)}}] - (n-R_1-\cdots-R_{i-1}-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} + (n-R_1-\cdots-R_{i-1}-i+1) \cdot \mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+1,R_{i+1},\ldots,R_m)^{(k+1)}}.$$

THEOREM 5.7. *For $2 \le m \le n-1$, $R_m \ge 1$ and $k \ge 0$,*

$$(5.7)\quad \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_m+1}\left\{(k+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} + \left(\frac{1}{P}-1\right) \cdot \left[\frac{A(n,m-1)}{A(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(k+1)}} - \frac{A(n,m-1)}{A(n-1,m-1)}R_m\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(k+1)}}\right] + (n-R_1-\cdots-R_{m-1}-m+1) \cdot \mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}}\right\}.$$

THEOREM 5.8. *For $2 \le m \le n$, $R_m = 0$ and $k \ge 0$,*

$$(5.8)\quad \mu_{m:m:n}^{(R_1,\ldots,R_{m-1},0)^{(k+1)}} = (k+1)\mu_{m:m:n}^{(R_1,\ldots,R_{m-1},0)^{(k)}} + (n-R_1-\cdots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+1)^{(k+1)}} - \left(\frac{1}{P}-1\right)\frac{A(n,m-1)}{A(n-1,m-2)}[P_1^{k+1} - \mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1})^{(k+1)}}].$$

Remark 7. The relations presented in this section are complete in the sense that they will enable one to compute all the single moments of progressive Type-II right censored order statistics from right truncated exponential distributions for all sample sizes and all censoring schemes. The recursive algorithm is presented in Section 7.

Remark 8. It is easily seen that by letting $P \to 1$, these results readily reduce to the recurrence relations given in Section 2 of this paper for the standard exponential distribution.

6. Recurrence relations for product moments—truncated exponential distribution

Following steps similar to those in Section 3 for product moments of progressive Type-II right censored order statistics from the exponential distribution, we may obtain the following recurrence relations for product moments of progressive Type-II right censored order statistics from the right truncated exponential distribution.

THEOREM 6.1. *For $1 \le i < j \le m-1$, $m \le n-1$ and $R_j \ge 1$,*

$$\begin{aligned}
(6.1)\quad & \mu_{i,j:m:n}^{(R_1,\ldots,R_m)} \\
&= \frac{1}{R_j+1}\Bigg\{\mu_{i:m:n}^{(R_1,\ldots,R_m)} - \left(\frac{1}{P}-1\right) \\
&\qquad \cdot\Bigg[\frac{A(n,j)}{A(n-1,j-1)}\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1},R_{j+2},\ldots,R_m)} \\
&\qquad - \frac{A(n,j-1)}{A(n-1,j-2)}\mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j,R_{j+1},\ldots,R_m)} \\
&\qquad + \frac{A(n,j-1)}{A(n-1,j-1)}R_j\mu_{i,j:m:n-1}^{(R_1,\ldots,R_{j-1},R_j-1,R_{j+1},\ldots,R_m)}\Bigg] \\
&\qquad - (n-R_1-\cdots-R_j-j) \\
&\qquad \cdot \mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)} \\
&\qquad + (n-R_1-\cdots-R_{j-1}-j+1) \\
&\qquad \cdot \mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)}\Bigg\}.
\end{aligned}$$

THEOREM 6.2. *For $1 \le i < j \le m-1$, $m \le n$ and $R_j = 0$,*

$$\begin{aligned}
(6.2)\quad & \mu_{i,j:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)} \\
&= \mu_{i:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)} \\
&\quad - \left(\frac{1}{P}-1\right)\frac{A(n,j-1)}{A(n-1,j-2)} \\
&\quad \cdot \left[\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_{j+1},\ldots,R_m)} - \mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_{j+1},\ldots,R_m)}\right] \\
&\quad - (n-R_1-\cdots-R_{j-1}-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_{j+1}+1,R_{j+2},\ldots,R_m)} \\
&\quad + (n-R_1-\cdots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+1,R_{j+1},\ldots,R_m)}.
\end{aligned}$$

THEOREM 6.3. *For $1 \leq i \leq m-1$, $m \leq n-1$ and $R_m \geq 1$,*

$$\begin{aligned}(6.3)\quad \mu_{i,m:m:n}^{(R_1,\dots,R_m)} = \frac{1}{R_m+1}\Bigg\{&\mu_{i:m:n}^{(R_1,\dots,R_m)} - \left(\frac{1}{P}-1\right)\\ &\cdot\left[-\frac{A(n,m-1)}{A(n-1,m-2)}\mu_{i,m-1:m-1:n-1}^{(R_1,\dots,R_{m-2},R_{m-1}+R_m)}\right.\\ &\left.+\frac{A(n,m-1)}{A(n-1,m-1)}R_m\mu_{i,m:m:n-1}^{(R_1,\dots,R_{m-1},R_m-1)}\right]\\ &+(n-R_1-\cdots-R_{m-1}-m+1)\\ &\cdot\mu_{i,m-1:m-1:n}^{(R_1,\dots,R_{m-2},R_{m-1}+R_m+1)}\Bigg\}.\end{aligned}$$

THEOREM 6.4. *For $1 \leq i \leq m-1$, $m \leq n$ and $R_m = 0$,*

$$\begin{aligned}(6.4)\quad &\mu_{i,m:m:n}^{(R_1,\dots,R_{m-1},0)}\\ &= \mu_{i:m:n}^{(R_1,\dots,R_{m-1},0)} - \left(\frac{1}{P}-1\right)\frac{A(n,m-1)}{A(n-1,m-2)}\\ &\cdot\left[P_1\mu_{i:m-1:n-1}^{(R_1,\dots,R_{m-1})} - \mu_{i,m-1:m-1:n-1}^{(R_1,\dots,R_{m-2},R_{m-1})}\right]\\ &+(n-R_1-\cdots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\dots,R_{m-2},R_{m-1}+1)}.\end{aligned}$$

Remark 9. Using these recurrence relations and those in Section 5, the means, variances and covariances of all the progressive Type-II right censored order statistics from a right truncated exponential distribution can be calculated for all sample sizes and all possible censoring schemes. This is described in detail in Section 7.

Remark 10. Letting $P \to 1$ in the above recurrence relations, we readily obtain the relations derived in Section 3 for the standard exponential distribution.

Remark 11. Following arguments similar to those in Remark 4, we may show that for the special case $R_1 = R_2 = \cdots = R_m = 0$, the recurrence relations in Sections 5 and 6 reduce to the following which are equivalent to those given by Joshi (1978, 1982) for the usual order statistics from the right truncated exponential distribution:

From equation (5.1), we obtain for $k \geq 0$,

$$\mu_{1:1}^{(k+1)} = (k+1)\mu_{1:1}^{(k)} - \left(\frac{1}{P}-1\right)P_1^{k+1}.$$

From equations (5.2) and (5.4), we have for $n \geq 2$ and $k \geq 0$,

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{n}\mu_{1:n}^{(k)} - \left(\frac{1}{P}-1\right)\mu_{1:n-1}^{(k+1)}.$$

From equation (5.6): For $2 \le i \le n-1$ and $k \ge 0$,

$$\mu_{i:n}^{(k+1)} = \mu_{i-1:n}^{(k+1)} + \frac{1}{n-i+1}\left\{(k+1)\mu_{i:n}^{(k)} - n\left(\frac{1}{P}-1\right)[\mu_{i:n-1}^{(k+1)} - \mu_{i-1:n-1}^{(k+1)}]\right\},$$

and from equation (5.8): For $n \ge 2$ and $k \ge 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+1)} - n\left(\frac{1}{P}-1\right)[P_1^{k+1} - \mu_{n-1:n-1}^{(k+1)}].$$

These recurrence relations are equivalent to those given by Joshi (1978) for the single moments of usual order statistics from the right truncated exponential distribution.

From equation (6.2), we obtain, for $1 \le i < j \le n-1$,

$$\mu_{i,j:n} = \mu_{i,j-1:n} + \frac{1}{n-j+1}\left\{\mu_{i:n} - n\left(\frac{1}{P}-1\right)[\mu_{i,j:n-1} - \mu_{i,j-1:n-1}]\right\},$$

and from equation (6.4): For $1 \le i \le n-1$,

$$\mu_{i,n:n} = \mu_{i,n-1:n} + \mu_{i:n} - n\left(\frac{1}{P}-1\right)[P_1\mu_{i:n-1} - \mu_{i,n-1:n-1}].$$

These recurrence relations are equivalent to those given by Joshi (1982) for the product moments of usual order statistics from the right truncated exponential distribution.

7. Recursive algorithm—truncated exponential distribution

Using the recurrence relations developed in Sections 5 and 6, the means, variances and covariances of all progressive Type-II right censored order statistics can be readily computed as follows:

Setting $k=0$, equation (5.1) will give us the value $\mu_{1:1:1}^{(0)}$, which in turn, again using (5.1) with $k=1$, will give us the value $\mu_{1:1:1}^{(0)^{(2)}}$. From these values, we can recursively compute $\mu_{1:1:n}^{(n-1)}$ and $\mu_{1:1:n}^{(n-1)^{(2)}}$ for $n = 2, 3, \ldots$ using (5.2). Thus, all first and second moments with $m=1$ for all sample sizes n will be obtained. Next, using (5.4), we can determine all moments of the form $\mu_{1:2:n}^{(0,n-2)}$, $n = 2, 3, \ldots$ which can in turn be used, with (5.4), to determine all moments of the form $\mu_{1:2:n}^{(0,n-2)^{(2)}}$, $n = 2, 3, \ldots$. Equation (5.3) can then be used to obtain $\mu_{1:2:n}^{(R_1,R_2)}$ for $R_1 = 1, 2, \ldots$ and $n \ge 3$, and these values can be used to obtain all moments of the form $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$ using (5.3) again. Now, equation (5.8) can be used again to obtain $\mu_{2:2:n}^{(n-2,0)}$, $\mu_{2:2:n}^{(n-2,0)^{(2)}}$ for all n and (5.7) can be used next to obtain, for $R_1, R_2 = 1, 2, \ldots$ and $n \ge 3$, all moments of the form $\mu_{2:2:n}^{(R_1,R_2)}$, $\mu_{2:2:n}^{(R_1,R_2)^{(2)}}$. This process can be continued until all desired first and second order moments (and therefore all variances) are obtained.

From (6.4), all moments of the form $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_{m-1},0)}$, $m = 2,3,\ldots,n$, can be determined, since only the single moments, which have already been computed, are needed to calculate them. Relation (6.3) can then be used to obtain $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_m)}$, for $R_m = 1,2,\ldots$. Then, using (6.2), all moments of the form $\mu_{j-1,j:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)}$, $j < m$, can be obtained, and using (6.1), all moments of the form $\mu_{j-1,j:m:n}^{(R_1,\ldots,R_m)}$, $j < m$, $R_j \geq 1$, can be calculated. From this point, using (6.4) and (6.3), we can obtain all moments of the form $\mu_{m-2,m:m:n}^{(R_1,\ldots,R_m)}$ and, subsequently, using (6.2) and (6.1), all moments of the form $\mu_{j-2,j:m:n}^{(R_1,\ldots,R_m)}$, $j < m$. Continuing this way, all the desired product moments (and therefore all covariances) can be obtained.

Remark 12. Along the lines of Joshi (1979), all the results presented in this section may also be generalized to the case of progressive Type-II right censored order statistics from a doubly truncated exponential distribution. However, we abstain from presenting those results for brevity.

Acknowledgements

The authors wish to thank the Natural Sciences and Engineering Research Council of Canada for funding this research, and a referee for suggesting some changes which led to an improvement in the presentation of this paper.

REFERENCES

Arnold, B. C. and Balakrishnan, N. (1989). Relations, bounds and approximations for order statistics, *Lecture Notes in Statist.*, **53**, Springer, New York.

Balakrishnan, N. and Sandhu, R. A. (1995*a*). A simple simulational algorithm for generating progressive Type-II censored samples, *Amer. Statist.*, **49**, 229–230.

Balakrishnan, N. and Sandhu, R. A. (1995*b*). Best linear unbiased and maximum likelihood estimation for exponential distributions under general progressive Type-II censored samples, *Sankhyā Ser. B* (to appear).

Cohen, A. C. (1963). Progressively censored samples in life testing, *Technometrics*, **5**, 327–329.

Cohen, A. C. (1966). Life testing and early failure, *Technometrics*, **8**, 539–549.

Cohen, A. C. (1975). Multi-censored sampling in the three parameter Weibull distribution, *Technometrics*, **17**, 347–351.

Cohen, A. C. (1976). Progressively censored sampling in the three parameter log-normal distribution, *Technometrics*, **18**, 99–103.

Cohen, A. C. (1991). *Truncated and Censored Samples: Theory and Applications*, Marcel Dekker, New York.

Cohen, A. C. and Whitten, B. J. (1988). *Parameter Estimation in Reliability and Life Span Models*, Marcel Dekker, New York.

Gibbons, D. I. and Vance, L. C. (1983). Estimators for the 2-parameter Weibull distribution with progressively censored samples, *IEEE Transactions on Reliability*, **R-32**, 95–99.

Joshi, P. C. (1978). Recurrence relations between moments of order statistics from exponential and truncated exponential distributions, *Sankhyā Ser. B*, **39**, 362–371.

Joshi, P. C. (1979). A note on the moments of order statistics from doubly truncated exponential distribution, *Ann. Inst. Statist. Math.*, **31**, 321–324.

Joshi, P. C. (1982). A note on the mixed moments of order statistics from exponential and truncated exponential distributions, *J. Statist. Plann. Inference*, **6**, 13–16.

Mann, N. R. (1969). Exact three-order-statistic confidence bounds on reliable life for a Weibull model with progressive censoring, *J. Amer. Statist. Assoc.*, **64**, 306–315.

Mann, N. R. (1971). Best linear invariant estimation for Weibull parameters under progressive censoring, *Technometrics*, **13**, 521–534.

Thomas, D. R. and Wilson, W. M. (1972). Linear order statistic estimation for the two-parameter Weibull and extreme-value distributions from Type-II progressively censored samples, *Technometrics*, **14**, 679–691.

Viveros, R. and Balakrishnan, N. (1994). Interval estimation of life characteristics from progressively censored data, *Technometrics*, **36**, 84–91.