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LOCAL LINEARIZATION METHOD FOR THE NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. The Local Linearization (LL) approach for the numerical solution of stochastic differential equations (SDEs) is extended to general scalar SDEs, as well as to non-autonomous multidimensional SDEs with additive noise. In case of autonomous SDEs, the derivation of the method introduced gives theoretical support to one of the previously proposed variants of the LL approach. Some numerical examples are given to demonstrate the practical performance of the method.

Key words and phrases: Stochastic differential equations, numerical solution, local linearization.

1. Introduction

There is a great variety of numerical schemes for the solution of stochastic differential equations (SDEs), such as those of Maruyama (1955), McShane (1974), Milshtein (1974), Kloeden and Platen (1989), Newton (1991), and Saito and Mitsui (1992).

The common theoretical basis of these methods is the stochastic Ito-Taylor expansion of the solution in terms of multiple Wiener integrals (Kloeden *et al.* (1993)). In spite of the well known convergence properties achieved by means of this approach, two limitations have been pointed out (e.g. Henrici (1962), Ozaki (1985*a*, 1992)). First, this approach does not give exact solutions in case of linear SDEs. Second, the numerical solution does not always preserve the qualitative characteristics of the exact solution. In particular, there are many examples of SDEs with bounded trajectories such that, for any fixed stepsize of the time discretization, the numerical solution results to be explosive when the initial value is in a certain region of the phase space (Ozaki (1985*a*)).

Ozaki (1985*a*, 1985*b*) has introduced a new approach for the numerical solution of SDEs, known as Local Linearization (LL), which overcomes these difficulties for the autonomous d-dimensional SDE of the form

$$\begin{aligned} dX(t) &= f(X)dt + dW(t) \qquad t \in [t_0, T], \\ X(t_0) &= X_0, \end{aligned} \tag{1.1}$$

where $W(t)$ is a Wiener process. The LL method seems to be inspired by heuristic considerations that attempt to obtain a numerical scheme in the form of a linear multivariate autoregressive time series with state-dependent coefficients. This approach does not involve a stochastic Taylor expansions of the solution. The LL scheme (for discrete times $t_n = t_0 + n\Delta$, $n = 0, 1, \ldots$) is:

$$X_{t_{n+1}} = A(X_{t_n})X_{t_n} + \xi_{t_{n+1}}, \tag{1.2}$$

where

$$\begin{aligned} &A(X_{t_n}) = \exp(L(X_{t_n})\Delta), \\ &L(X_{t_n}) = \ln(I + (J_f^x(t_n))^{-1}(\exp(J_f^x(t_n)\Delta) - I)F(X_{t_n}))/\Delta. \end{aligned} \tag{1.3}$$

Here $J_f^x(t_n)$ is the Jacobian matrix of the function f evaluated at X_{t_n}, $F(X_{t_n})$ is any matrix such that $F(X_{t_n})X_{t_n} = f(X_{t_n})$, and I is the d-dimensional identity matrix. The innovation $\xi_{t_{n+1}}$ is a time series of independent Gaussian random vectors with zero mean, and represents the stochastic part of the numerical solution.

The performance of the LL method has been illustrated in a number of papers (see e.g. Ozaki (1985*a*, 1985*b*, 1992)).

However, the current formulation of this method is ambiguous as to the proposed specification of the random term $\xi_{t_{n+1}}$. Originally, Ozaki (1985*a*) proposed $\xi_{t_{n+1}} = W(t_{n+1}) - W(t_n)$, and consequently with a covariance matrix equals to ΔI. In Ozaki (1985*b*, 1992) this time series was given by

$$\xi_{t_{n+1}} = \int_{t_n}^{t_{n+1}} \exp(L(X_{t_n})(t_{n+1} - u))dW(u), \tag{1.4}$$

with the covariance matrix

$$E(\xi_{t_{n+1}}\xi_{t_{n+1}}) = \int_{t_n}^{t_{n+1}} \exp(L(X_{t_n})(t_{n+1} - u)) \exp(L(X_{t_n})(t_{n+1} - u))'du, \tag{1.5}$$

where for any matrix M, M' denotes its transpose.

More recently, Ozaki (1994) has suggested that $\xi_{t_{n+1}}$ be defined by (1.4)–(1.5) but with $L(X_{t_n})$ substituted by $J_f^x(t_n)$.

In addition, the Ozaki's scheme (1.4)–(1.5) has the drawback of the non-uniqueness of the factor $F(X_{t_n})$ in case of a multidimensional SDE. It is not clear which of the admissible matrices $F(X_{t_n})$ (i.e. those ones which satisfy $F(X_{t_n})X_{t_n} = X_{t_n}$) is the most adequate to be used in (1.3).

In this paper the LL approach is extended to general scalar SDEs, as well as to non-autonomous multidimensional SDEs with additive noise. The resulting numerical scheme overcomes the shortcomings just mentioned of previous formulations of the LL approach. The method introduced is deduced neither from stochastic Taylor expansions of the solution nor following the heuristic arguments developed by Ozaki (1985*a*, 1985*b*). Instead, it is derived from (first order) deterministic Taylor expansions of the drift and diffusion coefficients of the SDE.

In the particular case of a SDE of the form (1.1), the method introduced essentially leads to the numerical scheme most recently suggested by Ozaki (1994), namely the scheme described by (1.2)–(1.5) but using $J_f^x(t_n)$ instead of $L(X_{t_n})$ in the expressions (1.4)–(1.5). Thus, our work constitutes a theoretical derivation of this variant of Ozaki's LL method, and also an extension of this scheme to cover more general, non-autonomous SDEs.

In Sections 2 and 3 the numerical scheme is introduced for scalar and multidimensional non-autonomous SDEs, respectively. Detailed formulae of the algorithms are given. Section 4 presents some simulation results that illustrate the performance of the method.

2. Local linearization method for scalar non-autonomous Ito stochastic differential equations

Consider a scalar non-autonomous SDE with multiplicative noise given by

$$\begin{aligned} &dX(t) = f(X,t)dt + g(X,t)dW(t) \qquad t \in [t_0, T], \\ &X(t_0) = X_0. \end{aligned} \tag{2.1}$$

Here $W(t)$ is a standard Wiener process. It is assumed that the functions $f(x,t)$ and $g(x,t)$ satisfy the standard conditions for the existence and uniqueness of a (strong) solution of (2.1) (see e.g. Arnold (1974), p. 105). In addition, these functions are assumed to have continuous derivatives with respect to both arguments x and t. The case of additive noise (i.e. $g(X,t) = g(t)$) will be considered in the next section.

The LL method for solving this SDE will be defined on the basis of the solution of the linear SDE resulting from the local linearization of the right member of (2.1).

Let Δ be the stepsize of the time discretization, and t be any fixed time instant in $[t_0, T)$. Consider the truncated (deterministic) Taylor expansion of the functions f and g in (2.1) at the point $(t, X(t))$,

$$\begin{aligned} f(X(s),s) &\approx f(X(t),t) + J_f^x(t)(X(s) - X(t)) + J_f^t(t)(s-t), \\ g(X(s),s) &\approx g(X(t),t) + J_g^x(t)(X(s) - X(t)) + J_g^t(t)(s-t), \end{aligned}$$

where only the first order terms of the expansions are retained. Using these expansions, the following SDE is obtained as an approximation to equation (2.1):

$$dX(s) = (A(t)X(s) + a(t,s))ds + (B(t)X(s) + b(t,s))dW(s), \tag{2.2}$$

$s \in [t, t+\Delta]$, with the initial condition $X(t) = X_t$. Here

$$\begin{aligned} &A(t) = J_f^x(t), \\ &a(t,s) = f(X(t),t) - J_f^x(t)X(t) + J_f^t(t)(s-t), \\ &B(t) = J_g^x(t), \\ &b(t,s) = g(X(t),t) - J_g^x(t)X(t) + J_g^t(t)(s-t). \end{aligned} \tag{2.3}$$

In these expressions, $J_f^x(t)$ and $J_g^x(t)$ denote, respectively, the derivatives of the functions $f(x,t)$ and $g(x,t)$ with respect to the variable x, evaluated at the point $(X(t),t)$. In a similar fashion, $J_f^t(t)$ and $J_g^t(t)$ are the derivatives of the functions $f(x,t)$ and $g(x,t)$ with respect to the variable t, evaluated at the point $(X(t),t)$.

The resulting linear SDE (2.2) has an explicit solution (Arnold (1974), p. 136) whose value at the time $t+\Delta$ is

$$X(t+\Delta) = \phi(t+\Delta)\Big\{X(t) + \int_t^{t+\Delta} \phi^{-1}(u)(a(t,u) - B(t)b(t,u))du + \int_t^{t+\Delta} \phi^{-1}(u)b(t,u)dW(u)\Big\}, \tag{2.4}$$

where $\phi(u) = \exp((A(t) - B^2(t)/2)(u-t) + B(t)(W(u) - W(t))$.

In order to compute this solution it is necessary to approximate the two stochastic integrals contained in the expression (2.4). One of them is an Ito integral, which can be approximated in different ways. A simple approximation follows directly from the definition of the Ito integral by means of Riemann-Stieltjes sums:

$$\begin{aligned}\int_t^{t+\Delta} \phi^{-1}(u)b(t,u)dW(u) &\approx \phi^{-1}(t)b(t,t)(W(t+\Delta) - W(t)) \\ &= (g(X(t),t) - J_t^x(t)X(t))(W(t+\Delta) - W(t)).\end{aligned}$$

However, with this naive approximation the information contained in the Jacobians $J_g^t(t)$, $J_f^t(t)$ and $J_f^x(t)$ is partially lost. To avoid this we transform the Ito integral in (2.4) by using the known identity (Schuss (1980), p. 71)

$$\begin{aligned}\int_t^{t+\Delta} h(W(u),u)dW(u) \\ = \int_t^{t+\Delta} h(W(u),u) \circ dW(u) - (1/2)\int_t^{t+\Delta} J_h^W(W(u),u)du,\end{aligned} \tag{2.5}$$

where the symbol $\circ$ denotes Stratonovich integration, and J_h^W is the derivative of the integrand function $h(W,u)$ with respect to W. Thus, expression (2.4) is transformed into

$$X(t+\Delta) = \phi(t+\Delta)\Big\{X(t) + \int_t^{t+\Delta} \phi^{-1}(u)(a(t,u) - B(t)b(t,u)/2)du + \int_t^{t+\Delta} \phi^{-1}(u)b(t,u) \circ dW(u)\Big\}. \tag{2.6}$$

The first integral in (2.5) is then approximated by the Trapezoidal rule,

$$\int_t^{t+\Delta} h(W(u),u)du \approx (h(W(t+\Delta),t+\Delta) + h(W(t),t))\Delta/2, \tag{2.7}$$

and the second (Stratonovich) integral is approximated in the form

$$(2.8)\quad \int_t^{t+\Delta} h(W(u),u)\circ dW(u) \approx h((W(t+\Delta)+W(t))/2,t)(W(t+\Delta)-W(t)).$$

Using these approximations, (2.4) is converted into

$$(2.9)\quad \begin{aligned} X(t+\Delta) \approx \phi(t+\Delta)\{&X(t)+(Q(t+\Delta)+Q(t))\Delta/2 \\ &+K(\Delta W(t+\Delta)/2,t)\Delta W(t+\Delta)\}, \end{aligned}$$

where $Q(u)=\phi^{-1}(u)(a(t,u)-B(t)b(t,u)/2)$, $K(u,v)=\exp(-B(t)u)b(t,v)$ and $\Delta W(t+\Delta)=W(t+\Delta)-W(t)$.

Expression (2.9) permits the calculation of the local solution at the time $t+\Delta$ from the local solution at time t. The numerical scheme of the LL method is defined by the iteration of this computation, starting from the initial condition $X_{t_0}=X_0$:

$$(2.10)\quad \begin{aligned} X_{t_{n+1}} = \phi(t_{n+1})\{&X_{t_n}+(Q(t_{n+1})+Q(t_n))\Delta/2 \\ &+K(\Delta W(t_{n+1})/2,t_n)\Delta W(t_{n+1})\}, \end{aligned}$$

where $t_n=t_0+n\Delta$ $(n=0,1,\ldots)$ are the step-points of time discretization, and $\Delta W(t_{n+1})=W(t_{n+1})-W(t_n)$.

3. Local linearization method for multidimensional non-autonomous Ito stochastic differential equations

This section introduces the local linearization method for multidimensional non-autonomous SDEs. The presentation is restricted to the case of additive noise, i.e., the diffusion term does not depend on X:

$$(3.1)\quad \begin{aligned} &dX(t)=f(X,t)dt+g(t)dW(t), \qquad t\in[t_0,T], \\ &X(t_0)=X_0, \end{aligned}$$

where $W(t)$ is a m-dimensional Wiener process, $X(t)$ and $f(X,t)$ are vectors in $\mathbb{R}^d$, and $g(t)$ is a $d\times m$ real matrix. The standard conditions for the existence and uniqueness of a (strong) solution are assumed (see e.g. Arnold (1974), p. 105). In addition, the components of $f(x,t)$ and $g(t)$ are assumed to have continuous derivatives with respect to the components of both arguments x and t.

The linearization of the function f in this SDE,

$$f(X(s),s)\approx f(X(t),t)+J_f^x(t)(X(s)-X(t))+J_f^t(t)(s-t),$$

leads to an equation of the type (2.2) but with $B(t)=0$ and $b(t,s)=g(s)$. The solution of the resulting equation (Arnold (1974), p. 129) is

$$(3.2)\quad \begin{aligned} X(t+\Delta)=\phi(t+\Delta)\Big\{&X(t)+\int_t^{t+\Delta}\phi^{-1}(u)a(t,u)du \\ &+\int_t^{t+\Delta}\phi^{-1}(u)g(u)dW(u)\Big\}, \end{aligned}$$

where $\phi(u) = \exp(A(t)(u-t))$. Notice that if the original system includes multiplicative noise, the resulting linearized multidimensional system does not have an explicit solution in general. Hence the restriction to additive noise in the case of multidimensional systems.

For any matrix M and any number $a > 0$, define

$$r_n(M, a) = \int_0^a \exp(Mu)u^n du.$$

(For details about the computation of $r_n(M, a)$, see Appendix.) From (3.2) and the definition of $a(t, s)$ given in (2.3) it follows that

$$\begin{aligned} X(t+\Delta) = X(t) &+ r_0(J_f^x(t), \Delta) f(X(t), t) \\ &+ (\Delta r_0(J_f^x(t), \Delta) - r_1(J_f^x(t), \Delta)) J_f^t(t) + \xi(t+\Delta), \end{aligned} \tag{3.3}$$

where

$$\xi(t+\Delta) = \int_t^{t+\Delta} \phi(2t+\Delta-u) g(u) dW(u) \tag{3.4}$$

is a stochastic process with zero mean and variance

$$E(\xi(t+\Delta)\xi'(t+\Delta)) = \int_t^{t+\Delta} \phi(2t+\Delta-s) g(t) g'(t) \phi'(2t+\Delta-s)) ds. \tag{3.5}$$

In order to approximate the Ito integral involved in the definition of $\xi(t+\Delta)$, the following property (Schuss (1980), p. 66) will be used

$$\int_a^b h(u) dW(u) = h(b)W(b) - h(a)W(a) - \int_a^b J_h^u(u) W(u) du, \tag{3.6}$$

which holds for any smooth deterministic function h with derivative $J_h^u(u)$. This leads to

$$\begin{aligned} \xi(t+\Delta) = g(t+\Delta)W(t+\Delta) &- \phi(\Delta) g(t) W(t) \\ &+ \int_t^{t+\Delta} \phi(2t+\Delta-u)(J_f^x(t) g(u) - J_g^u(u)) W(u) du. \end{aligned}$$

In turn, applying the Trapezoidal rule (2.7) to the integral in the last expression the following approximation is obtained

$$\begin{aligned} \xi(t+\Delta) \approx \{(J_f^x(t) g(t+\Delta) - J_g^t(t+\Delta))\Delta/2 + g(t+\Delta)\} W(t+\Delta) \\ + \phi(t+\Delta)\{(J_f^x(t) g(t) - J_g^t(t))\Delta/2 - g(t)\} W(t). \end{aligned} \tag{3.7}$$

Finally, the LL numerical scheme is defined by the iterative computation of expressions (3.3) and (3.6) evaluated at the discrete times t_n, starting at $X(t_0) = X_0$:

$$\begin{aligned} X_{t_{n+1}} = X_{t_n} &+ r_0(J_f^x(t_n), \Delta) f(X_{t_n}, t_n) \\ &+ (\Delta r_0(J_f^x(t_n), \Delta) - r_1(J_f^x(t_n), \Delta)) J_f^t(t_n) + \xi(t_{n+1}), \end{aligned} \tag{3.8}$$

where

$$\xi(t_{n+1}) = \{(J_f^x(t_n)g(t_{n+1}) - J_g^t(t_{n+1}))\Delta/2 + g(t_{n+1})\}W(t_{n+1}) + \phi(t_{n+1})\{(J_f^x(t_n)g(t_n) - J_g^t(t_n))\Delta/2 - g(t_n)\}W(t_n). \tag{3.9}$$

It is worth noting the relations of this scheme with those proposed by Ozaki in the particular case of a scalar SDE of the type (1.1). It is simple to show that in this case the expressions (3.3)–(3.5), evaluated at the discrete times, reduced to the numerical scheme proposed by Ozaki (1994) (i.e. equations (1.2)–(1.4) with $L(X_{t_n})$ substituted by $J_f^x(t_n)$ in (1.4)). The algorithm described by (3.8)–(3.9) constitutes an approximation to this scheme, as a consequence of the use of the approximation (3.6) to the random term.

4. Simulation results

The variant of the LL method introduced above gives exact solutions for linear autonomous SDEs. In particular the method is exact for the stiff linear autonomous SDEs, which are frequently used as critical tests to assess numerical schemes (e.g. the stochastic versions of the standard examples in Kloeden *et al.* (1993), pp. 132–133). Therefore, it is only of interest to carry out simulation studies of the method for nonlinear or non-autonomous equations.

The LL for nonlinear autonomous SDEs with additive noise has been illustrated in a number of papers (see e.g. Ozaki (1985*a*, 1985*b*, 1992)). The examples in this section will refer to the remaining class of SDEs, i.e. equations that are non-autonomous or with multiplicative noise.

To summarize the performance of the scheme, plots of the exact and approximate solutions, as well as global errors for different step sizes, are shown in each example.

To take into consideration the effect of the realization of the exact solution by means of pseudo-random numbers generated in a digital computer, the global error is decomposed as in Saito and Mitsui (1993):

$$\mathrm{E}(X(T) - \hat{X}(T))^2 \leq \mathrm{E}(X(T) - X_{\mathrm{N}}(T))^2 + \mathrm{E}(X_{\mathrm{N}}(T) - \hat{X}(T))^2,$$

where $X(T)$ is the exact solution, $X_{\mathrm{N}}(T)$ is the realized exact solution (which depends on the discrete times $t_0, \ldots, t_{\mathrm{N}} = T$), $\hat{X}(T)$ is the approximate solution, and E() denotes mathematical expectation. The quantity $DE = \mathrm{E}(X_{\mathrm{N}}(T) - \hat{X}(T))^2$, which is called the deterministic part of the error, is used to characterize the global error. The realized exact solution is computed from its explicit expression by substituting the stochastic integrals by the same types of discrete approximations used in the derivation of the numerical solution, i.e. the approximations (2.5), (2.7), (2.8) and (3.6). The mean value of 1000 simulations of $(X_{\mathrm{N}}(T) - \hat{X}(T))^2$ is used as an estimate of the deterministic error. These concepts are extended to the multidimensional case by its application to each component of the solution.

The order β of the global error, which is defined by

$$\mathrm{E}(X(T) - \hat{X}(T))^2 = O(\Delta^\beta) \quad (\Delta \to 0),$$

is estimated by the slope of the straight line fitted to the points $(\log_2(\Delta_i), \log_2(DE(\Delta_i))$, where $DE(\Delta_i)$ are the estimated deterministic errors corresponding to a set of selected step sizes Δ_i. This estimate of the global error order has been discussed extensively by Saito and Mitsui (1993).

The first example is an autonomous equation with multiplicative noise. This equation has been used by some authors to test a wide variety of numerical schemes (Gard (1988), Saito and Mitsui (1993)).

Example 1.

$$f(X,t) = X(t)(1 - X(t)),$$
$$g(X,t) = X(t),$$
$$t_0 = 0, \quad T = 9, \quad X(t_0) = 1/2.$$

The corresponding exact solution is

$$X(t) = \exp(t/2 + W(t))/\left(2 + \int_{t_0}^{t} \exp(u/2 + W(u))du\right).$$

Figure 1(a) shows the simulation of a trajectory of the exact solution and the corresponding approximate solution by the LL method. For reference, the solution obtained by the Explicit Euler scheme is also presented in Fig. 1(b) (a detailed study of the numerical performance of other schemes in this example can be consulted in Saito and Mitsui (1993)). Figures 1(a) and 2 demonstrate the satisfactory performance of the LL in this example, with a global error of order about 2.

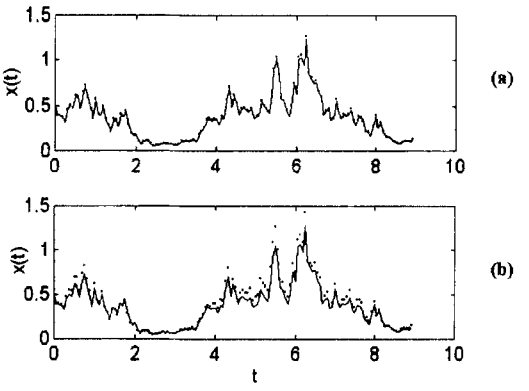

Fig. 1. Trajectories of the realized exact solution (continuous curve) of Example 1 and the approximated solution (points) obtained by the LL scheme (a), and the EE scheme (b). The step size is $\Delta = 2^{-4}$.

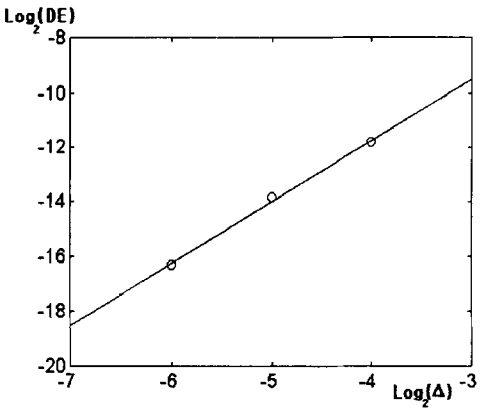

Fig. 2. Deterministic (global) errors of the LL solutions of Example 1 corresponding to the step sizes of time discretization $\Delta = 2^{-4}, 2^{-5}, 2^{-6}$. The slope of the fitted straight line is 2.3.

The performance of the LL method for scalar non-autonomous equations is illustrated by the next example.

Example 2.

$$f(X,t) = -t^2 X(t),$$
$$g(X,t) = (3/2)\exp(-(t^3 - t_0^3)/3)/(t+1),$$
$$t_0 = 0, \quad T = 9, \quad X(t_0) = 1.$$

The exact solution is

$$X(t) = \exp(-(t^3 - t_0^3)/3)\left(X(t_0) + (3/2)\int_{t_0}^{t} dW(u)/(u+1)\right).$$

The LL method leads to a good approximation of this solution with a global error order of about 2 (Figs. 3(a) and 4). The trajectory of the approximate solution converges to zero, which is the unique asymptotically stable point of this equation. In this sense, the LL solution preserves the qualitative properties of the exact solution. In contrast, the Explicit Euler (EE) scheme gives an explosive trajectory (Fig. 3(b)). Even the Explicit Taylor (ET) scheme of Kloeden and Platen (1989), which is the best existing scheme from the viewpoint of global error ($\beta = 3$) according to the comparative study of Saito and Mitsui (1993), also gives an explosive trajectory (Fig. 3(c)).

This illustrates that, in some examples, the LL method preserves the qualitative properties of non-autonomous equations while other explicit schemes fail to do this. This fact has been emphasized by Ozaki (1985*a*, 1985*b*, 1992) in the case of autonomous equations with additive noise.

As it is well known, implicit methods present better numerical stability than explicit ones (Kloeden and Platen (1992)). In particular, the Implicit Euler (IE)

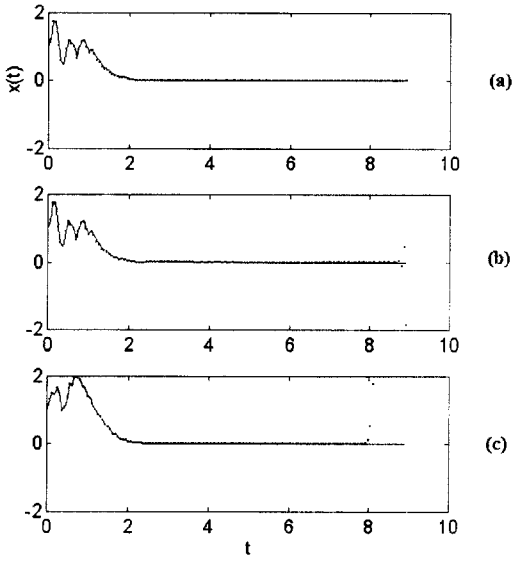

Fig. 3. Trajectories of the realized exact solution (continuous curve) of Example 2 and the approximated solution (points) obtained by the LL scheme (a), the EE scheme (b), and the ET scheme (c). The step size is $\Delta = 2^{-4}$. Notice that the trajectories obtained by the Euler and Taylor schemes rapidly diverge for $t > 9$.

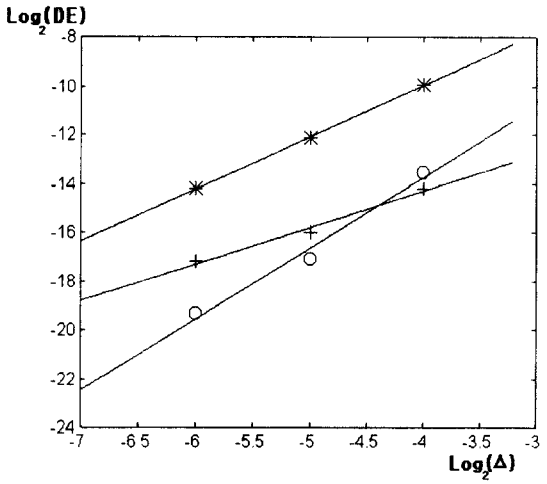

Fig. 4. Deterministic (global) errors of the LL (*), IE (+) and IRK (o) solutions of Example 2 corresponding to the step sizes of time discretization $\Delta = 2^{-4}, 2^{-5}, 2^{-6}$. The slope of the fitted straight lines are 2.1, 1.4 and 2.9 respectively.

scheme and the Implicit Order 1.5 Runge-Kutta (IRK) scheme ($\beta = 2 \times 1.5$) do not show exploding numerical solutions in this example (see Fig. 4). However, implicit methods involve a considerable additional computational effort because of they require the solution of a system of nonlinear equations at each step, which

implies more CPU time.

In contrast, the LL method shows not only numerical stability but also a low CPU time (see Table 1). The results presented in Table 1 demonstrate that the CPU time of the LL is about 10 times lower than the Implicit Euler scheme, and 24 times lower than the Implicit Runge-Kutta scheme. Thus, the LL method is a valuable alternative with respect to the trade-off between numerical stability and computational cost.

Table 1. Relative times of different numerical schemes (Explicit Euler (EE), Explicit Taylor (ET), Local Linearization (LL), Implicit Euler (IE), and Implicit Runge-Kutta (IRK)) when solving Example 2. The EE scheme showed the minimum CPU time (mCPU). Relative times are computed by dividing the actual CPU time of each scheme by the mCPU value.

Scheme	EE	ET	LL	IE	IRK
Relative Time	1	2.8	3.6	34.7	72.8

The last example is a multidimensional non-autonomous system:

Example 3.

$$
\begin{aligned}
&f_1(X,t) = C/\exp(X_1(t)/C),\\
&f_2(X,t) = C/(t^2\exp(X_1(t)/C)) + 4(X_2(t) - X_1(t)/t^2)\cos(t)/(2+\sin(t))\\
&\qquad\quad - 2X_1(t)/t^3,\\
&g_1(X,t) = \sqrt{2}C,\\
&g_2(X,t) = \sqrt{2}C(1/t^2 + D(2+\sin(t))^4/(t+1)),\\
&t_0 = 2, \quad T = 10, \quad X_1(t_0) = 1, \quad X_2(t_0) = 50,\\
&C = 30/\sqrt{2}, \quad D = 1/(2+\sin(t_0))^4, \quad W(t_0) = 0.
\end{aligned}
$$

The exact solution is

$$
\begin{aligned}
X_1(t) &= \sqrt{2}C\left(W(t) + (1/\sqrt{2})\ln\left|\exp(X_1(t_0)/C) + \int_{t_0}^{t}\exp(-\sqrt{2}W(u))du\right|\right),\\
X_2(t) &= X_1(t)/t^2\\
&\quad + D(2+\sin(t))^4\left(X_2(t_0) - X_1(t_0)/t_0^2 + \sqrt{2}C\int_{t_0}^{t} dW(u)/(u+1)\right).
\end{aligned}
$$

Figures 5(a), 5(b) and 6 show the results of the application of the LL scheme to this example. As in the previous examples, also an estimate of the order of the global error about 2 is obtained.

The programs that implement the methods described in this article are available in MATLAB from the authors.

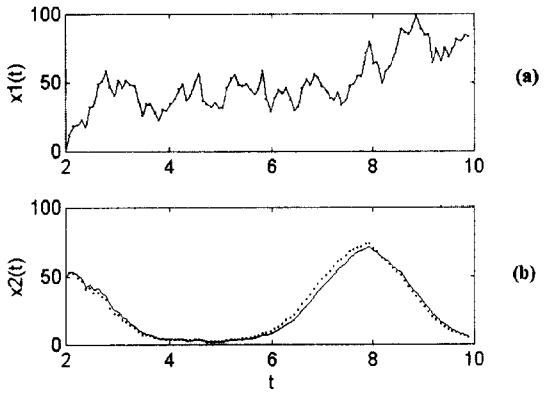

Fig. 5. a) Trajectories of the first component of the realized exact solution (continuous curve) and the LL solution (points) of Example 3. b) Trajectories of the second component of the realized exact solution (continuous curve) and the LL solution (points) of the same example. The step size is $\Delta = 2^{-4}$.

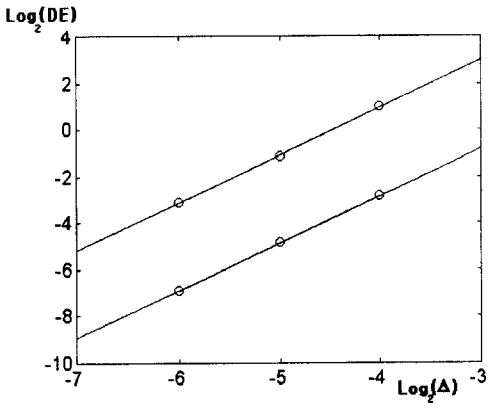

Fig. 6. Deterministic (global) errors of the two components (X_1 and X_2) of the LL solutions of Example 3 corresponding to the step sizes of time discretization $\Delta = 2^{-4}$, 2^{-5}, 2^{-6}. The slopes of the fitted straight lines are 2.0.

5. Conclusions

The LL approach has been extended to general scalar SDEs, as well as to non-autonomous multidimensional SDEs with additive noise. This extension is derived from (first order) deterministic Taylor expansions of the drift and diffusion coefficients of the SDEs. The resulting numerical scheme overcomes some ambiguities of the previous formulations of the LL approach.

The LL scheme is a one-step explicit method that gives exact solutions for linear autonomous SDEs.

Simulation results demonstrate that the method has a satisfactory performance in a wide variety of SDEs, with an estimate of the global error of order

about 2.

Interestingly, there are examples of asymptotically stable SDEs for which the method introduced gives solutions that converge to the equilibrium points while other classical explicit numerical schemes give explosive trajectories.

Both implicit methods and the LL method show numerical stability in the examples studied, but the LL approach has the advantage of requiring a considerably lower computational cost.

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Appendix

Let M be a square matrix, $a \in \mathbb{R}_+$ and

$$r_n(M,a) = \int_0^a \exp(Mu)u^n du.$$

If M is not singular, it can be shown by induction that

$$\text{(A.1)} \qquad r_n(M,a) = a^{n+1}n!\left\{(-Ma)^{-n-1}(I-\exp(Ma)) - \exp(Ma)\sum_{k=0}^{n-1}(-Ma)^{-k-1}/(n-k)!\right\}.$$

In general, if M is singular, then $r_n(M,a)$ can be defined as the $\lim r_n(M_k,a)$ as M_k tends to M within the class of nonsingular matrices.

This limit can be computed using the well known Schur decomposition approach for evaluating matrix functions (Golub and Van Loan (1989), pp. 542–545). Specifically, let $M = QTQ'$ be the Schur decomposition of M. Define $M_k = QT_kQ'$, where T_k is defined by $T_k = T + D_k$, and D_k are invertible diagonal matrices such that D_k tends to the zero matrix as k tends to infinity. For simplicity, denote $r_n(M,a)$ by $f(M)$. Then $f(M) = Q(\lim f(T_k))Q'$. By means of the recursive algorithm of Parlett (Golub and Van Loan (1989), pp. 543–545), the computation of $f(T_k)$ can be expressed in terms of $f([T_k]_{ii})$, where $[T_k]_{ii}$ are the diagonal entries of T_k. In turn, if $[T]_{ii} \neq 0$, then $\lim f([T_k]_{ii}) = f([T]_{ii})$, where $f([T]_{ii})$ is computed by (A1); if $[T]_{ii} = 0$, it can be shown that $\lim f([T_k]_{ii}) = a^{n+1}/(n+1)$.

References

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications*, Wiley-Interscience Publications, New York.

Gard, T. C. (1988). *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York.

Golub, G. H. and Van Loan, C. F. (1989). *Matrix Computations*, 2nd ed., The Johns Hopkins University Press, Maryland.

Henrici, P. (1962). *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York.

Kloeden, P. E. and Platen, E. (1989). A survey of numerical methods for stochastic differential equations, *Stochastic Hydrology & Hydraulics*, **3**, 155–178.

Kloeden, P. E. and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin.

Kloeden, P. E., Platen, E. and Scurz, H. (1993). *Numerical Solution of SDE Through Computer Experiments*, Springer, Berlin.

Maruyama, G. (1955). Continuous Markov processes and stochastic equations, *Rend. Circ. Mat. Palermo*, **4**, 48–90.

McShane, E. J. (1974). *Stochastic Calculus and Stochastic Models*, Academic Press, New York.

Milshtein, G. N. (1974). Approximate integration of stochastic differential equations, *Theory Probab. Appl.*, **19**, 557–562.

Newton, N. J. (1991). Asymptotically efficient Runge-Kutta methods for a class of Ito and Stratonovich equations, *SIAM J. Appl. Math.*, **52**, 542–567.

Ozaki, T. (1985*a*). Nonlinear time series models and dynamical systems (eds. E. J. Hannan, P. R. Krishnaiah and M. M. Rao), *Handbook of Statistics 5*, North Holland, Amsterdam.

Ozaki, T. (1985*b*). Statistical identification of storage models with application to stochastic hydrology, *Water Resource Bull.*, **21**, 663–675.

Ozaki, T. (1992). A bridge between nonlinear time series models and nonlinear stochastic dynamical systems: a local linearization approach, *Statistica Sinica*, **2**, 113–135.

Ozaki, T. (1994). Personal communication.

Saito, Y. and Mitsui, T. (1992). Discrete approximation for stochastic differential equations, *Transactions of the Japan Society for Industrial and Applied Mathematics*, **2**, 1–16 (in Japanese).

Saito, Y. and Mitsui, T. (1993). Simulation of stochastic differential equations, *Ann. Inst. Statist. Math.*, **45**, 419–432.

Schuss, Z. (1980). *Theory and Applications of Stochastic Differential Equations*, Wiley, New York.