Ann. Inst. Statist. Math.
Vol. 48, No. 4, 645–662 (1996)

STOCHASTIC COMPARISONS AND BOUNDS FOR AGING RENEWAL PROCESS SHOCK MODELS AND THEIR APPLICATIONS

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(Received July 27, 1994; revised October 16, 1995)

Abstract. Sharp comparisons between aging renewal process shock models and the corresponding Esary-Marshall-Proschan (*EMP*) shock model are considered. The usefulness of such comparisons derive from the simplicity of the latter models. Simple conditions under which such aging renewal process shock models are stochastically ordered relative to a corresponding *EMP*-model are derived. Applications to renewal functions and single server queues are indicated.

Key words and phrases: Shock models, aging renewal processes, bounds, stochastic comparisons.

1. Introduction and summary

A shock model survival distribution S is governed by a counting process $\{N(t) : t > 0\}$ of shocks over time and a sequence of shock-resistance probabilities $\{\bar{P}_k : k = 0, 1, 2, \dots\}$ with $1 \geq \bar{P}_0 \geq \bar{P}_k \downarrow 0$. The latter is the tail $\bar{P}_k := P\{J > k\}$, $k = 0, 1, 2, \dots$ of the number J of shocks to failure. Set

$$p_k := \bar{P}_{k-1} - \bar{P}_k = P\{J = k\}, \qquad k = 0, 1, 2, \dots$$

$(\bar{P}_{-1} = 1)$. Then the survival probability $\bar{S}(t) := P\{N(t) < J\}$ has the two equivalent representations:

$$\bar{S}(t) = E\bar{P}_{N(t)} = \sum_{k=0}^{\infty} \bar{P}_k P\{N(t) = k\},$$

$$S(t) := 1 - \bar{S}(t) = P\{J \leq N(t)\} = p_0 + \sum_{k=1}^{\infty} p_k P\{N(t) \geq k\}. \tag{1.1}$$

Even for relatively simple counting processes $N(t)$ such as a renewal process, for which $S(t)$ is called a *renewal process shock model* (*RPSM*); (1.1) is typically hard to compute and usually cannot be expressed in a closed form. The only exceptions are when $N(t)$ is *either* (i) a Poisson process or (ii) a *Phase*(PH)-type renewal process together with discrete PH-type shock resistance probabilities. In the latter case, $\bar{S}(t)$ is continuous PH-type (Neuts and Bhattacharjee (1981)) and thus lends itself to computational manipulation. When $N(t)$ is a Poisson process (homogeneous or nonstationary), (1.1) leads to the the well known shock model of Esary *et al.* ((1973); *henceforth abbreviated as the EMP-model*), and the nonstationary Poisson shock model due to A-Hameed and Proschan (1973) respectively. Block and Savits (1978) considered conditions on the renewal process which are sufficient for the *RPSM* probability in (1.1) to inherit different aging properties of the number of shocks to failure. Pellerey (1994) has investigated some properties of shock models with different underlying counting processes. In contrast, our focus in this paper is on comparisons between shock models driven by suitably aging renewal processes and a corresponding *EMP*-survival model where shocks occur at the same rate. The main interest in and justification of looking for such pointwise comparisons is that *EMP* survival distributions are conceptually and computationally simplest of such models.

The possibility of comparing an aging RPSM to a corresponding Poisson process shock model is implicit in some results in Barlow and Proschan (1975) on bounds for the distribution of the renewal quantity when the distribution F driving the renewal process has aging characteristics. The following preliminary results provided our motivation to investigate the relationship between RPSM-survival distributions and corresponding *EMP*-shock models under appropriately weak conditions on the RPSM.

If $N(t)$ is a NBU-renewal process (i.e., the interarrival time distribution F between shocks is NBU), then since F is NBU implies (Barlow and Proschan (1975), p. 162)

$$P\{N(t) < k\} \geq 1 - Q(k-1, \Lambda(t)) \tag{1.2}$$

for $k = 1, 2, \ldots$; where $\Lambda := -\ln(1 - F)$,and $Q(k, \lambda)$ is the Poisson tail

$$Q(k, \lambda) := \sum_{j>k} e^{-\lambda} \frac{\lambda^j}{j!} = \int_0^\lambda e^{-x} \frac{x^k}{k!} dx; \qquad k \geq 0;$$

we see that the corresponding survival probability in (1.1) satisfies:

$$\begin{aligned}
\bar{S}(t) &= 1 - p_0 - \sum_{k=1}^{\infty} p_k P\{N(t) \geq k\} \\
&= \bar{P}_0 - \sum_{k=1}^{\infty} (\bar{P}_{k-1} - \bar{P}_k) P\{N(t) \geq k\} \\
&\geq \bar{P}_0 + \sum_{k=1}^{\infty} (\bar{P}_k - \bar{P}_{k-1}) Q(k-1, \Lambda(t)) \\
&= \bar{P}_0 \bar{F}(t) + \sum_{k=1}^{\infty} \bar{P}_k \{Q(k-1, \Lambda(t)) - Q(k, \Lambda(t))\}
\end{aligned} \tag{1.3}$$

$$= \sum_{k=0}^{\infty} \bar{P}_k e^{-\Lambda(t)} \frac{\Lambda^k(t)}{k!}, \qquad t > 0.$$

Thus, for NBU-renewal process shock models, the survival time distribution function (d.f.) S *stochastically dominates* the corresponding d.f. of the survival time in a non-homogeneous Poisson process ($NHPP$) shock model of A-Hameed and Proschan (1973). If we strengthen the assumption on $N(t)$ to an IFR (DFR) renewal process, then (1.3) can be replaced by the bound

$$\bar{S}(t) \geq (\leq) \sum_{k=0}^{\infty} \bar{P}_k e^{-t/\mu} \frac{(t/\mu)^k}{k!}, \qquad 0 < t < \mu \tag{1.4}$$

using a well known bound analogous to (1.2), which is valid only in the range $0 < t < \mu$, under IFR assumption (Barlow and Proschan (1975)).

Although these results ((1.3) and (1.4)) are fairly easily obtained and the former *holds without any assumption on the shock-resistance probabilities* $\bar{P}_k$, they do not go far enough in the sense that they are not able to establish a pointwise comparison of an RPSM-survival probability to the corresponding simple EMP-survival model over the entire time axis. It is thus of legitimate interest to seek appropriate weak conditions under which such pointwise comparisons with the EMP-model can be made.

Section 2 summarizes our basic results. Section 3 is devoted to proofs. We establish sharp pointwise comparisons between a RPSM survival probability, driven by a $NBUE$ or $NWUE$ renewal process of shocks with convex or concave shock resistance probabilities, and the corresponding EMP-shock model survival probability in which shocks occur at the same rate (Theorems 2.1–2.3). When the aging renewal process assumption on shocks is relaxed from $NBUE$ ($NWUE$) to $HNBUE$ ($HNWUE$), a weaker but still sharp stochastic comparison holds without any assumption on the distribution on the number of shocks to failure (Theorem 2.4). In Section 4 devoted to applications, we (i) derive an improved upper bound for $NBUE$-renewal functions and (ii) show that the well known stochastic domination of the stationary waiting time in single server queues with $NBUE$ service and arrivals by the waiting time in $M/M/1$ queues with the same service utilization factor follows as an easy consequence of our results.

In the sequel, $*$ denotes convolutions and ^{*n} the n-fold self-convolution of a distribution with itself. $M(t) = EN(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ is the renewal function associated with the renewal process generated by a given survival distribution function F (d.f. of the 'shock interarrival time') with tail $\bar{F} = 1 - F$ and a finite mean μ. Corresponding to F, G will always denote the exponential d.f. $G(t) := 1 - e^{-t/\mu}$, $t \geq 0$, with the same mean as the mean of F. Also $\leq_{st}$ and $=_{st}$ will denote the usual stochastic ordering and stochastic equivalence respectively. To avoid trivial situations, we assume $p_0 \neq 1$; i.e., there is a positive probability that failure occurs only after a shock.

2. Main results

To state our results, let,

$$\bar{S}_{\exp}(t) := \sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \qquad t > 0 \tag{2.1}$$

denote the *Esary-Marshall-Proschan* (*EMP*) *shock model* survival probability, when the shocks are driven by a Poisson process with rate $\lambda \equiv \mu^{-1}$, and where μ is the mean of the d.f. F of the time between shocks.

THEOREM 2.1. *Let the shock resistance probabilities $\bar{P}_k$ be convex.*

(i) *If the successive shock times constitute a NBUE-renewal process, then*

$$\bar{S}(t) \le \bar{S}_{\exp}(t) + p_1 \left(\frac{t}{\mu} - M(t) \right). \tag{2.2}$$

(ii) *If the shock times constitute a NWUE-renewal process with a finite mean time between shocks, then*

$$\bar{S}(t) \ge \bar{S}_{\exp}(t) - p_1 \left(M(t) - \frac{t}{\mu} \right). \tag{2.3}$$

Both bounds are sharp, with equality iff F is exponential.

THEOREM 2.2. *Let $\bar{P}_k$ be convex. If the incidence of shocks is a stationary NBUE (NWUE)-renewal process, then*

$$\bar{S}(t) \le (\ge) \bar{S}_{\exp}(t), \tag{2.4}$$

i.e., $S \le_{st} (\ge_{st}) S_{\exp}$. The bound is sharp. Equality holds ($S = S_{\exp}$) iff F is exponential.

THEOREM 2.3. *For the NBUE (NWUE* resp.*) renewal process shock model (1.1), let $\bar{P}_k$ be concave. Then*

$$\bar{S}(t) \ge (\le) \bar{S}_{\exp}(t) - p_N \Delta_N^{F,G}(t), \tag{2.5}$$

where $\Delta_k^{F,G}(t) := \sum_{n>k} G^{*n}(t) - \sum_{n>k} F^{*n}(t)$, $k = 0, 1, 2, \dots$ and $N = \max\{k : p_k > 0\}$. The bound is sharp. Equality holds ($S = S_{\exp}$) iff F is exponential.

Clearly, $\Delta_k^{F,G}(t) \in [0, 1)$ ($[-1, 0)$ respectively) when F is *NBUE* (*NWUE*) for all $k \ge 0$. We note however that, because of the difficulty in computing $\Delta_k^{F,G}(t)$ for an arbitrary $k \ge 1$, the bound in (2.5) cannot be easily computed in general. In particular, note when $\bar{P}_k$ is concave in k as in Theorem 2.3, under the non-triviality assumption $p_0 \ne 1$, we have $N \ge 1$. We may also note that the bounds in Theorem 2.1 can be described in a manner analogous to (2.5), by rewriting (2.2)–(2.3) as:

$$\bar{S}(t) \leq (\geq)\bar{S}_{\exp}(t) + p_1\Delta_0^{F,G}(t),$$

whenever the renewal process of shocks is *NBUE* (*NWUE* resp.) and $\bar{P}_k$ is convex in k, since $\Delta_0^{F,G}(t) = (t/\mu) - M(t)$.

If we make no assumptions on the shock resistance probabilities $\bar{P}_k$, we can get a weaker but still sharp comparison (Theorem 2.4 below) in terms of a partial ordering. For this result, the aging renewal process assumption can be relaxed from *NBUE* (*NWUE*) to *HNBUE* (*HNWUE*) renewal process of shocks.

DEFINITION. (i) For nonnegative r.v.s, let $\leq_c$ denote the *convex ordering* (see, Stoyan (1983)), defined by:

$$X \leq_c Y \quad \text{if} \quad \int_t^\infty P(X > x)dx \leq \int_t^\infty P(Y > x)dx, \qquad \text{all} \quad t \geq 0.$$

(ii) A stronger partial ordering $\leq_v$ (convex ordering with equal means) is defined by:

$$X \leq_v Y \quad \text{if} \quad X \leq_c Y \quad \text{and} \quad EX = EY.$$

We denote the d.f.s of X and Y to be correspondingly partially ordered, without any ambiguity. The reverse partial orderings $\geq_c$ and $\geq_v$ are defined by reversing the integral inequality.

Remarks. 1. The partial orderings $\leq_c$ and $\leq_v$ are also known as *variability orderings* in the literature (e.g., Ross (1983), Bhattacharjee (1991)), although often in different notations. (The orderings $\leq_c$, $\leq_v$ defined above are denoted in Bhattacharjee (1991) by $<^{(1)}$ and $<^1$ respectively, while Ross (1983) uses $\leq_v$ for $\leq_c$.) Our notation for $\leq_c$ thus follows Stoyan (1983), but differs from Ross (1983).

2. Clearly, $X \leq_v Y \Rightarrow X \leq_c Y$, but not conversely unless $EX = EY$. Further, $X \leq_c (\leq_v) Y \Leftrightarrow Eh(X) \leq Eh(Y)$ for all convex increasing (convex, resp.) functions h (see Corollary 8.5.2 in Ross (1983) or, Theorem 2.4 in Bhattacharjee (1991)). In our context, the distinction between and applicability of the orderings $\leq_c$ and $\leq_v$ is illustrated by Corollary 3.1.

THEOREM 2.4. *If the shocks are governed by a HNBUE (HNWUE) renewal process, and the shock resistance probabilities are arbitrary; then the stationary RPSM and the corresponding EMP shock model survival d.f.s satisfy*

$$S \leq_v (\geq_v) S_{\exp}. \tag{2.6}$$

The stochastic comparison is sharp. Equality ($S = S_{\exp}$) holds iff F is exponential.

Remarks. 1. Since F is exponential iff the renewal process $N(t)$ and, the stationary renewal process $\tilde{N}(t)$ driven by F is a Poisson process; the sharpness results in the above theorems can be viewed as characterizations of Poisson processes within RPSM models.

2. Note $\bar{P}_k$ is convex (concave) if and only if $\bar{P}_k - \bar{P}_{k-1}$ is $\uparrow$ ($\downarrow$), $k \geq 0$ ($\bar{P}_{-1} := 1$) and $0 \leq p_k = \bar{P}_{k-1} - \bar{P}_k$. Hence $\bar{P}_k$ is convex (concave) if and only if $0 \leq p_k$ is $\downarrow$ ($\uparrow$). Thus, when $\bar{P}_k$ is convex, we must have $p_1 \neq 0$, since $p_0 \neq 1$. On the other hand; if $\bar{P}_k$ is concave, then the number J of shocks to failure must have a finite support. In this case, the representations of $\bar{S}(t)$ in (1.1) are finite sums over $0 \leq k \leq N-1$ and $1 \leq k \leq N$ respectively, where $N = \max\{k : p_k > 0\}$.

3. Since $M(t) \leq (\geq) t/\mu$ under *NBUE* (*NWUE*); the second term in the *upper* (*lower*) bound in (2.2) and ((2.3) resp.) is non-negative, and is moreover strictly positive for a nonempty set of t; unless the shock process is Poisson, since $\bar{P}_k$ convex requires $p_1 > 0$.

3. Proofs

To prove our theorems in Section 2, we will need the following preliminary results. They are parallel to and are variants of some similar observations in Barlow and Proschan (1975). Lemma 3.2 is not directly used in our arguments, but is potentially useful and of independent interest.

LEMMA 3.1. (cf. Barlow and Proschan (1975)) *If F_i, $i = 1, 2, \ldots$ are NBUE (NWUE) with a common mean $\mu < \infty$ and $\bar{G}(t) = e^{-t/\mu}$, then*

$$\sum_{n \geq j} F_1 * F_2 * \cdots * F_n(t) \leq (\geq) \sum_{n \geq j} G^{*n}(t), \qquad j \geq 1, \;\; t > 0. \tag{3.1}$$

LEMMA 3.2. *Let F be a survival distribution with finite mean μ, and $\{h(n) : n = 1, 2, \ldots\}$ a monotone infinite sequence of reals.*

(i) *If F is NBUE and $0 \leq h(n) \uparrow$ (or, F is NWUE and $0 \geq h(n) \downarrow$) for $n \geq 1$, then*

$$0 \leq \sum_{n=1}^{\infty} h(n) F^{*n}(t) \leq \sum_{n=1}^{\infty} h(n) \int_0^{t/\mu} e^{-x} \frac{x^{n-1}}{(n-1)!} dx = \sum_{n=1}^{\infty} h(n) G^{*n}(t). \tag{3.2}$$

(ii) *If F is NBUE and $0 \geq h(n) \downarrow$ (or, F is NWUE and $0 \leq h(n) \uparrow$) for $n \geq 1$, then (3.2) holds with the inequality reversed.*

The validity of (3.1) for the *NWUE* case with a finite mean follows immediately as a dual to the *NBUE* case, due to Barlow and Proschan (1975), Theorem 3.15, p. 171.

Barlow and Proschan ((1975), p. 173) proved (3.2) when F is *NBUE* and the sequence $h(n)$ is nonnegative nondecreasing. To prove Lemma 3.2 in the three remaining cases, use the decomposition

$$\sum_{n=1}^{\infty} c_n F_n(t) = c_1 \sum_{n=1}^{\infty} F_n(t) + \sum_{j=2}^{\infty} \Delta c_{j-1} \sum_{n=j}^{\infty} F_n(t),$$
$$\Delta c_{j-1} := c_j - c_{j-1}, \qquad j \geq 2,$$

for any sequence of d.f.s F_n and nonnegative reals c_n; then choose $F_n = F^{*n}$ and $h(n) = -c_n$ (c_n resp.) according as $0 \geq h(n) \downarrow$ ($0 \leq h(n) \uparrow$ resp.) and apply Lemma 3.1 when F is *NWUE* (*NBUE* resp.). Note that to apply Lemma 3.1 to the above decomposition, we need the sequence c_n to be monotone over the *entire set of positive integers*, since the decomposition is no longer valid for a finite sum $\sum_{n=1}^{N} c_n F_n(t)$.

We now turn to proving the main results. For clarity, the arguments for exponentiality to be necessary and sufficient for sharpness of the bounds are collected together at the end.

PROOF OF THEOREM 2.1. Suppose $\bar{P}_k$ is convex; equivalently $0 \leq p_k \downarrow$ in $k = 0, 1, 2, \ldots$. If F is *NBUE* with mean μ and $\bar{G}(t) = e^{-t/\mu}$, then

$$S(t) = p_0 + \sum_{k=1}^{\infty} p_k F^{*k}(t) = p_0 + p_1 \sum_{k\geq 1} F^{*k}(t) + \sum_{j=2}^{\infty} (p_j - p_{j-1}) \sum_{k \geq j} F^{*k}(t). \tag{3.3}$$

Apply Lemma 3.1 to the third group of terms above, noting $p_j - p_{j-1} \leq 0$. Since $\sum_{k=1}^{\infty} F^{*k}(t) = M(t)$ and $\sum_{k=1}^{\infty} G^{*k}(t) = t/\mu$, the renewal functions of $N(t)$ and the corresponding Poisson process; we get

$$\begin{aligned} S(t) &\geq p_0 + p_1 \left(M(t) - \frac{t}{\mu} \right) + p_1 \sum_{k=1}^{\infty} G^{*k}(t) + \sum_{j=2}^{\infty} (p_j - p_{j-1}) \sum_{k\geq j} G^{*k}(t) \\ &= p_0 + \sum_{k=1}^{\infty} p_k G^{*k}(t) + p_1 \left(M(t) - \frac{t}{\mu} \right) \\ &= S_{\exp}(t) - p_1 \left(\frac{t}{\mu} - M(t) \right), \end{aligned}$$

which yields (2.2). The proof for the *NWUE* case leading to (2.3) follows analogously using the *NWUE* version of (3.1). □

PROOF OF THEOREM 2.2. If $\tilde{N}(t)$ is the stationary renewal process induced by F, then

$$P\{\tilde{N}(t) \geq k\} = (TF) * F^{*(k-1)}(t), \qquad k \geq 1$$

since the time to first shock has distribution

$$TF(t) := \mu^{-1} \int_0^t \bar{F}(x)dx. \tag{3.4}$$

The basic argument is to show that (2.2) and (2.3) remain valid when the underlying counting process of shocks is a *stationary renewal process* with *NBUE* (*NWUE* resp.) survival and hinges on extending Lemma 3.1 in an appropriate direction. We show that F is *NBUE* (*NWUE* resp.) implies

$$\sum_{k\geq j} (TF) * F^{*(k-1)}(t) \leq (\geq) \sum_{k\geq j} G^{*k}(t), \qquad j \geq 1, \ t \geq 0. \tag{3.5}$$

If this holds, then we can proceed as in the proof of Theorem 2.1, by replacing $F^{*k}(t)$ in each term of (3.3) by the left hand side of (3.5) and mimic the remaining computational steps to conclude

$$\bar{S}(t) = E\bar{P}_{\tilde{N}(t)} \le (\ge)\bar{S}_{\exp}(t) + p_1\left(\frac{t}{\mu} - E\tilde{N}(t)\right) \equiv \bar{S}_{\exp}(t),$$

when F is *NBUE* (*NWUE* resp.). Here is the argument for (3.5) to complete the proof, when F is *NBUE*. The *NWUE* case is analogously obtained by reversing all inequalities below. Since $E\tilde{N}(t) = \frac{t}{\mu}$; (3.5) is trivially true as an equality when $j = 1$. If it holds up to some $j \ge 1$ and all $t > 0$; then

$$\begin{aligned}
\sum_{k \ge j+1} (TF)^* F^{*(k-1)}(t) &= \sum_{k \ge j} (TF)^* F^{*k}(t) \\
&= \int_0^t \left\{ \sum_{k=j}^{\infty} ((TF) * F^{*(k-1)})(t-x) \right\} F(dx) \\
&\le \int_0^t \left\{ \sum_{k=j}^{\infty} G^{*k}(t-x) \right\} F(dx) \\
&= \int_0^t \left\{ \sum_{k=j}^{\infty} F * G^{*(k-1)}(t-x) \right\} G(dx) \\
&\le \int_0^t \left\{ \sum_{k=j}^{\infty} G^{*k}(t-x) \right\} G(dx) \\
&= \sum_{k \ge j+1} G^{*k}(t),
\end{aligned}$$

where the first inequality uses the inductive hypothesis, and the last one follows from the version of Lemma 3.1 obtained by choosing $F_1 = F$; $F_n = G$, all $n \ge 2$. □

Let $N(t)$ ($N^*(t)$ respectively) denote the renewal counting process induced by F (by TF respectively). Let $\tilde{N}(t)$ be the *stationary renewal* counting process induced by F, $N_{\exp}(t)$ the Poisson process generated by the exponential survival distribution with mean $\mu = \int_0^\infty \bar{F}(x)dx$ and $\le_v$ ($\le_c$ resp.) the 'variability ordering' *with* (*without*, resp.) the condition of equal means—defined earlier. From Lemma 3.1 and the arguments for Theorem 2.2, we also have

COROLLARY 3.1. *If F is NBUE, then*

$$\begin{aligned}
&N(t) \le_{st} \tilde{N}(t) \le_{st} N^*(t), \\
&N(t) \le_c N_{\exp}(t) \quad \text{and} \quad \tilde{N}(t) \le_v N_{\exp}(t).
\end{aligned} \tag{3.6}$$

PROOF. F is $NBUE \Leftrightarrow TF \leq^{st} F \Rightarrow N(t) \leq^{st} \tilde{N}(t) \leq^{st} N^*(t)$, since

$$\begin{aligned} P(N(t) \geq n) &= F^{*n}(t) \leq TF * F^{*(n-1)}(t) \\ &\equiv P(\tilde{N}(t) \geq n) \leq TF^{*n}(t) \equiv P(N^*(t) \geq n), \end{aligned}$$

by using (3.5) and (3.1) with $F_n \equiv F$ for all $n = 1, 2, \ldots$. To check the remaining claims, note that for any $x > 0$, $P(N(t) > x) = F^{*n}(t)$ if $n - 1 \leq x < n$. The tails of $\tilde{N}(t)$ and $N_{\text{exp}}(t)$ can be analogously expressed. For any $u \geq 0$, let $\lceil u \rceil$ be the *ceiling* of u. (i.e., $\lceil u \rceil = j$, where $j \geq 1$ is the integer such that $j - 1 \leq u < j$. Note $\lceil 0 \rceil = 1$.) Hence, for any $u \geq 0$, using Lemma 3.1 with $F \equiv F_1 = F_2 = \cdots$, we have

$$\begin{aligned} \int_u^\infty P\{N(t) > x\}dx &= \sum_{n=\lceil u \rceil}^{\infty} F^{*n}(t) \leq \sum_{n=\lceil u \rceil}^{\infty} G^{*n}(t) \\ &= \int_u^\infty P\{N_{\text{exp}}(t) > x\}dx. \end{aligned} \tag{3.7}$$

Thus $N(t) \leq_c N_{\text{exp}}(t)$. We note that this ordering between $N(t)$ and $N_{\text{exp}}(t)$ cannot be strengthed to $\leq_v$, since equality does *not* hold in (3.7) for $u = 0$; viz., F is $NBUE$ implies $\{t : \sum_{n=1}^\infty F^{*n}(t) < \sum_{n=1}^\infty G^{*n}(t)\} = \{t : M(t) < t/\mu\}$ is not empty unless F is exponential.

An analogous computation using (3.5) similarly yields

$$\begin{aligned} \int_u^\infty P\{\tilde{N}(t) > x\}dx &= \sum_{n=\lceil u \rceil}^{\infty} (TF) * F^{*(n-1)}(t) \leq \sum_{n=\lceil u \rceil}^{\infty} G^{*n}(t) \\ &= \int_u^\infty P\{N_{\text{exp}}(t) > x\}dx, \end{aligned}$$

for all nonnegative u with equality at $u = 0$, since $E\tilde{N}(t) = t/\mu$. Hence $\tilde{N}(t) \leq_v N_{\text{exp}}(t)$. □

If F is $NWUE$ with a finite mean, the orderings in Corollary 3.1 are reversed. These results supplement those of Barlow and Proschan (1975) contained in Lemma 3.1.

PROOF OF THEOREM 2.3. $\bar{P}_k$ concave and $p_0 \neq 1$ implies $0 < p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_N$ for some $N \geq 1$; $p_k = 0$ for all $k > N$. Since $(p_j - p_{j-1})$ is thus nonnegative for $2 \leq j \leq N$; using Lemma 3.1 for equal components (i.e., $N(t) \leq_c N_{\text{exp}}(t)$) and the representation (1.1), we get

$$\begin{aligned} S(t) &= p_0 + \sum_{k=1}^{\infty} p_k F^{*k}(t) \\ &= p_0 + p_1 \sum_{k \geq 1} P\{N(t) \geq k\} + \sum_{j=2}^{N} (p_j - p_{j-1}) \sum_{k \geq j} P\{N(t) \geq k\} \end{aligned}$$

$$
\begin{aligned}
&- p_N \sum_{k \geq N+1} P\{N(t) \geq k\} \\
&\leq \left\{ p_0 + p_1 \sum_{k \geq 1} P\{N_{\exp}(t) \geq k\} \right. \\
&\quad \left. + \sum_{j=2}^{N} (p_j - p_{j-1}) \sum_{k \geq j} P\{N_{\exp}(t) \geq k\} - p_N \sum_{k \geq N+1} P\{N_{\exp}(t) \geq k\} \right\} \\
&\quad + p_N \sum_{k > N} [P\{N_{\exp}(t) \geq k\} - P\{N(t) \geq k\}] \\
&= S_{\exp}(t) + p_N \Delta_N^{F,G}(t).
\end{aligned}
$$

This proves (2.5), when F is *NBUE*. The *NWUE* case follows analogously. □

PROOF OF THEOREM 2.4. Using (1.1), express the survival probability as

$$\bar{S}(t) = \sum_{k=1}^{\infty} p_k P\{N(t) < k\} = \sum_{k=1}^{\infty} p_k \bar{F}^{*k}(t).$$

Since F with mean μ is *HNBUE* (Klefsjö (1982)) if and only if $\int_t^\infty \bar{F}(x)dx \leq \mu e^{-t/\mu}$ for all $t \geq 0$ and equality obviously holds at $t = 0$; it follows that the *HNBUE* property can be defined in terms of the $\leq_v$-ordering; viz.,

$$F(\text{with mean } \mu) \text{ is } HNBUE \Leftrightarrow F \leq_v G \quad \text{where} \quad \bar{G}(t) = e^{-t/\mu}.$$

We note in passing that since G, by definition, has the same mean as that of F; the *HNBUE* property of F is also equivalent to requiring $F \leq_c G$. Since $\leq_c$, as well as the $\leq_v$, ordering is preserved under convolutions (see Stoyan (1983) and the remark following Theorem 3.2 in Bhattacharjee (1991)), it follows that F is *HNBUE* implies $F^{*k} \leq_v G^{*k}$ for all $k = 1, 2, \ldots$. Hence, for $x \geq 0$,

$$\int_x^\infty \bar{S}(t)dt = \sum_{k=1}^{\infty} p_k \int_x^\infty \bar{F}^{*k}(t)dt \leq \sum_{k=1}^{\infty} p_k \int_x^\infty \bar{G}^{*k}(t)dt = \int_x^\infty \bar{S}_{\exp}(t)dt,$$

with equality at $x = 0$, since

$$\int_0^\infty \bar{S}(t)dt = \int_0^\infty \bar{S}_{\exp}(t)dt = \sum_{k=1}^{\infty} p_k(k\mu) = \mu(EJ).$$

Thus, $S \leq_v S_{\exp}$. The dual theorem, when the interarrival times between shocks is *HNWUE* with a finite mean, follows by interchanging the roles of F and G in the above argument; since, F is *HNWUE* $\Leftrightarrow F \geq_v G \Leftrightarrow G \leq_v F$. □

The sharpness of the bounds in Theorems 2.1–2.4 follow by choosing $N(t)$ to be a Poisson process. Below, we prove its necessity when F is *NBUE* (*HNBUE* in

Theorem 2.4), under the respective assumptions on $\bar{P}_k$ in Theorems 2.1–2.4. The dual arguments in the *NWUE* (*HNWUE* resp.) case follow analogously.

To prove necessity, we will show that for the respective bounds to be attained in each case; the interarrival times in the renewal process of shocks must be exponential (i.e., $N(t)$ must be Poisson) and then $S = S_{\text{exp}}$. In fact, as the following lemma, which is of independent interest shows: *among RPSM survival probabilities*; $S = S_{\text{exp}}$ *characterizes the Poisson processes*. Further note that when the corresponding bound is attained, the second term in the bounds in Theorems 2.1 and 2.3 vanishes and we have $S = S_{\text{exp}}$.

LEMMA 3.3. *A RPSM survival probability* (1.1) *reduces to comparable EMP-model* (2.1) *if and only if the shock process $N(t)$ is a Poisson process.*

PROOF. From (1.1); a RPSM survival distribution

$$S(t) = p_0 + \sum_{k=1}^{\infty} p_k F^{*k}(t), \qquad t \geq 0$$

has a possible discontinuity at zero, with $P(X = 0) = S\{0\} = \phi(F\{0\}) \geq p_0 \geq 0$, where $\phi(z) = \sum_{k=0}^{\infty} p_k z^k$, $0 < z < 1$ is the p.g.f. of the number of shocks to failure. Elementary computations show that the Laplace transform of S is

$$L_S(s) = \int_0^{\infty} e^{-st} S(dt) = \phi(L_F(s)), \qquad s > 0.$$

For the corresponding *EMP*-model, $L_{S_{\text{exp}}}(s) = \phi(\frac{\lambda}{\lambda+s})$. Hence a RPSM survival distribution reduces to an *EMP*-model ($S = S_{\text{exp}}$ pointwise) iff

$$L_S(s) = L_{S_{\text{exp}}}(s) \Leftrightarrow L_F(s) = \phi^{-1} L_{S_{\text{exp}}}(s) = \frac{\lambda}{\lambda + s},$$

for all $s \geq 0$; i.e., F is exponential; or iff $N(t)$ is a Poisson process. □

We now turn to proving that $N(t)$ is necessarily a Poisson process for the bounds to be sharp.

In Theorem 2.1. Recall $\bar{P}_k$ is convex implies $p_1 > 0$, since otherwise p_k decreasing would imply $p_0 = 1$ which contradicts the nontriviality assumption.

Case (i). $p_0 + p_1 = 1$. Here, failure occurs either before or at the first shock. Hence, from (1.1)

$$\begin{aligned} S(t) &= p_0 + (1 - p_0) F(t), \\ S_{\text{exp}}(t) &= p_0 + (1 - p_0)(1 - e^{-t/\mu}). \end{aligned}$$

If equality holds in (2.2), then we must have

$$(1 - p_0)\bar{F}(t) = \bar{S}_{\text{exp}}(t) + (1 - p_0)\left(\frac{t}{\mu} - M(t)\right)$$

or

$$\bar{F}(t) = e^{-t/\mu} + \frac{t}{\mu} - M(t) \geq e^{-t/\mu}, \tag{3.8}$$

since F is *NBUE* implies $M(t) \leq t/\mu$. This is enough to imply exponentiality of F (Bhattacharjee (1993)). Simply note that if $g(t)$ denotes the *mean residual life* function of F, then using the *NBUE* property $g(t) \leq \mu$ pointwise; (3.8) implies the derived distribution in (3.4) must satisfy

$$e^{-t/\mu} \leq \bar{TF}(t) = \exp\left\{-\int_0^t \frac{dx}{g(x)}\right\} \leq e^{-t/\mu}.$$

Case (ii). $p_0 + p_1 < 1$. Taking Laplace transforms, (2.2) implies

$$\begin{aligned} s^{-1}[1-\phi(L(s))] \leq s^{-1}\left[1 - \phi\left(\frac{1}{1+s\mu}\right)\right] \\ + p_1 \int_0^\infty \left(\frac{t}{\mu} - M(t)\right) e^{-st} dt, \qquad s > 0 \end{aligned}$$

where, for brevity, we write $L(s) \equiv L_F(s)$ for the Laplace transform of F. Equivalently,

$$0 \leq \phi(L(s)) - \phi\left(\frac{1}{1+s\mu}\right) + p_1 s \int_0^\infty e^{-st}\left(\frac{t}{\mu} - M(t)\right) dt.$$

Since $s\int_0^\infty e^{-st} M(t)dt = -\int_0^\infty M(t)d(e^{-st}) = \int_0^\infty e^{-st} M(dt) = L(s)/\{1-L(s)\}$, using integration by parts, $M(0) = 0$, and $e^{-st}M(t) \to 0$ as $t \to \infty$ with $s > 0$; we get

$$0 \leq \phi\left(\frac{1}{1+s\mu}\right) - \phi(L(s)) \leq p_1\left[(s\mu)^{-1} - \frac{L(s)}{1-L(s)}\right].$$

The first inequality above is a consequence of (i) the nondecreasing nature of ϕ and (ii) the *NBUE* property which implies $L(s) \leq (1+s\mu)^{-1}$. (The last statement can be verified in several ways; e.g., F is *NBUE* iff $TF \leq^{st} F$ which implies $\{1-L(s)\}/s\mu = E_{TF}(e^{-sX}) \geq E_F(e^{-sX}) \equiv L(s)$, $s > 0$.) Thus if equality holds in (2.2), then

$$\phi\left(\frac{1}{1+s\mu}\right) - p_1(s\mu)^{-1} = \phi(L(s)) - p_1\frac{L(s)}{1-L(s)}. \tag{3.9}$$

Since $\phi(z) = \sum_{j=0}^\infty p_j z^j$, $0 < z < 1$ and convexity of $\bar{P}_j$ implies $\alpha_j := (p_1 - p_j)$ is nonnegative for $j \geq 1$ with $\alpha_j > 0$ for some $j \geq 2$, the left hand side of (3.9) can be expanded as

$$\phi(z) - \frac{p_1 z}{1-z} = p_0 - \sum_{j=1}^\infty \alpha_j z^j,$$

where $z = (1+s\mu)^{-1}$. Hence (3.9) can be written as

$$\sum_{j=2}^\infty \alpha_j \left(\frac{1}{1+s\mu}\right)^j = \sum_{j=2}^\infty \alpha_j \{L(s)\}^j, \qquad s > 0. \tag{3.10}$$

Since F is $NBUE$ implies $L(s) \le (1+s\mu)^{-1}$; it is clear that (3.10) would be violated if there exists an $s_0 > 0$ such that $L(s_0) < (1+s_0\mu)^{-1}$. Hence $L(s) = (1+s\mu)^{-1}$, all $s > 0$; i.e., F is exponential.

In Theorem 2.2. The RPSM survival probability, under shocks governed by a *stationary* renewal process $\tilde{N}(t)$ is

$$\begin{aligned}\bar{S}(t) = 1 - S(t) = 1 - p_0 - \sum_{k=1}^{\infty} p_k P\{\tilde{N}(t) \ge k\} &= \sum_{k=1}^{\infty} p_k P\{\tilde{N}(t) < k\} \\ &= \sum_{k=1}^{\infty} p_k\{1 - TF * F^{*(k-1)}(t)\}.\end{aligned}$$

Standard computations then yield:

$$s^{-1}[1 - L_S(s)] \equiv \int_0^{\infty} e^{-st}\bar{S}(t)dt = s^{-1}\sum_{k=1}^{\infty} p_k\left[1 - \frac{1-L(s)}{s\mu}L^{k-1}(s)\right],$$

or,

$$L_S(s) = p_0 + \frac{1-L(s)}{s\mu L(s)}[\phi(L(s)) - p_0], \qquad s > 0$$

as the Laplace transform of the *stationary* RPSM distribution $S(t)$. Now, if equality holds in (2.4); then taking Laplace transforms on both sides implies $L_S(s) = L_{S_{\exp}}(s) \equiv \phi(\frac{1}{1+s\mu})$. Using this in the previous step, we get,

$$\text{(3.11)} \qquad \frac{1-L(s)}{L(s)}[\phi(L(s)) - p_0] = s\mu\left[\phi\left(\frac{1}{1+s\mu}\right) - p_0\right].$$

But (i) $L(s) \le (1+s\mu)^{-1}$, by $NBUE$ property and (ii) ϕ is nondecreasing, together imply

$$\frac{\phi\left(\dfrac{1}{1+s\mu}\right) - p_0}{\phi(L(s)) - p_0} \le 1 \le \frac{1-L(s)}{s\mu L(s)}.$$

Hence, if (3.11) holds, then both sides above collapse to 1, so that from the second equality in the collapsed chain, we must have $L(s) = (1+s\mu)^{-1}$, all $s > 0$; implying exponentiality of F.

In Theorem 2.3. If equality holds in (2.5), then

$$\bar{S}(t) = \bar{S}_{\exp}(t) - p_N\Delta_N^{F,G}(t) = \bar{S}_{\exp}(t) - p_N\sum_{n>N}\{\bar{F^{*n}}(t) - \bar{G^{*n}}(t)\}.$$

Taking Laplace transforms, we get

$$s^{-1}[1 - \phi(L(s))]$$

$$
\begin{aligned}
&= \int_0^\infty e^{-st}\bar{S}(t)dt \\
&= s^{-1}\left[1-\phi\left(\frac{\lambda}{\lambda+s}\right)\right] \\
&\quad - p_N \sum_{n>N}\left\{s^{-1}[1-L^n(s)] - s^{-1}\left[1-\left\{\phi\left(\frac{\lambda}{\lambda+s}\right)\right\}^n\right]\right\}
\end{aligned}
$$

or,

$$
\phi(L(s)) = \phi\left(\frac{\lambda}{\lambda+s}\right) + p_N \sum_{n>N}\left\{\left(\frac{\lambda}{\lambda+s}\right)^n - L^n(s)\right\} \geq \phi\left(\frac{\lambda}{\lambda+s}\right), \tag{3.12}
$$

since $p_N > 0$ and F is *NBUE* implies every term in the sum is non-negative. Since ϕ is nondecreasing, this implies

$$
L(s) \geq \frac{\lambda}{\lambda+s} \equiv \frac{1}{(1+s\mu)} \geq L(s);
$$

where the last inequality is implied by the *NBUE* hypothesis. Hence, $L(s) = (1+s\mu)^{-1}$, all $s > 0$; i.e., F is exponential, and we have $\Delta_k^{F,G}(t) \equiv 0$, all $k = 0, 1, 2, \ldots$; $S = S_{\text{exp}}$.

Remark. A direct argument is also possible. Since $p_j = 0$ for $j > N$, and

$$
\begin{aligned}
\phi(z) + p_N \sum_{n>N} z^n &= \sum_{j=0}^{N} p_j z^j + p_N \sum_{n>N} z^n \\
&= \sum_{j=0}^{N-1} p_j z^j + p_N(z^N + z^{N+1} + \cdots), \qquad 0 < z < 1
\end{aligned}
$$

rewrite the identity in (3.12) as

$$
\sum_{j=0}^{\infty} \alpha_j \left\{\left(\frac{\lambda}{\lambda+s}\right)^j - L^j(s)\right\} = 0, \tag{3.13}
$$

where $\alpha_j = p_j$ for $j < N$ and $= p_N$ for $j \geq N$. The left hand side is nonnegative since the multiplier of each α_j is nonnegative by the *NBUE* hypothesis and $\alpha_j > 0$ at least for all $j \geq N$ since $p_N > 0$ by concavity of the shock resistance probabilities. Hence (3.13) implies that the coefficient of every positive α_j vanishes; i.e., $L(s) = \lambda/(\lambda+s)$, $s > 0$; or F is exponential.

To prove that if the bound in Theorem 2.4 is attained, then the shock interarrival times must be exponential; we will find the following result, which is a special case of a more general result (see Theorem 2.2 in Bhattacharjee and Sethuraman (1990)) useful.

THEOREM 3.1. (Bhattacharjee and Sethuraman (1990)) *Let nonnegative r.v.s X, Y be ordered by $X \leq_v Y$. If $EX^r = EY^r < \infty$ for some $r \neq 0$, $r \neq 1$, then $X =_{st} Y$.*

Let U (V resp.) denote the survival time under the renewal process shock model distribution S ($S_{\exp}$ resp.). If $\{X_n : n \geq 1\}$ ($\{Y_n : n \geq 1\}$ resp.) are the corresponding i.i.d. interarrival times between shocks with distribution F (G resp., where G is exponential with mean $\mu = EX_1$); $S_k := \sum_{i=1}^k X_i$, $\tilde{S}_k := \sum_{i=1}^k Y_i$ and J is the number of shocks to failure; then $U = S_J$ and $V = \tilde{S}_J$.

PROOF OF NECESSITY IN THEOREM 2.4. Under the assumption in Theorem 2.4, we have $U \leq_v V$. Further, for any $r > 0$,

$$EU^r = r\int_0^\infty t^{r-1}\bar{S}(t)dt = \sum_{k=1}^\infty p_k ES_k^r, \qquad \text{and similarly} \quad EV^r = \sum_{k=1}^\infty p_k E\tilde{S}_k^r.$$

If equality holds (i.e., $U =^{st} V$) in Theorem 2.4, then,

$$0 = EV^r - EU^r = \sum_{k=1}^\infty p_k(E\tilde{S}_k^r - ES_k^r), \qquad \text{all} \quad r > 0. \tag{3.14}$$

Since (i) $X_i \leq_v Y_i$ for all $i \geq 1$ (i.e., F is *HNBUE*) implies $S_k \leq_v \tilde{S}_k$ for all $k \geq 1$ (viz., $\leq_v$ is closed under convolutions) and (ii) x^r is convex in $x \geq 0$ for $r \geq 1$; it follows from known characterization of $\leq_v$-ordering (see Theorem 2.2 in Bhattacharjee (1991), or Corollary 8.5.2 in Ross (1983)) that $ES_k^r \leq E\tilde{S}_k^r$, all $k \geq 1$ so that every term in the right hand side of (3.14) is nonnegative and hence must vanish. Choose any $k \in \{j \geq 1 : p_j > 0\}$. Then for such a k, we must have $ES_k^r = E\tilde{S}_k^r$, all $r \geq 1$. This is more than enough; since choosing *any* such $r > 1$, Theorem 3.1 guarantees $S_k =_{st} \tilde{S}_k$, fixed $k \geq 1$. This in turn implies

$$\left(\frac{1}{1+s\mu}\right)^k = E^k(e^{-sY_1}) = E(e^{-s\tilde{S}_k}) = E(e^{-sS_k}) = E^k(e^{-sX_1}) = \{L_F(s)\}^k,$$

for all $s > 0$ and fixed $k \geq 1$; or, F is exponential. □

4. Applications

4.1 *Improved bounds on NBUE renewal functions*

When any shock is independently survived, or leads to failure with fixed probabilities s and $(1-s)$ respectively $(0 < s < 1)$, the corresponding geometric shock resistance probabilities $\bar{P}_k = s^k$, $k = 0, 1, 2, \ldots$ are convex and decreasing in k. The associated *RPSM* survival probability (1.1) is then the probability generating function (p.g.f.) of the renewal counting process induced by F and the corresponding *EMP* survival probability

$$\bar{S}_{\exp}(t) = \sum_{k=0}^\infty s^k e^{-t/\mu}\frac{(t/\mu)^k}{k!} = e^{-(1-s)t/\mu},$$

is the p.g.f. of the corresponding Poisson process. When F is *NBUE*, an application of Theorem 2.1 yields

$$Es^{N(t)} \leq e^{-(1-s)t/\mu} + (1-s)\left(\frac{t}{\mu} - M(t)\right).$$

As $s \to 0^+$, we get

$$\bar{F}(t) = \lim_{s\to 0^+} Es^{N(t)} \leq e^{-t/\mu} + \frac{t}{\mu} - M(t).$$

Together with the known bound $M(t) \leq t/\mu$, when F is *NBUE*, we get

COROLLARY 4.1. *For an NBUE renewal process,*

$$M(t) \leq \min\left(\frac{t}{\mu}, \frac{t}{\mu} - \bar{F}(t) + e^{-t/\mu}\right) = \frac{t}{\mu} - (\bar{F}(t) - e^{-t/\mu})^+.$$

Since F is *NBUE* with mean μ implies $I(t) := \int_t^\infty \{e^{-x/\mu} - \bar{F}(x)\}dx = \mu^{-1}(e^{-t/\mu} - \overline{TF}(t)) \geq 0$ for $t \geq 0$ with $I(0) = 0$ (viz., Barlow and Proschan (1975)) and thus implies $\{t > 0 : \bar{F}(t) > e^{-t/\mu}\}$ is not empty unless F is exponential; the bound in Corollary 4.1 is tighter than the bound (t/μ) under *NBUE* aging. Clearly the extent of tightening over t/μ is no more than 1, and the improvement is usually strict for all t up to a critical value if F is aging in an appropriately stronger sense. Such is the case for example if F is *IFRA*, since $\bar{F}(t)$ then crosses $e^{-t/\mu}$ exactly once from above.

4.2 *Application to waiting time in $M/GI/1$ and $GI/G/1$ queues*

As a further simple application of our results, we show (Corollary 4.3) that the stationary waiting time in a stable $GI/G/1$ queue is stochastically dominated by the stationary waiting time in the corresponding $M/M/1$ queue, when the $GI/G/1$ queue has *NBUE* interarrival times and service time distributions (cf. Miyazawa (1976) and Stoyan (1977)). This is obtained by first deriving the corresponding result (Corollary 4.2) for $M/GI/1$ queues with *NBUE* service as an application of Corollary 3.1, which together with a result of Rolski (1979) on comparing the waiting time in a $GI/GI/s$ queue ($s \geq 1$) with *NBUE* arrivals to the waiting time in an appropriate $M/GI/1$ queue then readily yields the desired conclusion.

For stable $M/GI/1$ queue, the classical Khinchin-Pollackzek formula $W(t) \equiv P(T \leq t) = (1-\rho)\sum_{n=0}^\infty \rho^n (TF)^{*n}(t) = \sum_{n=0}^\infty \rho^n P\{N^*(t) = n\}$, where $N^*(t)$ is the renewal quantity generated by the equilibrium distribution TF defined in (3.4), clearly shows that the distribution of the stationary waiting time T in a stable $M/GI/1$ queue (with service time distribution F and service utilization factor $\rho < 1$), is a *RPSM* where each '*shock*' corresponds to a service completion. Indeed, since $P(T > 0) = \rho$, we can further write,

(4.1) $\bar{S}(t) := P(T > t \mid T > 0)$

$$
\begin{aligned}
&= \bar{W}(t)/\rho \\
&= \rho^{-1}\left\{(1-\rho)\sum_{n=0}^{\infty}\rho^n P\{N^*(t)\geq n\}\right\} \\
&= \rho^{-1}\left\{1-(1-\rho)\left[\sum_{n=0}^{\infty}\rho^n P\{N^*(t)=n\}+\sum_{n=0}^{\infty}\rho^n P\{N^*(t)>n\}\right]\right\} \\
&= \rho^{-1}\left\{1-(1-\rho)\left[E\rho^{N^*(t)}+\frac{1-E\rho^{N^*(t)}}{1-\rho}\right]\right\} \\
&= E\rho^{N^*(t)}, \qquad t\geq 0.
\end{aligned}
$$

Hence in a stable $M/GI/1$ queue, not only $W(t)$, but the conditional distribution of the waiting time, given a customer does wait, is also a shock model survival probability with *geometric* "shock resistance".

COROLLARY 4.2. *With NBUE service times, the stationary waiting time in a stable $M/GI/1$ queue is stochastically dominated by the stationary waiting time in the corresponding $M/M/1$ queue with the same traffic intensity.*

PROOF. If the service time distribution F is *NBUE* (i.e., the mean residual service time function is pointwise no larger than the mean service time), then since ρ^n is $\downarrow$ in n $(0<\rho<1)$, and is convex in n; using Corollary 3.1 to bound the right hand side in (4.1), we get

$$P(T>t)\equiv \bar{W}(t)=\rho E\rho^{N^*(t)}\leq \rho E\rho^{\tilde{N}(t)}\leq \rho E\rho^{N_{\exp}(t)}=\rho e^{-\mu(1-\rho)t},$$

where μ denotes the *service rate* (consistent with notations standard in queueing theory, but *in contrast with* the notation in earlier sections). For the first inequality, we use $\tilde{N}(t)\leq_{st} N^*(t)$ (from (3.6)) and ρ^n decreasing in n. The second inequality follows by using (i) $\tilde{N}(t)\leq_v N_{\exp}(t)$ from (3.6), (ii) convexity of ρ^n in n and a well known characterization of the variability ordering $\leq_v$ (see Remark 2 following the definition of the $\leq_v$-ordering in Section 2). □

Rolski (1979) (see Stoyan (1983), p. 110) proved that in a queue with s servers $(s\geq 1)$, *NBUE* arrivals and arbitrary service times and $\rho<1$; the stationary waiting time is stochastically dominated by the stationary waiting time in a $M/GI/1$ system with the same service time distribution and arrival rate (λ/s), i.e.,

$$W(t)\geq (1-\rho/s)\sum_{n=0}^{\infty}(\rho/s)^n\overline{TF}^{*n}(t), \qquad t\geq 0.$$

If the arrival and service time d.f.s are both arbitrary with *NBUE* property, we can then apply Corollary 4.2 to Rolski's result with $s=1$ to obtain the following result.

COROLLARY 4.3. *In $GI/GI/1$ queues with arbitrary NBUE interarrival and service times, the stationary waiting time is stochastically dominated by the corresponding $M/M/1$ stationary waiting time.*

PROOF. Simply note,

$$T_{GI/GI/1} \leq_{st} T_{M/GI/1} \leq_{st} T_{M/M/1},$$

where the subscripts indicate the appropriate queues to which the waiting times refer, use Corollary 4.2 and the transitiveness of the partial ordering $\leq_{st}$. $\square$

Acknowledgements

The author gratefully acknowledges several suggestions of the associate editor and an anonymous referee, leading to an improved presentation.

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