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LIMIT THEOREMS FOR THE MAXIMUM LIKELIHOOD ESTIMATE UNDER GENERAL MULTIPLY TYPE II CENSORING*

FANHUI KONG[1] AND HELIANG FEI[2]

[1] *Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228, U.S.A.*
[2] *Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China*

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Abstract. Assume n items are put on a life-time test, however for various reasons we have only observed the r_1-th, $\dots$, r_k-th failure times $x_{r_1,n}, \dots, x_{r_k,n}$ with $0 \le x_{r_1,n} \le \cdots \le x_{r_k,n} < \infty$. This is a multiply Type II censored sample. A special case where each $x_{r_i,n}$ goes to a particular percentile of the population has been studied by various authors. But for the general situation where the number of gaps as well as the number of unobserved values in some gaps goes to ∞, the asymptotic properties of MLE are still not clear. In this paper, we derive the conditions under which the maximum likelihood estimate of θ is consistent, asymptotically normal and efficient. As examples, we show that Weibull distribution, Gamma and Logistic distributions all satisfy these conditions.

Key words and phrases: Maximum likelihood estimation, multiply Type II censoring, law of large numbers, central limit theorem, order statistic.

1. Introduction

Assume we are sampling from a population with density function $f(x, \theta)$, where θ is the parameter of interest in R^q. The estimation of θ has been extensively studied both for complete and censored data. In the case of different censoring scheme, the maximum likelihood estimate (MLE) has been proved to be consistent, asymptotically normal and asymptotically efficient under certain regularity conditions, see Cramér (1946), Halperin (1952), Basu and Ghosh (1980), Bhattacharyya (1985), etc.

In this paper, we assume that n items are put on a life test, but only r_1-th, $\dots$, r_k-th failures are observed, the rest are unobserved, where $r_1, \dots, r_k$ are considered to be fixed. That is, for some items, we may not know their exact failure times,

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not even their orders, but for each of these items, we have observed the r_{i-1}-th and r_i-th failure times $x_{r_{i-1},n}$ and $x_{r_i,n}$ such that it fails between these two failures. This is the multiply Type II censoring. Multiply Type II censoring is a generalization of Type II censoring where only the first k failure times are observed. It is a frequently practiced censoring scheme, particularly if one fails to record the failure time of every subject, only several failure times and the number of failures between them are recorded. For examples of such situation, see Mann and Fertig (1973), Balasubramanian and Balakrishnan (1992), Balakrishnan *et al.* (1992), and Fei *et al.* (1995).

A special case as a simple generalization of Type II censoring has been studied by many authors. For instance, for $1 < r_1 < r_2 < n$, only $x_{r_1,n} \le x_{r_1+1,n} \le \cdots \le x_{r_2,n}$ are observed. The derivation of the asymptotic properties of the MLE is similar as that for Type II censored sample. This has been described by Halperin (1952), Bhattacharyya (1985) and others. For more general case of Multiply Type II censoring, it is yet unclear under what conditions the MLE of the parameters is consistent, asymptotically normal and asymptotically efficient. Especially, when some gaps of unobserved failures go to ∞ along with n, it is interesting to know under what conditions the desired properties are still valid. This is the main objective of the paper. For the desired properties to be still valid, we need to impose some extra restrictions on the density function, especially at the tails of the distribution. In addition, we need to restrict the speed at which the maximum gap goes to ∞.

Multiply Type II censoring has been studied for several special populations such as exponential, normal, logistic, Weibull and extreme-value distributions. Various estimations such as MLE, approximate MLE, BLUE and BLIE have been derived and compared, see Balasubramanian and Balakrishnan (1992), Balakrishnan *et al.* (1992, 1995*a*, 1995*b*), and Fei *et al.* (1995). For one- and two-parameter exponential distributions, Fei and Kong (1994) have provided several approximate and accurate interval estimations for the parameters.

The assumptions and results about the asymptotic normality and efficiency are stated in Section 2. In Section 3 we give some examples. The proofs are provided in Section 4.

2. Assumptions and theorems

Suppose $x_{r_1,n}, \ldots, x_{r_k,n}$ with $0 \le x_{r_1,n} < \cdots < x_{r_k,n} < \infty$ is a multiply Type II censored sample from a population with *p.d.f.* $f(x,\theta)$ and *c.d.f.* $F(x,\theta)$ for $\theta \in R^q$. Since our problem is of prominence in the field of life-data analysis, here we make the assumption that the random variable is defined on $[0,\infty)$ although this is in no way critical to our proofs.

Denote the likelihood function of the sample as $h(x,\theta) = h(x_{r_1,n}, \ldots, x_{r_k,n}, \theta)$, then with $x_{0,n} = 0$ we have

$$h(x,\theta) = C \prod_{i=1}^{k} \{F(x_{r_i,n},\theta) - F(x_{r_{i-1},n},\theta)\}^{r_i - r_{i-1} - 1} \tag{2.1}$$

$$\cdot \{1 - F(x_{r_k,n}, \theta)\}^{n-r_k} \prod_{i=1}^{k} f(x_{r_i,n}, \theta),$$

where C is a constant not depending on θ. The likelihood equation becomes

$$\begin{aligned} \frac{\partial \log h(x,\theta)}{\partial \theta} &= \sum_{i=1}^{k} (r_i - r_{i-1} - 1) \frac{\partial \log\{F(x_{r_i,n}, \theta) - F(x_{r_{i-1},n}, \theta)\}}{\partial \theta} \\ &\quad + (n - r_k) \frac{\partial \log\{1 - F(x_{r_k,n}, \theta)\}}{\partial \theta} + \sum_{i=1}^{k} \frac{\partial \log f(x_{r_i,n}, \theta)}{\partial \theta} \\ &= 0. \end{aligned} \tag{2.2}$$

We shall derive the conditions under which (2.2) has a solution being consistent, asymptotically normal and asymptotically efficient.

For the multiply Type II censored data, define the gap between $x_{r_{i-1},n}$ and $x_{r_i,n}$ as $r_i - r_{i-1} - 1$, which is the total number of unobserved failures, and

$$g = \max_i (r_i - r_{i-1} - 1)$$

as the maximum gap. To obtain our results, let's introduce the following assumptions. To avoid the extra complexity in stating the assumptions, results and proofs, we present for most part one dimensional parameter case, and if it's necessary, we indicate the modifications for multiple parameters.

ASSUMPTION 1. For almost all x, the derivatives

$$\frac{\partial^i \log f(x,\theta)}{\partial \theta^i}, \quad i = 1, 2 \qquad \text{and} \qquad \frac{\partial^{i+1} \log f(x,\theta)}{\partial x \partial \theta^i}, \quad i = 1, 2, 3$$

exist, and are piecewise continuous for every θ belonging to a nondegenerate interval I and x in $[0, \infty)$.

ASSUMPTION 2. There exist positive numbers A_1, A_2, γ_{ij}, $i = 1, 2$, $j = 1, \ldots, 5$, such that when θ is in some neighborhood of true value θ_0, and x is large enough,

$$\begin{aligned} &\left| \frac{\partial^i \log f(x,\theta)}{\partial \theta^i} \right| \le A_1 x^{\gamma_{1i}}, \quad i = 1, 2, \\ &\left| \frac{\partial^{i+1} \log f(x,\theta)}{\partial x \partial \theta^i} \right| \le A_1 x^{\gamma_{1,i+2}}, \quad i = 1, 2, 3, \end{aligned} \tag{2.3}$$

and when x is small enough, meaning being close enough to zero,

$$\begin{aligned} &\left| \frac{\partial^i \log f(x,\theta)}{\partial \theta^i} \right| \le A_2 x^{-\gamma_{2i}}, \quad i = 1, 2, \\ &\left| \frac{\partial^{i+1} \log f(x,\theta)}{\partial x \partial \theta^i} \right| \le A_2 x^{-\gamma_{2,i+2}}, \quad i = 1, 2, 3. \end{aligned} \tag{2.4}$$

Also assume there exists a function $H(x)$ that for every θ in R,

$$\left|\frac{\partial^3 \log f(x,\theta)}{\partial\theta^3}\right| \le H(x), \qquad \text{for} \quad -\infty < x < \infty,$$

and there exists M independent of θ such that

$$\int_{-\infty}^{\infty} H(x) f(x,\theta)dx \le M < \infty.$$

For simplicity, we define

$$\gamma_1 = \max\{2\gamma_{11} + \gamma_{13}, \gamma_{11} + \gamma_{14}, \gamma_{12} + \gamma_{13}, \gamma_{15}\}$$

and

$$\gamma_2 = \max\{2\gamma_{21} + \gamma_{23}, \gamma_{21} + \gamma_{24}, \gamma_{22} + \gamma_{23}, \gamma_{25}\}.$$

ASSUMPTION 3. For x large enough, there exist positive numbers C_1 and α such that

$$f(x,\theta) \ge C_1\{1 - F(x,\theta)\}^{\alpha}. \tag{2.5}$$

For x small enough, there exists a real number β and a positive number C_2 such that

$$f(x,\theta) \ge C_2\{F(x,\theta)\}^{\beta}. \tag{2.6}$$

ASSUMPTION 4. For every θ in I, the integral

$$M_1^2 = \int_{-\infty}^{\infty} \left(\frac{\partial \log f(x,\theta)}{\partial\theta}\right)^2 f(x,\theta)dx \tag{2.7}$$

is finite and positive.

The above assumptions are easy to be modified for multiple parameter case. Note for different parameters, the values of γ_1, γ_2 could be different. Under these assumptions, we have the following theorems.

THEOREM 2.1. *For constants α, β, γ_1, γ_2 defined as above, assume Assumptions 1–4 are valid. Further assume there are small positive numbers D, ϕ, τ_1 and τ_2 with $0 < \phi < 1$, such that*

$$\int_0^D \{F^{-1}(1 - \phi x, \theta)\}^{\gamma_1} x^{-(\alpha-1)}dx < \infty, \tag{2.8}$$

$$\int_0^D \{F^{-1}[(1 - \phi)x, \theta]\}^{-\gamma_2} x^{-(\beta-1)}dx < \infty, \tag{2.9}$$

and for some positive y,

$$n^{\alpha-2}\left\{F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right\}^{\gamma_1} \to 0, \tag{2.10}$$

$$n^{\beta-2}\left\{F^{-1}\left(\frac{y}{n(\log n)^{1+\tau_2}},\theta\right)\right\}^{-\gamma_2} \to 0. \tag{2.11}$$

Then if the maximum gap g *is always bounded the likelihood equation* (2.2) *has a solution converging in probability to the true value* θ_0 *as* $n \to \infty$.

To derive the asymptotic normality, we only need to add very limited assumptions.

THEOREM 2.2. *In Theorem* 2.1, *instead of conditions* (2.8)–(2.11), *if we have*

$$\int_0^D \{F^{-1}(1-\phi x,\theta)\}^{\gamma_1} x^{-(\alpha-1/2)}dx < \infty, \tag{2.12}$$

$$\int_0^D \{F^{-1}[(1-\phi)x,\theta]\}^{-\gamma_2} x^{-(\beta-1/2)}dx < \infty, \tag{2.13}$$

and

$$n^{\alpha-3/2}\left\{F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right\}^{\gamma_1} \to 0, \tag{2.14}$$

$$n^{\beta-3/2}\left\{F^{-1}\left(\frac{y}{n(\log n)^{1+\tau_2}},\theta\right)\right\}^{-\gamma_2} \to 0, \tag{2.15}$$

then when g *is bounded, the solution of* (2.2) *is an asymptotically normal and asymptotically efficient estimate of* θ_0.

Consider the case that $g \to \infty$ along with n.

THEOREM 2.3. *Under Assumptions* 1–4 *for the distribution, assume as* $n \to \infty$, $ne^{-(n/g)\epsilon} \to 0$ *for any* $\epsilon > 0$. *At two tails of the order statistics, we assume* $(r_{j+1}-r_j-1)/(r_j-1)$ *on the left tail or* $(r_{j+1}-r_j-1)/(n-r_{j+1}-1)$ *on the right tail are bounded. Further, instead of* (2.10) *and* (2.11) *we have*

$$n^{\alpha-2}(n-r_k)^2\left\{F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right\}^{\gamma_1} \to 0, \tag{2.16}$$

$$n^{\beta-2}r_1^2\left\{F^{-1}\left(\frac{y}{n(\log n)^{1+\tau_2}},\theta\right)\right\}^{-\gamma_2} \to 0, \tag{2.17}$$

and (2.8), (2.9) *in Theorem* 2.1. *Then the MLE is consistent. Furthermore, if together with* (2.12) *and* (2.13) *in Theorem* 2.2, (2.16) *is modified by replacing* $\alpha-2$ *by* $\alpha-3/2$ *and* (2.17) *is modified by replacing* $\beta-2$ *by* $\beta-3/2$. *Then the derived MLE is asymptotically normal and efficient.*

3. Examples and discussions

Before giving the proofs, let's consider some examples.

Example 1. First, consider the Gamma distribution. It has the density function

$$f(x,k,\theta)=\frac{1}{\theta^k\Gamma(k)}x^{k-1}e^{-x/\theta},\qquad k>0,\quad \theta>0,\quad x>0. \tag{3.1}$$

Take k as a constant and only consider θ as a parameter, Assumption 2 is valid and we have $\gamma_1=2$, and $\gamma_2=0$. Choose $1<\alpha<2$ and $1-1/k<\beta<2$, both Assumption 3 and conditions (2.8)–(2.11) are satisfied. Theorem 2.1 is valid. Further let $0<\alpha<3/2$ and $1-1/k<\beta<3/2$, Theorem 2.2 is also valid. For the case that the maximum gap $g\to\infty$ under conditions in Theorem 2.3, this theorem is also valid.

Example 2. As the second example, consider the two-parameter Weibull distribution, which has the following *c.d.f.*

$$F(x,\theta,\mu)=1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\mu}\right\},\qquad \mu>0,\quad \theta>0,\quad x\geq 0, \tag{3.2}$$

and the log-density function is

$$\log f(x,\theta,\mu)=\log\mu-\mu\log\theta+(\mu-1)\log x-\left(\frac{x}{\theta}\right)^{\mu}.$$

By taking derivatives with respect to μ, we find that for any $\epsilon>0$ the values of γ's satisfying Assumption 2 are

$$\begin{aligned}&\gamma_{11}=\mu+\epsilon,\qquad \gamma_{12}=\mu+\epsilon,\qquad \gamma_{13}=\mu-1+\epsilon,\\&\gamma_{14}=\mu-1+\epsilon,\qquad \gamma_{15}=\mu-1+\epsilon.\end{aligned}$$

So we have $\gamma_1=3\mu-1+\epsilon$. Similarly, noticing

$$\begin{aligned}&\gamma_{21}=\epsilon,\qquad \gamma_{22}=0,\qquad \gamma_{23}=1,\\&\gamma_{24}=1-\mu+\epsilon\quad\text{if}\quad \mu<1,\qquad \gamma_{24}=0\quad\text{otherwise},\\&\gamma_{25}=1-\mu+\epsilon\quad\text{if}\quad \mu<1,\qquad \gamma_{25}=0\quad\text{otherwise},\end{aligned}$$

so we have $\gamma_2=1+\epsilon$.

Apparently, Assumption 3 is satisfied if $\alpha>1$ and $\beta>1-1/\mu$. Note when $\mu<1$, β could be negative. Furthermore, if α, β also satisfy $\alpha<2$ and $\beta<2-1/\mu$, one can find a positive ϵ small enough that conditions (2.8)–(2.11) are satisfied. For (2.12)–(2.15) to be satisfied, one needs to choose α, β such that $1<\alpha<3/2$, and $1-1/\mu<\beta<3/2-1/\mu$.

By taking derivatives with respective to θ, we can easily determine those γ_{ij}'s satisfying Assumption 2. Similar as above, we have $\gamma_1=3\mu-1$ and $\gamma_2=1-\mu$

if $\mu < 1$, and $\gamma_2 = 0$ otherwise. For $\mu < 1$, we choose $1 < \alpha < 2$ and $1 - 1/\mu < \beta < 3 - 1/\mu$. For $\mu > 1$, we choose $1 < \alpha < 2$ and $1 - 1/\mu < \beta < 2$. Then Assumption 3 and (2.8)–(2.11) are satisfied. For (2.12)–(2.15) to be satisfied, we further let $1 < \alpha < 3/2$ and $1 - 1/\mu < \beta < 5/2 - 1/\mu$ if $\mu < 1$, $1 < \alpha < 3/2$ and $1 - 1/\mu < \beta < 3/2$ if $\mu \geq 1$.

Put these together, we are able to find common values of α and β such that all the assumptions of Theorem 2.1 and Theorem 2.2 are satisfied, therefore the MLE of (θ, μ) is consistent, asymptotically normal and efficient. Theorem 2.3 is also valid if the gaps satisfy the assumptions there. Notice here the values of γ_1 and γ_2 need not to be the same for different parameters, but the values of α and β must be the same. This can be seen in proofs.

In the theorems, we point out that at the upper tail of $[0, \infty)$, we can replace γ_1 by some easier to obtain γ_1' although this might make the assumption more restrictive. For example, we can define γ_1' in Assumption 2 as follows: *When x is large enough, there exists a positive number γ_0 that*

$$\left| \frac{\partial^{i+j} \log f(x, \theta)}{\partial x^i \partial \theta^j} \right| \leq A x^{\gamma_0}, \qquad i = 1, 2, \quad j = 1, 2, 3, \tag{3.3}$$

and let $\gamma_1' = 3\gamma_0$. Under this modification γ_1' is usually larger, therefore (2.8) and (2.10) become more restrictive. Nevertheless, this will not eliminate some important distributions because for many distributions $F^{-1}(1 - \phi x, \theta)$ goes to ∞ at the order of $(\log x)^m$ as $x \to \infty$, for some $m > 0$. So if α satisfies Assumption 3 and (2.8), (2.10) in Theorem 2.1 and also (2.12), (2.14) in Theorem 2.2 for γ_1, it will also satisfy these conditions for γ_1'. But at the lower tail of $[0, \infty)$ where x is near zero, one needs to be much more careful. For example, for the Weibull distribution we have just seen,

$$F^{-1}((1 - \phi)x, \theta) = \theta\{-\log[1 - (1 - \phi)x]\}^{1/\mu}.$$

As $x \to 0$, $\{F^{-1}((1 - \phi)x, \theta)\}^{-1}$ goes to ∞ at the order of $x^{-1/\mu}$. If we replace the value of γ_2 as we did for γ_1 in (3.3), then (2.9) and (2.11) will no longer be satisfied.

If the *r.v.*'s are defined on $(-\infty, \infty)$, the assumptions need to be modified correspondingly. In this case, it is easy to see that the corresponding assumptions for (2.9), (2.11), (2.13) and (2.15) are derived simply by replacing $-\gamma_2$ by γ_2. Let's take the logistic distribution as an example.

Example 3. Consider the density function as follows

$$f(x, \theta) = \frac{1}{\theta} e^{-(x-\eta)/\theta} / (1 + e^{-(x-\eta)/\theta})^2, \qquad -\infty < x < \infty. \tag{3.4}$$

Taking derivatives with respect to θ, one has $\gamma_{11} = \gamma_{12} = 1$, and $\gamma_{1i} = 0$, $i = 3, 4, 5$ in Assumption 2, so one has $\gamma_1 = 2$. Choose $1 < \alpha < 3/2$, Assumption 3 and (2.8), (2.10), (2.12), (2.14) in Theorem 2.1 are satisfied. By the symmetry of the distribution at two tails, Assumption 3 is satisfied for $1 < \beta < 3/2$, so are the corresponding assumptions at the lower tail of $(-\infty, \infty)$. At the same time, by taking derivatives with respective to η, one has $\gamma_1 = 0$, $\gamma_2 = 0$, and the values for α and β to satisfy Theorems 2.1 and 2.2 are $1 < \alpha < 3/2$ and $1 < \beta < 3/2$. So for parameter (θ, η), we can find common values of α and β that both Theorem 2.1 and Theorem 2.2 are valid. If the gaps satisfy the assumptions in Theorem 2.3, this theorem is also valid.

Note that if μ is fixed then the one-parameter Weibull distribution becomes exponential after a power transformation. Therefore the conditions restricted on these two distributions should be equivalent. This is true because our theorems include both the family of exponential distributions and that of the Weibull distributions for all values of μ. So they are equivalent in terms of including these two families.

Here we assume the natural parameter space. However, from the proofs we will see that the theorems are still valid if the assumptions are satisfied for θ only in a neighborhood of the true parameter θ_0. So if the natural parameter space is truncated and the assumptions are satisfied in the truncated one, then the theorems are also valid for θ_0 inside the space.

So for the situation where there exist some gaps between observed failure times in a life test experiment, we have derived the conditions under which the MLE is consistent, asymptotically normal and efficient. This generalizes the results of regularly understood multiply Type II censoring, where the last $[p_1 n]$ failures are unobserved or the first $[p_2 n]$ failures are unobserved for $0 < p_1, p_2 < 1$, or even the case of several points of truncation, each being defined as a particular sample percentage point. In these cases, the derived MLE is not asymptotically efficient since part of information is totally lost.

Besides the regularity conditions for the regular Type II censoring in Halperin (1952), we have introduced (2.3)–(2.6) and the assumptions included in the theorems. Although these additional conditions seem complicated and sometime difficult to verify, they are all restrictions on the tails of the population distribution. Under these conditions, the scores at two tails of the order statistics do not fluctuate too dramatically such that the unobserved scores are different from the observed scores in a small scale. Therefore the score statistic of the incomplete sample is essentially equivalent to that of the complete sample. Similar assumptions are used in Hall (1984) and Khashimov (1988).

4. Derivations

LEMMA 4.1. *Let $U_1, \ldots, U_n$ be independent r.v.'s with uniform distribution on $(0,1)$. Denote $0 \le U_{1,n} \le \cdots \le U_{n,n} \le 1$ as the order statistics of these variables. Then for any $y > 0$ and $\epsilon > 0$, we have*

$$P\left\{U_{n,n} \ge 1 - \frac{y}{n(\log n)^{1+\epsilon}}, i.o.\right\} = 0 \tag{4.1}$$

and

$$P\left\{U_{1,n} < \frac{y}{n(\log n)^{1+\epsilon}}, i.o.\right\} = 0, \tag{4.2}$$

where i.o. stands for "infinitely often".

PROOF. Notice $U_1, \ldots, U_n$ have a common distribution function $F(u) = u$ for $0 \le u \le 1$. Define $u_n = 1 - y/[n(\log n)^{1+\epsilon}]$ for any positive y and ϵ. Then u_n is increasing and

$$\sum_2^\infty [1 - F(u_n)] = y \sum_2^\infty \frac{1}{n(\log n)^{1+\epsilon}} < \infty. \tag{4.3}$$

By Corollary 4.3.1 in Galambos (1978), we have proved (4.1). Similar arguments lead to (4.2) via using Theorem 4.3.3 in Galambos (1978). □

LEMMA 4.2. *For $0 < \delta < 1/2$ denote $E_n = \{\delta n \le i \le (1-\delta)n\}$. Assume positive integers $j, j+k \in E_n$, as $n \to \infty$, $ne^{-(n/k)\epsilon} \to 0$ for any $\epsilon > 0$, then for order statistics $X_{1,n} \le \cdots \le X_{n,n}$ from a population with continuous density*

$$P\left(\max_{j,j+k \in E_n} \{X_{n-j+1,n} - X_{n-j-k+1,n}\} > \epsilon\right) \to 0. \tag{4.4}$$

PROOF. From Khashimov (1988), one has

$$\begin{aligned} X_{n-j+1,n} - X_{n-j-k+1,n} &= \frac{1}{n-j}\left(\sum_{j+1}^{j+k} Z_i\right)\left(1 + O_p\left(\frac{k}{n}\right)\right) \\ &\quad \times \left[\frac{1 - j/n}{f\{F^{-1}(1-j/n)\}} + O_p\left(\frac{k}{n}\right)\right], \end{aligned} \tag{4.5}$$

where Z_j's are *i.i.d.* standard exponential random variables. For fixed $\delta > 0$, and f continuous, there exist finite $K_1, K_2 > 0$ such that for $j, j+k \in E_n$

$$\frac{K_1}{n} \max_j \left(\sum_{j+1}^{j+k} Z_i\right) \le \max_j \{X_{n-j+1,n} - X_{n-j-k+1,n}\} \le \frac{K_2}{n} \max_j \left(\sum_{j+1}^{j+k} Z_i\right).$$

Therefore to prove (4.4), it suffices to prove that as $ne^{-(n/k)\epsilon} \to 0$,

$$P\left\{\max_j \sum_{j+1}^{j+k} Z_i > n\epsilon\right\} \to 0, \tag{4.6}$$

for $j, j+k \in E_n$. To do that, we define $X_j = \sum_{j+1}^{j+k} Z_i$ and $Y_n = \max_{1 \le j \le n} X_j$. Then X_j has an exponential distribution with density $p(x) = \frac{1}{k}e^{-x/k}$, and $P\{X_1 \ge n\epsilon\} = e^{-(n/k)\epsilon}$. So we have

$$\begin{aligned} P\left\{\max_j \sum_{j+1}^{j+k} Z_i \le n\epsilon\right\} &\ge P\{Y_n \le n\epsilon\} \\ &\ge 1 - nP\{X_1 \ge n\epsilon\} = 1 - ne^{-(n/k)\epsilon} \to 1. \end{aligned}$$

Thus we have proved (4.6) and the lemma. □

PROOF OF THEOREM 2.1. For simplicity let's prove a special case, where the sample size n is an odd number, and the observed survival times are $x_{2,n}, \dots, x_{2k,n}$, with $k = [n/2]$. The proof of the general case is given in the proof of Theorem 2.3. Although this looks like a rather special case, many parts in the proof can be used in more general case only if the notations are modified. Denote the likelihood function of the sample as $h(x,\theta) = h(x_{2,n}, \dots, x_{2k,n}, \theta)$, then

$$h(x,\theta) = C \prod_{i=1}^{k} \{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\} f(x_{2i,n},\theta), \tag{4.7}$$

where $x_{0,n} = 0$, and C is a constant not depending on θ. The likelihood equation is

$$\begin{aligned} &\frac{\partial \log h(x,\theta)}{\partial\theta} \\ &= \sum_{i=1}^{k} \left\{ \frac{\partial \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta} + \frac{\partial \log f(x_{2i,n},\theta)}{\partial\theta} \right\} \\ &= 0. \end{aligned} \tag{4.8}$$

We expand

$$\begin{aligned} \frac{1}{n}\frac{\partial \log h(x,\theta)}{\partial\theta} &= \frac{1}{n}\left(\frac{\partial \log h(x,\theta)}{\partial\theta}\right)_{\theta_0} + \frac{(\theta-\theta_0)}{n}\left(\frac{\partial^2 \log h(x,\theta)}{\partial\theta^2}\right)_{\theta_0} \\ &\quad + \frac{(\theta-\theta_0)^2}{2n}\left(\frac{\partial^3 \log h(x,\theta)}{\partial\theta^3}\right)_{\theta^*} \\ &= B_0 + (\theta-\theta_0)B_1 + \frac{1}{2}(\theta-\theta_0)^2 B_2. \end{aligned} \tag{4.9}$$

Here θ^* is between θ and θ_0, and B_0, B_1 are functions of the sample $x_{2,n}, \dots, x_{2k,n}$ and θ_0 only. We shall prove that in probability $B_0 \to 0$, $B_1 \to -M_1^2$ and $B_2 \to M_2$, where M_1^2 is defined in (2.7) and M_2 is defined as $E(\frac{\partial^3 \log h(x,\theta_0)}{\partial\theta^3})$.

First, let's consider B_0. For the simplicity of notation, we use θ instead of θ_0.

$$B_0 = \frac{1}{n}\sum_{i=1}^{k} \left\{ \frac{\partial \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta} + \frac{\partial \log f(x_{2i,n},\theta)}{\partial\theta} \right\}. \tag{4.10}$$

Assume $\tilde{x}_{1,n}, \tilde{x}_{3,n}, \dots, \tilde{x}_{2k+1,n}$ are those unobserved failure times, then

$$\begin{aligned} B_0 &= \frac{1}{n}\sum_{i=1}^{k} \left\{ \frac{\partial \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta} + \frac{\partial \log f(x_{2i,n},\theta)}{\partial\theta} \right\} \\ &\quad + \frac{1}{n}\sum_{i=1}^{k} \left\{ \frac{\partial \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta} \right. \\ &\qquad\qquad \left. - \frac{\partial \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta} \right\} \\ &= L_1^{(1)} + L_2^{(1)}. \end{aligned} \tag{4.11}$$

Adding one term, $L_1^{(1)}$ becomes the full log-likelihood derivative, or score statistic of the complete sample, $L_1^{(1)'}$, say. By the regular law of large numbers, $L_1^{(1)'} \to 0$ in probability, so does $L_1^{(1)}$. So it suffices only to prove $L_2^{(1)} \to 0$ in probability. For multiple parameter case, B_0, $L_1^{(1)}$ and $L_2^{(1)}$ all become q dimensional vectors. The following proof is valid for each component of the vector, therefore the conclusion for B_0 is valid for the q dimensional case.

$$
\begin{aligned}
|L_2^{(1)}| &\le \frac{1}{n}\sum_{i=1}^{k}\left|\frac{\partial \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta}\right. \\
&\qquad\qquad \left. - \frac{\partial \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta}\right| \\
&= \frac{1}{n}\sum_{i=1}^{k}\left|\frac{\partial \log f(x^*_{2i,n},\theta)}{\partial\theta} - \frac{\partial \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta}\right| \\
&\le \frac{1}{n}\sum_{i=1}^{k}\left|\frac{\partial^2 \log f(x^{**}_{2i,n},\theta)}{\partial x\partial\theta}\right|\{x_{2i,n} - x_{2(i-1),n}\},
\end{aligned}
\tag{4.12}
$$

where $x^*_{2i,n}$ and $x^{**}_{2i,n}$ are both in $[x_{2(i-1),n}, x_{2i,n}]$. For a fixed δ small enough, we separate the right side of (4.12) into three parts, and according to Assumption 2, it becomes

$$
\begin{aligned}
&\frac{1}{n}\sum_{i<\delta k} + \frac{1}{n}\sum_{\delta k\le i\le(1-\delta)k} + \frac{1}{n}\sum_{i>(1-\delta)k} \\
&\le \frac{A_2}{n}\sum_{i<\delta k} x_{2(i-1),n}^{-\gamma_2}\{x_{2i,n} - x_{2(i-1),n}\} \\
&\quad + \frac{1}{n}\sum_{i=\delta k}^{(1-\delta)k}\left|\frac{\partial^2 \log f(x^{**}_{2i,n},\theta)}{\partial x\partial\theta}\right|\{x_{2i,n} - x_{2(i-1),n}\} \\
&\quad + \frac{A_1}{n}\sum_{i>(1-\delta)k} x_{2i,n}^{\gamma_1}\{x_{2i,n} - x_{2(i-1),n}\},
\end{aligned}
\tag{4.13}
$$

where $0 < \delta < 1/2$. The second term on the right hand side is the easiest to deal with, we consider it first. The value of δ will be determined while considering the first and third term of (4.13). At this moment, we assume it is fixed and known. From Lemma 4.2, for any $\delta < 1/2$, on set $E_n = \{\delta k \le i \le (1-\delta)k\}$, as $n \to \infty$, $\max_{\{i\in E_n\}}\{x_{2i,n} - x_{2(i-1),n}\} \to 0$ in probability. Therefore

$$
\sum_{i=\delta k}^{(1-\delta)k} \xrightarrow{P} \int_{\lambda_\delta}^{\lambda_{1-\delta}} \left|\frac{\partial^2 \log f(x,\theta)}{\partial x\partial\theta}\right| dx.
\tag{4.14}
$$

Here λ_δ, $\lambda_{1-\delta}$ are the 100δ-th and $100(1-\delta)$-th percentile of the population distribution. From below, one will see that δ can be selected uniformly for n. In the light of Assumption 1, the integral (4.14) exists and is finite. So as $n \to \infty$, the second term of (4.13) goes to zero in probability.

Consider the third term of (4.13). It is obvious that with possible difference of one term, when n is an odd number, the following equation is valid.

$$\begin{aligned}\frac{A_1}{n} \sum_{i>(1-\delta)k} & x_{2i,n}^{\gamma_1}\{x_{2i,n} - x_{2(i-1),n}\} \\ &= \frac{A_1}{n} \sum_{i=1}^{\delta k} x_{n-2i+1,n}^{\gamma_1}\{x_{n-2i+1,n} - x_{n-2i-1,n}\}.\end{aligned} \tag{4.15}$$

If n is an even number, this equation needs some small modification, so does the following proof, but the conclusion is still valid. Let $H(x) = F^{-1}(e^{-x}, \theta)$, use Rényi's representation for order statistics, we have

$$x_{n-l+1,n} = H\left\{\sum_{j=1}^{l} Z_j/(n-j+1)\right\}, \qquad l = 1, \ldots, n, \tag{4.16}$$

where Z_i's are *i.i.d.* standard exponential random variables. Therefore

$$\begin{aligned} & x_{n-2i+1,n} - x_{n-2i-1,n} \\ &= -\sum_{j=2i+1}^{2i+2} Z_j/(n-j+1) \\ &\quad \times H'\left\{\sum_{j=1}^{2i} Z_j/(n-j+1) + \psi_1 \sum_{j=2i+1}^{2i+2} Z_j/(n-j+1)\right\},\end{aligned} \tag{4.17}$$

where $0 \le \psi_1 \le 1$. In view of Assumption 3, for small values of x,

$$-H'(x) = e^{-x}/[f\{F^{-1}(e^{-x}, \theta)\}] \le 1/[C_1(1-e^{-x})^{\alpha}] \le K_1 x^{-\alpha},$$

for certain positive number K_1. Similar as Hall (1984), for positive number η, we define

$$E_{n1} = \left\{\sum_{j=1}^{i} Z_j > \eta i \quad \text{for} \quad i = 1, \ldots, n\right\}. \tag{4.18}$$

We may choose η so small such that $P\{E_{n1}\} > 1 - \epsilon/2$, and η does not have to depend on n. On set E_{n1},

$$\begin{aligned} & x_{n-2i+1,n} - x_{n-2i-1,n} \\ &\le K_1 \left\{\sum_{j=2i+1}^{2i+2} Z_j/(n-j+1)\right\} \left(\sum_{j=1}^{2i} Z_j/n\right)^{-\alpha} \\ &\le K_2 \eta^{-\alpha} \left(\frac{n}{i}\right)^{\alpha-1} \left(\sum_{j=2i+1}^{2i+2} Z_j\right).\end{aligned} \tag{4.19}$$

So the right hand side of (4.15) is dominated by

$$\frac{K_3\eta^{-\alpha}}{n}\sum_{i=1}^{\delta k} H^{\gamma_1}\left\{\sum_{j=1}^{2i} Z_j/(n-j+1)\right\}\left(\frac{n}{i}\right)^{\alpha-1}\left(\sum_{j=2i+1}^{2i+2} Z_j\right). \tag{4.20}$$

Since Z_i's are *i.i.d.* standard exponential random variables, $\sum_1^{2i}(Z_j-1)/2i$ converges to zero almost surely as $n\to\infty$, that means for $0<\phi<1$,

$$\lim_{h\to\infty} P\left\{\max_{i\geq h}\left|\sum_1^{2i}(Z_j-1)/2i\right|>\phi\right\}=0. \tag{4.21}$$

Take i_0 large enough such that as $i\geq i_0$ with $1-\epsilon/2$ probability that

$$\left|\sum_1^{2i}(Z_j-1)/2i\right|\leq\phi. \tag{4.22}$$

For any such i_0, define a positive integer $N_1=[2i_0/\delta]$, as $n\geq N_1$, for $i\geq i_0$,

$$\begin{aligned}\sum_{j=1}^{2i} Z_j/(n-j+1) &\geq \frac{2i}{n}\left\{1+\sum_1^{2i}(Z_j-1)/2i\right\}\\ &\geq(1-\phi)\frac{2i}{n}\geq -K_\delta\log\left\{1-(1-\phi)\frac{2i}{n}\right\}\\ &\geq -\log\left\{1-\phi_1\frac{2i}{n}\right\},\end{aligned} \tag{4.23}$$

for some $0<\phi_1<1$ and $2i/n\leq\delta$ which is satisfied by i in (4.20). Besides, we may let δ and ϕ be small enough, such that ϕ_1 is very close to 1. Hence

$$H^{\gamma_1}\left\{\sum_{j=1}^{2i} Z_j/(n-j+1)\right\}\leq\left\{F^{-1}\left(1-\phi_1\frac{2i}{n},\theta\right)\right\}^{\gamma_1}. \tag{4.24}$$

For i_0 determined as above, separate (4.20) into two terms

$$\sum_{i=1}^{i_0}+\sum_{i=i_0+1}^{\delta k} \tag{4.25}$$

and only consider the second term. By (4.24), its expectation is dominated by

$$\frac{K_4\eta^{-\alpha}}{n}\sum_{i=i_0+1}^{\delta k}\left\{F^{-1}\left(1-\phi_1\frac{2i}{n},\theta\right)\right\}^{\gamma_1}\left(\frac{n}{i}\right)^{\alpha-1}. \tag{4.26}$$

As $n \to \infty$, it converges to integral

$$K_4\eta^{-\alpha}\int_0^\delta \{F^{-1}(1-\phi_1 x,\theta)\}^{\gamma_1} x^{-(\alpha-1)}dx.$$

By (2.8), for each η this can be arbitrarily small simply by choosing δ sufficiently small. Therefore we have proved that for i_0 fixed as earlier, as n large enough,

$$P\left\{\frac{1}{n}\sum_{i=i_0+1}^{\delta k} x_{n-2i+1,n}^{\gamma_1}(x_{n-2i+1,n}-x_{n-2i-1,n}) > \epsilon\right\} \le \epsilon. \tag{4.27}$$

For this i_0, the rest part of (4.15) is bounded by

$$\frac{A_1}{n} x_{n,n}^{\gamma_1}\{x_{n-1,n}-x_{n-2i_0+1,n}\}. \tag{4.28}$$

On set E_{n1} again,

$$x_{n-1,n}-x_{n-2i_0+1,n} \le K_5\eta^{-\alpha}n^{\alpha-1}\left(\sum_{j=1}^{2i_0} Z_j\right). \tag{4.29}$$

For any positive integer n and $r.v.$'s $X_1,\ldots,X_n$ with distribution $F(x,\theta)$, $U_{1,n} = F(X_{1,n}),\ldots,U_{n,n} = F(X_{n,n})$ are the order statistics from a uniform population. From Lemma 4.1, for any $\tau_1 > 0$,

$$P\left\{X_{n,n} \ge F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right), i.o.\right\} = 0. \tag{4.30}$$

That is

$$\lim_{n_1\to\infty} P\left\{\bigcup_{n\ge n_1}\left[X_{n,n} \ge F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right]\right\} = 0. \tag{4.31}$$

So there is an event E_{n2} with $P\{E_{n2}\} \ge 1-\epsilon/2$ and a positive integer N_2, such that on E_{n2}, when $n \ge N_2$,

$$X_{n,n} \le F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right). \tag{4.32}$$

Combining (4.29) and (4.32), we have

$$\begin{aligned}\frac{A_1}{n} & x_{n,n}^{\gamma_1}\{x_{n-1,n}-x_{n-2i_0+1,n}\} \\ & \le K_5\eta^{-\alpha}n^{\alpha-2}\left\{F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right\}^{\gamma_1}\left(\sum_{j=1}^{2i_0} Z_j\right).\end{aligned} \tag{4.33}$$

The expectation is dominated by

$$K_6\eta^{-\alpha}n^{\alpha-2}\left\{F^{-1}\left(1-\frac{y}{n(\log n)^{1+\tau_1}},\theta\right)\right\}^{\gamma_1}. \tag{4.34}$$

By (2.10) this can be arbitrarily small if n is large enough. So we have proved that for i_0 fixed,

$$P\left\{\frac{A_1}{n}x_{n,n}^{\gamma_1}(x_{n-1,n}-x_{n-2i_0+1,n})>\epsilon\right\}\leq\epsilon. \tag{4.35}$$

Together with (4.27), we have proved that the third term of (4.13) converges to zero in probability.

Let's consider the first term of (4.13), which is

$$\frac{A_2}{n}\sum_{i<\delta k}x_{2(i-1),n}^{-\gamma_2}\{x_{2i,n}-x_{2(i-1),n}\}. \tag{4.36}$$

Let $H_1(x)=F^{-1}(1-e^{-x},\theta)$. Using Rényi's representation again, we have

$$x_{l,n}=H_1\left\{\sum_{j=1}^{l}Z_j/(n-j+1)\right\},\qquad l=1,\dots,n, \tag{4.37}$$

where Z_j's are *i.i.d.* standard exponential random variables. Therefore

$$\begin{aligned}(4.38)\quad x_{2i,n}&-x_{2(i-1),n}\\ &=\sum_{j=2i-1}^{2i}Z_j/(n-j+1)\\ &\quad\times H_1'\left\{\sum_{j=1}^{2(i-1)}Z_j/(n-j+1)+\psi_2\sum_{j=2i-1}^{2i}Z_j/(n-j+1)\right\},\end{aligned}$$

where $0\leq\psi_2\leq 1$. Notice that when x is between 0 and $1/2$, $x/2\leq 1-e^{-x}\leq x$ is true, so by Assumption 3, for $x\leq 1/2$ there are certain positive numbers C_2 and C_3,

$$H_1'(x)=e^{-x}/[f\{F^{-1}(1-e^{-x},\theta)\}]\leq 1/[C_2(1-e^{-x})^{\beta}]\leq C_3x^{-\beta}.$$

Determine i_0 again by (4.22), for this i_0, we separate summation (4.36) into two terms as we did in (4.25). Since $H_1^{-\gamma_2}(x)$ is a nonincreasing function for $\gamma_2\geq 0$, for the second term, use (4.23) again, we have

$$H_1^{-\gamma_2}\left\{\sum_{j=1}^{2(i-1)}Z_j/(n-j+1)\right\}\leq\left\{F^{-1}\left(\phi_1\frac{2(i-1)}{n},\theta\right)\right\}^{-\gamma_2}, \tag{4.39}$$

for $2(i-1)/n < \delta$ and some $0 < \phi_1 < 1$. If $\beta > 0$, we use (4.38) and earlier arguments and find that on set E_{n1} defined in (4.18),

$$x_{2i,n} - x_{2(i-1),n} \le C_4 \eta^{-\beta} \left(\frac{n}{i}\right)^{\beta-1} \left(\sum_{j=2i-1}^{2i} Z_j\right), \tag{4.40}$$

for η defined as earlier. So the second term is dominated by

$$\frac{C_5 \eta^{-\beta}}{n} \sum_{i=i_0+1}^{\delta k} \left\{F^{-1}\left(\phi_1 \frac{2(i-1)}{n}, \theta\right)\right\}^{-\gamma_2} \left(\frac{n}{i}\right)^{\beta-1} \left(\sum_{j=2i-1}^{2i} Z_j\right). \tag{4.41}$$

Taking expectation and using the same arguments as we did for (4.26), with (2.9), for η arbitrarily small, we can choose δ small enough and n large enough such that (4.41) is smaller than ϵ with $1-\epsilon$ probability. If $\beta < 0$, (4.38) leads to

$$x_{2i,n} - x_{2(i-1),n} \le C_6 \sum_{j=2i-1}^{2i} Z_j/(n-j+1) \left\{\sum_{j=1}^{2i} Z_j/(n-j+1)\right\}^{-\beta}. \tag{4.42}$$

Instead of E_{n1}, we define

$$E_{n3} = \left\{\sum_{j=1}^{i} Z_j < \eta i \quad \text{for} \quad i = 1, \ldots, n\right\}, \tag{4.43}$$

and choose η large enough that $P\{E_{n3}\} > 1 - \epsilon/2$. On E_{n3}, (4.42) leads to the same formula as (4.40). So we have an almost same formula as (4.41), and the conclusion is valid via (2.9). As for the first term of (4.36), using Lemma 4.1 and previous arguments, one can define an event E_{n4} with $P\{E_{n4}\} > 1 - \epsilon/2$, such that on $E_{n1} \cap E_{n4}$,

$$\begin{aligned} &\frac{A_2}{n} \sum_{i=2}^{i_0} x_{2(i-1),n}^{-\gamma_2} \{x_{2i,n} - x_{2(i-1),n}\} \\ &\le \frac{A_2}{n} x_{1,n}^{-\gamma_2} \{x_{2i_0,n} - x_{1,n}\} \\ &\le K \eta^{-\beta} n^{\beta-2} \left\{F^{-1}\left(\frac{y}{n(\log n)^{1+\tau_2}}, \theta\right)\right\}^{-\gamma_2} \left(\sum_{j=2}^{2i_0} Z_j\right). \end{aligned} \tag{4.44}$$

It converges to zero in probability according to (2.11). Put (4.14), (4.27), (4.35), (4.41) and (4.44) together, we have proved that B_0 converges to zero in probability. Similar techniques can be used to prove the convergence of B_1 and B_2. However, both B_1 and B_2 are more complicated, and more assumptions have to be imposed to ensure the convergence. So before declaring the consistency of the MLE, let's clarify these conditions.

First, consider B_1, where

$$(4.45)\qquad \begin{aligned} B_1 &= \frac{1}{n}\frac{\partial^2 \log h(x,\theta)}{\partial\theta^2} \\ &= \frac{1}{n}\sum_{i=1}^{k}\left\{\frac{\partial^2 \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta^2} + \frac{\partial^2 \log f(x_{2i,n},\theta)}{\partial\theta^2}\right\} \\ &\quad + \frac{1}{n}\sum_{i=1}^{k}\left\{\frac{\partial^2 \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta^2}\right. \\ &\qquad\qquad\qquad \left. - \frac{\partial^2 \log f(\tilde{x}_{2i-1,n},\theta)}{\partial\theta^2}\right\} \\ &= L_1^{(2)} + L_2^{(2)}. \end{aligned}$$

Here $L_1^{(2)}$ is almost the log-likelihood derivative of the complete sample. By the law of large numbers and Assumption 4,

$$(4.46)\qquad L_1^{(2)} \to -M_1^2 = -\int_{-\infty}^{\infty}\left(\frac{\partial \log f(x,\theta)}{\partial\theta}\right)^2 f(x,\theta)dx.$$

Since

$$(4.47)\qquad \begin{aligned} &\frac{\partial^2 \log\{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)\}}{\partial\theta^2} \\ &= \frac{F''(x_{2i,n},\theta) - F''(x_{2(i-1),n},\theta)}{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)} \\ &\quad - \left\{\frac{F'(x_{2i,n},\theta) - F'(x_{2(i-1),n},\theta)}{F(x_{2i,n},\theta) - F(x_{2(i-1),n},\theta)}\right\}^2 \\ &= \frac{f''(x_{2i,n}^{(1)},\theta)}{f(x_{2i,n}^{(1)},\theta)} - \left\{\frac{f'(x_{2i,n}^{(2)},\theta)}{f(x_{2i,n}^{(2)},\theta)}\right\}^2 \\ &= \frac{f''(x_{2i,n}^{(1)},\theta)}{f(x_{2i,n}^{(1)},\theta)} - \left\{\frac{f'(x_{2i,n}^{(1)},\theta)}{f(x_{2i,n}^{(1)},\theta)}\right\}^2 \\ &\quad + \left\{\frac{f'(x_{2i,n}^{(1)},\theta)}{f(x_{2i,n}^{(1)},\theta)}\right\}^2 - \left\{\frac{f'(x_{2i,n}^{(2)},\theta)}{f(x_{2i,n}^{(2)},\theta)}\right\}^2 \\ &= \frac{\partial^2 \log f(x_{2i,n}^{(1)},\theta)}{\partial\theta^2} \\ &\quad + 2\left(\frac{\partial \log f(x_{2i,n}^{(3)},\theta)}{\partial\theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(3)},\theta)}{\partial x\partial\theta}\right)(x_{2i,n}^{(1)} - x_{2i,n}^{(2)}), \end{aligned}$$

where $x_{2i,n}^{(1)}$, $x_{2i,n}^{(2)}$, $x_{2i,n}^{(3)}$ are middle values and are all in $[x_{2(i-1),n}, x_{2i,n}]$. So

$$(4.48)\qquad |L_2^{(2)}| \le \frac{1}{n}\sum_{i=1}^{k}\left|\frac{\partial^3 \log f(x_{2i,n}^{(4)},\theta)}{\partial x\partial\theta^2}\right|\{x_{2i,n} - x_{2(i-1),n}\}$$

$$+\frac{2}{n}\sum_{i=1}^{k}\left|\frac{\partial \log f(x_{2i,n}^{(3)},\theta)}{\partial\theta}\right| \times\left|\frac{\partial^2 \log f(x_{2i,n}^{(3)},\theta)}{\partial x\partial\theta}\right|\{x_{2i,n}-x_{2(i-1),n}\},$$

where $x_{2i,n}^{(4)}$ is in $[x_{2(i-1),n}, x_{2i,n}]$. Previous arguments proving the convergence of (4.11) leads to the conclusion that (4.48) converges to zero in probability. So $B_1 \to M_1^2$ in probability. For q dimensional parameter θ, B_1 is a $q\times q$ matrix. The elements on the diagonal obviously satisfy (4.47) and (4.48), therefore converge to the corresponding elements in M_1^2. Very similar results as (4.47) and (4.48) are valid for the elements off the diagonal of B_1 where the derivatives are taken with respect to θ_i, θ_j. So $B_1 \to M_1^2$ in probability in general.

Finally, let's consider B_2. Similar as (4.45), we separate B_2 into

$$B_2 = L_1^{(3)} + L_2^{(3)}. \tag{4.49}$$

By Lebesgue dominated convergence theorem, $L_1^{(3)} \to M_2$. To establish the convergence of $L_2^{(3)}$, we need to use the middle value theorem three times in the following expression

$$\begin{aligned}&\frac{\partial^3 \log\{F(x_{2i,n},\theta)-F(x_{2(i-1),n},\theta)\}}{\partial\theta^3}\\ &=\frac{f'''(x_{2i,n}^{(1)},\theta)}{f(x_{2i,n}^{(1)},\theta)}\\ &\quad-\frac{f'(x_{2i,n}^{(2)},\theta)}{f(x_{2i,n}^{(2)},\theta)}\left\{3\frac{f''(x_{2i,n}^{(3)},\theta)}{f(x_{2i,n}^{(3)},\theta)}-2\left[\frac{f'(x_{2i,n}^{(2)},\theta)}{f(x_{2i,n}^{(2)},\theta)}\right]^2\right\},\end{aligned} \tag{4.50}$$

where $x_{2i,n}^{(1)}$, $x_{2i,n}^{(2)}$, $x_{2i,n}^{(3)}$ are different from those in (4.47), and all in $[x_{2(i-1),n}, x_{2i,n}]$. So (4.50) becomes

$$\begin{aligned}&\frac{\partial^3 \log f(x_{2i,n}^{(1)},\theta)}{\partial\theta^3}+3\left(\frac{\partial \log f(x_{2i,n}^{(1)},\theta)}{\partial\theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(1)},\theta)}{\partial\theta^2}\right)\\ &\quad-3\left(\frac{\partial \log f(x_{2i,n}^{(2)},\theta)}{\partial\theta}\right)\left[\frac{f''(x_{2i,n}^{(3)},\theta)}{f(x_{2i,n}^{(3)},\theta)}-\left\{\frac{f'(x_{2i,n}^{(3)},\theta)}{f(x_{2i,n}^{(3)},\theta)}\right\}^2\right.\\ &\qquad\left.+\left\{\frac{f'(x_{2i,n}^{(3)},\theta)}{f(x_{2i,n}^{(3)},\theta)}\right\}^2-\left\{\frac{f'(x_{2i,n}^{(2)},\theta)}{f(x_{2i,n}^{(2)},\theta)}\right\}^2\right]\\ &\quad+\left\{\frac{\partial \log f(x_{2i,n}^{(1)},\theta)}{\partial\theta}\right\}^3-\left\{\frac{\partial \log f(x_{2i,n}^{(2)},\theta)}{\partial\theta}\right\}^3\end{aligned} \tag{4.51}$$

$$
\begin{aligned}
&= \frac{\partial^3 \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta^3} + 3\left(\frac{\partial \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta^2}\right) \\
&\quad - 3\left(\frac{\partial \log f(x_{2i,n}^{(2)}, \theta)}{\partial \theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(3)}, \theta)}{\partial \theta^2}\right) \\
&\quad - 6\left(\frac{\partial \log f(x_{2i,n}^{(2)}, \theta)}{\partial \theta}\right)\left(\frac{\partial \log f(x_{2i,n}^{(4)}, \theta)}{\partial \theta}\right) \\
&\quad \times \left(\frac{\partial^2 \log f(x_{2i,n}^{(4)}, \theta)}{\partial x \partial \theta}\right)(x_{2i,n}^{(3)} - x_{2i,n}^{(2)}) \\
&\quad + 3\left\{\frac{\partial \log f(x_{2i,n}^{(5)}, \theta)}{\partial \theta}\right\}^2 \left(\frac{\partial^2 \log f(x_{2i,n}^{(5)}, \theta)}{\partial x \partial \theta}\right)(x_{2i,n}^{(1)} - x_{2i,n}^{(2)}),
\end{aligned}
$$

where $x_{2i,n}^{(4)}$, $x_{2i,n}^{(5)}$ are all in $[x_{2(i-1),n}, x_{2i,n}]$. Easily we have

$$
\begin{aligned}
(4.52)\quad & 3\left(\frac{\partial \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta^2}\right) \\
&\quad - 3\left(\frac{\partial \log f(x_{2i,n}^{(2)}, \theta)}{\partial \theta}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(3)}, \theta)}{\partial \theta^2}\right) \\
&= 3\left(\frac{\partial \log f(x_{2i,n}^{(1)}, \theta)}{\partial \theta}\right)\left(\frac{\partial^3 \log f(x_{2i,n}^{(6)}, \theta)}{\partial x \partial \theta^2}\right)(x_{2i,n}^{(1)} - x_{2i,n}^{(3)}) \\
&\quad + 3\left(\frac{\partial^2 \log f(x_{2i,n}^{(3)}, \theta)}{\partial \theta^2}\right)\left(\frac{\partial^2 \log f(x_{2i,n}^{(7)}, \theta)}{\partial x \partial \theta}\right)(x_{2i,n}^{(1)} - x_{2i,n}^{(2)}),
\end{aligned}
$$

where $x_{2i,n}^{(6)}$, $x_{2i,n}^{(7)}$ are all in $[x_{2(i-1),n}, x_{2i,n}]$. The previous arguments lead to the conclusion that $L_2^{(3)} \to 0$. That is $B_2 \to M_2$ in probability. Just as B_1, the proof of multidimensional case is similar with some minor modifications. Following precisely the arguments in Cramér ((1946), 502–503), we have shown that under a special case of multiply Type II censoring where we have only observed failures of even order, the likelihood equation has a solution $\hat{\theta}$ which is consistent. One will see from the proof of Theorem 2.3 that under more general case of multiply Type II censoring where the biggest jump is bounded, this conclusion is still true.

PROOF OF THEOREM 2.2. Assume $\theta^* = \theta^*(x_2, \ldots, x_{2k})$ is a solution of the likelihood equation (4.8), by the definition of B_0, B_1 and B_2, we obtain

$$
(4.53)\qquad M_1\sqrt{n}(\theta^* - \theta_0) = \frac{\sqrt{n}B_0/M_1}{-B_1/M_1^2 - \frac{1}{2}B_2(\theta^* - \theta_0)/M_1^2}.
$$

It follows from the proof of Theorem 2.1 that the denominator of the right hand side of (4.53) converges to 1 in probability. By (4.11), $B_0 = L_1^{(1)} + L_2^{(1)}$. Since

$nL_1^{(1)}$ is almost the full log-likelihood derivative, by the central limit theorem, $\sqrt{n}L_1^{(1)}/M_1$ has asymptotically $N(0,1)$ distribution. To show that the numerator also has $N(0,1)$ distribution, it suffices to show that $\sqrt{n}L_2^{(1)}$ converges to zero in probability. Notice (4.19) can be replaced by

$$n^{1/2}\{x_{n-2i+1} - x_{n-2i-1}\} \le K\eta^{-\alpha}\left(\frac{n}{i}\right)^{\alpha-1/2}\left(\sum_{j=2i+1}^{2i+2} Z_j\right). \tag{4.54}$$

The previous arguments along with (2.8)–(2.11) lead to the conclusion.

It is obvious that θ^* is asymptotically efficient.

PROOF OF THEOREM 2.3. For the general likelihood function (2.1), the likelihood equation (4.8) becomes

$$\begin{aligned}\frac{\partial \log h(x,\theta)}{\partial\theta} = \sum_{i=1}^{k}\Bigg[(r_i - r_{i-1} - 1)\frac{\partial\log\{F(x_{r_i,n},\theta) - F(x_{r_{i-1},n},\theta)\}}{\partial\theta}& \\ + \frac{\partial \log f(x_{r_i,n},\theta)}{\partial\theta}\Bigg]& \\ + (n - r_k)\frac{\partial\log\{1 - F(x_{r_k,n},\theta)\}}{\partial\theta} = 0.&\end{aligned} \tag{4.55}$$

With the same expansion as (4.9) we shall prove that in probability $B_0 \to 0$, $B_1 \to -M_1^2$ and $B_2 \to M_2$, for M_1, M_2 defined as earlier. In expansion (4.9), we again use θ instead of θ_0. Suppose $\tilde{x}_{r_{i-1}+1,n},\ldots,\tilde{x}_{r_i-1,n}$ for $i = 1,\ldots,k$ and $\tilde{x}_{r_k+1,n},\ldots,\tilde{x}_{n,n}$ are those unobserved failure times, then

$$\begin{aligned} B_0 &= \frac{1}{n}\sum_{i=1}^{k}\left\{\frac{\partial\log f(x_{r_i,n},\theta)}{\partial\theta} + \sum_{j_i=r_{i-1}+1}^{r_i-1}\frac{\partial \log f(\tilde{x}_{j_i,n},\theta)}{\partial\theta}\right\} \\ &\quad + \frac{1}{n}\sum_{j_k=r_k+1}^{n}\frac{\partial\log f(\tilde{x}_{j_k,n},\theta)}{\partial\theta} \\ &\quad + \frac{1}{n}\sum_{i=1}^{k}\Bigg\{(r_i - r_{i-1} - 1)\frac{\partial\log\{F(x_{r_i,n},\theta) - F(x_{r_{i-1},n},\theta)\}}{\partial\theta} \\ &\qquad - \sum_{j_i=r_{i-1}+1}^{r_i-1}\frac{\partial\log f(\tilde{x}_{j_i,n},\theta)}{\partial\theta}\Bigg\} \\ &\quad + \frac{1}{n}\Bigg[(n - r_k)\frac{\partial\log\{1 - F(x_{r_k,n},\theta)\}}{\partial\theta} \\ &\qquad - \sum_{j_k=r_k+1}^{n}\frac{\partial\log f(\tilde{x}_{j_k,n},\theta)}{\partial\theta}\Bigg] \\ &= L_1^{(1)} + L_2^{(1)}, \end{aligned} \tag{4.56}$$

where $L_1^{(1)}$ is the score statistic of the complete sample therefore goes to zero in probability. Using the middle value theorem twice we have the following inequality,

$$
\begin{aligned}
(4.57)\qquad |L_2^{(1)}| &\le \frac{1}{n}\sum_{i=1}^{k}\{x_{r_i,n}-x_{r_{i-1},n}\}\sum_{j_i=r_{i-1}+1}^{r_i-1}\left|\frac{\partial^2 \log f(x^{**}_{j_i,n},\theta)}{\partial x\partial\theta}\right| \\
&\quad + \frac{1}{n}\sum_{j_k=r_k+1}^{n}\left|\frac{\partial^2 \log f(x^{**}_{j_k,n},\theta)}{\partial x\partial\theta}\right|\{\tilde{x}_{n,n}-x_{r_k,n}\} \\
&= M^{(1)}+M^{(2)},
\end{aligned}
$$

where $x^{**}_{j_i,n}\in[x_{r_{i-1},n},x_{r_i,n}]$ and $x^{**}_{j_k,n}\in[x_{r_k,n},\tilde{x}_{n,n}]$, and $\tilde{x}_{n,n}$ is the unobserved largest value of the order statistic. For $0<\delta<1/2$, as $n\to\infty$ choose $r_{k'}$ and $r_{k''}$ such that $r_{k'}/n\to\delta$ and $r_{k''}/n\to(1-\delta)$. As we did in (4.13), separate $M^{(1)}$ into three parts,

$$
M^{(1)} = \frac{1}{n}\sum_{1}^{k'} + \frac{1}{n}\sum_{k'+1}^{k''} + \frac{1}{n}\sum_{r_{k''}+1}^{k}.
$$

Since $\frac{\partial^2}{\partial x\partial\theta}\log f(x,\theta)$ is bounded when x is between 100δ-th and $100(1-\delta)$-th percentile of the distribution, using Lemma 4.2 under the condition that $ne^{(n/g)\epsilon}\to 0$ as $n\to\infty$, the second term converges to zero in probability. Consider the third term, it is bounded by

$$
\begin{aligned}
(4.58)\qquad & \frac{1}{n}\sum_{i=k''+1}^{k}\{x_{r_i,n}-x_{r_{i-1},n}\}\sum_{j_i=r_{i-1}+1}^{r_i-1}(x^{**}_{j_i,n})^{\gamma_1} \\
&\le \frac{1}{n}\sum_{k''+1}^{k} x^{\gamma_1}_{r_i,n}(r_i-r_{i-1}-1)\{x_{r_i,n}-x_{r_{i-1},n}\} \\
&= \frac{1}{n}\sum_{j=1}^{k-k''} x^{\gamma_1}_{r_{k-j+1},n}(r_{k-j+1}-r_{k-j}-1)\{x_{r_{k-j+1},n}-x_{r_{k-j},n}\}.
\end{aligned}
$$

In order to use (4.16) more easily, let's denote

$$
x_{n-\tilde{r}_j,n}=x_{r_{k-j+1},n} \qquad \text{or} \qquad n-r_{k-j+1}=\tilde{r}_j,
$$

then (4.58) becomes

$$
\frac{1}{n}\sum_{j=1}^{k-k''} x^{\gamma_1}_{n-\tilde{r}_j,n}(\tilde{r}_{j+1}-\tilde{r}_j-1)\{x_{n-\tilde{r}_j,n}-x_{n-\tilde{r}_{j+1},n}\}.
$$

Similar as (4.17), we have

$$
\begin{aligned}
(4.59)\qquad & x_{n-\tilde{r}_j,n}-x_{n-\tilde{r}_{j+1},n} \\
&= -\sum_{l=\tilde{r}_j+2}^{\tilde{r}_{j+1}+1}\frac{z_l}{n-l+1} \\
&\quad\times H'\left\{\sum_{l=1}^{\tilde{r}_j+1}\frac{z_l}{n-l+1}+\psi_1\sum_{l=\tilde{r}_j+2}^{\tilde{r}_{j+1}+1}\frac{z_l}{n-l+1}\right\}.
\end{aligned}
$$

Notice $-H'(x) \le K_1 x^{-\alpha}$ on event E_{n1} defined in (4.18), one has

$$
\begin{aligned}
(4.60)\quad x_{n-\tilde r_j,n} - x_{n-\tilde r_{j+1},n} &\le K_1 \left\{ \sum_{l=\tilde r_j+2}^{\tilde r_{j+1}+1} \frac{z_l}{n-l+1} \right\} \left(\sum_{l=1}^{\tilde r_j+1} z_l/n \right)^{-\alpha} \\
&\le K\eta^{-\alpha} \left(\frac{n}{\tilde r_j+1} \right)^{\alpha-1} \cdot \frac{1}{\tilde r_j+1} \left(\sum_{l=\tilde r_j+2}^{\tilde r_{j+1}+1} z_l \right).
\end{aligned}
$$

Put all these as well as (4.16) together, (4.58) is bounded by

$$
\begin{aligned}
(4.61)\qquad & \frac{K\eta^{-\alpha}}{n} \sum_{j=1}^{k-k''} H^{\gamma_1} \left\{ \sum_{l=1}^{\tilde r_j+1} \frac{z_l}{n-l+1} \right\} \\
& \quad \times \left(\frac{n}{\tilde r_j+1} \right)^{\alpha-1} \left(\frac{\tilde r_{j+1} - \tilde r_j - 1}{\tilde r_j+1} \right) \left(\sum_{l=\tilde r_j+2}^{\tilde r_{j+1}+1} z_l \right) \\
& \le \frac{K\eta^{-\alpha}}{n} \sum_{j=1}^{k-k''} H^{\gamma_1} \left\{ \sum_{l=1}^{\tilde r_j+1} \frac{z_l}{n-l+1} \right\} \\
& \quad \times \left(\frac{n}{\tilde r_j+1} \right)^{\alpha-1} \left(\sum_{l=\tilde r_j+2}^{\tilde r_{j+1}+1} z_l \right) \\
& = \sum_{j=1}^{i_0} + \sum_{j=i_0+1}^{k-k''},
\end{aligned}
$$

where i_0 is selected according to (4.22). Here we use the assumption that $(\tilde r_{j+1} - \tilde r_j - 1)/(\tilde r_j - 1)$ is bounded. Similar as (4.24), when $\tilde r_j \ge i_0$, one has

$$
H^{\gamma_1} \left\{ \sum_{l=1}^{\tilde r_j+1} \frac{z_l}{n-l+1} \right\} \le \left[F^{-1} \left\{ 1 - \phi_1 \left(\frac{\tilde r_j+1}{n} \right), \theta \right\} \right]^{\gamma_1}.
$$

To prove the second term of (4.61) converges to zero in probability, we see its expectation is bounded by

$$
\begin{aligned}
(4.62)\qquad & \frac{K\eta^{-\alpha}}{n} \sum_{j=1}^{k-k''} \left[F^{-1} \left\{ 1 - \phi_1 \left(\frac{\tilde r_j+1}{n} \right), \theta \right\} \right]^{\gamma_1} \\
& \quad \times \left(\frac{n}{\tilde r_j+1} \right)^{\alpha-1} (\tilde r_{j+1} - \tilde r_j - 1) \\
& \to K\eta^{-\alpha} \int_0^\delta \{ F^{-1}(1 - \phi_1 x, \theta) \}^{\gamma_1} x^{-(\alpha-1)} dx.
\end{aligned}
$$

It can be arbitrarily small as long as δ is small enough. Consider the first term of (4.61). Notice the way we choose i_0 in (4.22), the first term of (4.61) contains only finite terms of order statistics, therefore can be handled as the first term of (4.25). Finally, let's consider $M^{(2)}$ in (4.57). Same as (4.58), we have

$$M^{(2)} \le \frac{1}{n}(n-r_k)\tilde{x}_{n,n}^{\gamma_1}\{\tilde{x}_{n,n} - x_{r_k,n}\}. \tag{4.63}$$

If $n - r_k$ is finite, it becomes (4.28) and no more assumption is needed to prove it converges to zero in probability. If otherwise $n - r_k \to \infty$ along with n, we need to use similar results as (4.29) and (4.32), and derive

$$\begin{aligned}
&\frac{1}{n}(n-r_k)\tilde{x}_{n,n}^{\gamma_1}\{\tilde{x}_{n,n} - x_{r_k,n}\} \\
&\quad \le K\eta^{-\alpha}n^{\alpha-2}(n-r_k) \\
&\quad\quad \times \left\{F^{-1}\left(1 - \frac{y}{n(\log n)^{1+\tau_1}}, \theta\right)\right\}^{\gamma_1}\left(\sum_{j=1}^{n-r_k+1} Z_j\right).
\end{aligned} \tag{4.64}$$

It's expectation is dominated by

$$K\eta^{-\alpha}n^{\alpha-2}(n-r_k)^2\left\{F^{-1}\left(1 - \frac{y}{n(\log n)^{1+\tau_1}}, \theta\right)\right\}^{\gamma_1}.$$

With Assumption (2.16), we can prove $M^{(2)}$ converges to zero in probability. All these can be proved for multiple parameter case.

The proofs for the convergence of B_1 and B_2 are very similar as before. As an illustration, let's consider B_1. Just as (4.45), we can separate B_1 as $B_1 = L^{(1)} + L^{(2)}$ where $L^{(1)}$ is the complete yet unobserved negative Fisher information which converges in probability to the true negative Fisher information, and $L^{(2)}$ is similar as (4.45). In fact,

$$\begin{aligned}
L_2^{(2)} = \frac{1}{n}\sum_{i=1}^{k}\Bigg\{&(r_i - r_{i-1} - 1)\frac{\partial^2 \log\{F(x_{r_i,n},\theta) - F(x_{r_{i-1},n},\theta)\}}{\partial\theta^2} \\
&- \sum_{r_{i-1}+1}^{r_i-1}\frac{\partial^2 \log f(\tilde{x}_{j_i,n},\theta)}{\partial\theta^2}\Bigg\} \\
+ \frac{1}{n}\Bigg[&(n-r_k)\frac{\partial^2\log\{1 - F(x_{r_k,n},\theta)\}}{\partial\theta^2} \\
&- \sum_{r_k+1}^{n}\frac{\partial^2\log f(\tilde{x}_{j_k,n},\theta)}{\partial\theta^2}\Bigg].
\end{aligned} \tag{4.65}$$

Using similar result as (4.47), we have

$$|L_2^{(2)}| \le \frac{1}{n}\sum_{i=1}^{k}\{x_{r_i,n} - x_{r_{i-1},n}\}\sum_{r_{i-1}+1}^{r_i-1}\left|\frac{\partial^3\log f(x_{j_i,n}^{(4)},\theta)}{\partial x\partial\theta^2}\right| \tag{4.66}$$

$$+\frac{2}{n}\sum_{i=1}^{k}(r_i - r_{i-1} - 1)\left|\frac{\partial \log f(x_{j_i,n}^{(3)},\theta)}{\partial\theta}\right|$$
$$\times\left|\frac{\partial^2 \log f(x_{j_i,n}^{(3)},\theta)}{\partial x\partial\theta}\right|\{x_{r_i,n} - x_{r_{i-1},n}\}$$
$$+\frac{1}{n}\{\tilde{x}_{n,n} - x_{r_k,n}\}\sum_{r_k+1}^{n}\left|\frac{\partial^3 \log f(x_{j_k,n}^{(4)},\theta)}{\partial x\partial\theta^2}\right|$$
$$+\frac{2}{n}(n - r_k - 1)\left|\frac{\partial \log f(x_{j_k,n}^{(3)},\theta)}{\partial\theta}\right|$$
$$\times\left|\frac{\partial^2 \log f(x_{j_k,n}^{(3)},\theta)}{\partial x\partial\theta}\right|\{\tilde{x}_{n,n} - x_{r_k,n}\},$$

where $x_{j_i,n}^{(4)}$ and $x_{j_i,n}^{(3)}$ are in $[x_{r_{i-1},n}, x_{r_i,n}]$ and $x_{j_k,n}^{(4)}$ and $x_{j_k,n}^{(3)}$ are in $[x_{r_k,n}, \tilde{x}_{n,n}]$. So we can use previous method to prove that $L_2^{(2)} \to 0$. The proof for the asymptotic normality and efficiency is also the same as the proof of Theorem 2.2.

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