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QUENOUILLE-TYPE THEOREM ON AUTOCORRELATIONS

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Abstract. The central result is a limit theorem for not necessarily stationary processes resembling AR(p). Assumption of a vector limit distribution for standardized sample autocorrelations leads to the convergence of a vector limit distribution for ordinary sample partial autocorrelations, and to a clear relationship between the two limit distributions. The motivation is the study of the case $p = 1$ by Mills and Seneta (1989, *Stochastic Process Appl.*, **33**, 151–161). The central result is used to explain the nature of the relationship between the two results of Quenouille in the classical stationary AR(p) setting.

Key words and phrases: Limit theorem, sample autocorrelations, sample partial autocorrelations, Quenouille's test.

1. Introduction

For any stochastic process $X(t)$, $t = 0, \pm 1, \dots$, we define $R(0) = 1$, and for $1 \le k \le N-1$, arbitrary fixed integer N,

$$R(k) = C(k)/C(0)$$
$$C(k) = \sum_{t=k+1}^{N} (X(t) - \bar{X})(X(t-k) - \bar{X})$$

where $\bar{X} = \sum_{t=1}^{N} X(t)/N$. Put $R(-k) = R(k)$. In the standard case where $\{X(t)\}$ is a second-order stationary process, the $R(t)$'s are the sample autocorrelations on the basis of an observed sample $X(1), X(2), \dots, X(N)$. Thus we can also define, parallelling the definition of sample partial autocorrelations for a second-order stationary process, the quantities $\hat{\beta}_k$, by

$$\hat{\beta}_1 = R(1)$$
$$\hat{\beta}_k = \frac{\det(U_k)}{\det(L_k)} \qquad k \ge 2 \tag{1.1}$$

where

$$U_k = \begin{pmatrix} R(0) & \cdots & R(k-2) & R(1) \\ \vdots & \cdots & \vdots & \vdots \\ R(k-1) & \cdots & R(1) & R(k) \end{pmatrix}$$

$$L_k = \begin{pmatrix} R(0) & \cdots & R(k-1) \\ \vdots & \ddots & \vdots \\ R(k-1) & \cdots & R(0) \end{pmatrix}.$$

We shall henceforth refer to $R(k)$, $\hat{\beta}_k$ as sample autocorrelations and sample partial autocorrelations, respectively, whether the process $\{X(t)\}$ is second-order stationary or not.

We need to recall, for convenient reference, some properties of a second-order stationary AR(p) process $\{Y(t)\}$, $t = 0, \pm 1, \ldots$, satisfying, for $p \geq 1$,

$$Y(t) = \phi_1 Y(t-1) + \phi_2 Y(t-2) + \cdots + \phi_p Y(t-p) + Z(t) \tag{1.2}$$

where $\{Z(t)\}$ are independent identically distributed (i.i.d.) random variables with zero mean and variance σ_z^2. Then the autocorrelation function $\rho(i)$, $i = 0, \pm 1, \ldots$, with $\rho(0) = 1$ and $\rho(-i) = \rho(i)$ satisfies the Yule-Walker equations

$$\sum_{j=0}^{p} \phi_j \rho(i-j) = 0 \qquad i \geq 1 \tag{1.3}$$

where $\phi_0 = -1$. Denoting the partial autocorrelations by π_i, $i \geq 1$, it is known (e.g. Barndorff-Nielsen and Schou (1973)), that $\pi_i = \phi_{i,i}$, where for fixed i, $\phi_{i,r}$, $r = 1, 2, \ldots, i$ are uniquely determined from the $\rho(k)$'s, by the linear system

$$\rho(k) = \sum_{r=1}^{i} \rho(k-r)\phi_{i,r} \qquad k = 1, 2, \ldots, i \tag{1.4}$$

from which, in particular, it follows that $\pi_k = 0$ for $k > p$.

Our intention is to study, for a process $\{X(t)\}$ with certain features in common with a stationary AR(p) process, the limiting joint distribution of the $\hat{\beta}_k$'s defined by (1.1), when a limiting joint distribution of the $R(k)$'s is known to exist. The motivation comes from Theorem 1 of Mills and Seneta (1989), who studied the case $p = 1$ with a view to application to a branching (Bienaymé-Galton-Watson) process with immigration. Our general result, which we state now contains (and makes formal the proof of) the well-known result behind Quenouille's (1947, 1949) classical test. This classical aspect will be discussed in Section 3.

THEOREM 1.1. *Let $\phi_0 = -1$, and $\phi_1, \phi_2, \ldots, \phi_p$ be real numbers such that the roots of $\Phi(z) = 0$ are all outside the unit circle, where $\Phi(z) = -\sum_{i=0}^{p} \phi_i z^i$. Let $\rho(i)$, $i = 0, \pm 1, \ldots$ be real numbers satisfying $\rho(0) = 1$, $\rho(i) = \rho(-i)$, and (uniquely) (1.3). Write for each $i \geq 1$, $\pi_i = \phi_{i,i}$, where $\phi_{i,r}$, $r = 1, 2, \ldots, i$ are given (uniquely) by (1.4), so $\pi_k = 0$, $k > p$. For a process $\{X(t)\}$, assume $\{N^{1/2}(R(k) - \rho(k))\}$, $k = 1, \ldots, p+q$ (for arbitrary integer $q \geq 1$), converges in distribution as $N \to \infty$ to the distribution of a random vector $\{V(k)\}$, $k = 1, 2, \ldots, p+q$. Define $V(0) = 0$, $V(-k) = V(k)$. Then the vector $\{N^{1/2}\hat{\beta}_k\}$,*

$k = p+1, \dots, p+q$ converges in distribution as $N \to \infty$ to the distribution of a random vector $\{W(k)\}$, $k = p+1, \dots, p+q$, where

$$W(k) = \sum_{i=0}^{p} \sum_{j=0}^{p} \phi_i \phi_j V(k-i-j) / \prod_{i=1}^{p} (1-\pi_i^2). \tag{1.5}$$

2. Proof of Theorem 1.1

Define a linear operator H such that for any sequence $\{d(t)\}$, $t = 0, \pm 1, \dots$,

$$Hd(t) = -\sum_{s=0}^{p} \phi_s d(t-s). \tag{2.1}$$

From (1.3), then

$$H\rho(i) = 0 \qquad i \geq 1. \tag{2.2}$$

Write $\epsilon(i) = R(i) - \rho(i)$, $0 \leq i \leq N-1$, define $\epsilon(-i) = \epsilon(i)$, and express all quantities in the numerator of (1.1) in terms of $\epsilon(i)$'s and $\rho(i)$'s.

Now partition the numerator as

$$\det \begin{pmatrix} A & E_1 \\ E_2 & D \end{pmatrix}$$

with A being $p \times p$, D being $(k-p) \times (k-p)$, keeping in mind that within the theorem we are considering $k > p$. Perform the following row and column operations (which do not affect A) on the matrix involved:

$$\begin{aligned}
&H \operatorname{row}(i) \to \operatorname{row}(i) \qquad i = k, k-1, \dots, p+1 \qquad (\text{since} \quad k \geq p+1) \\
&H \operatorname{col}(j) \to \operatorname{col}(j) \qquad j = k-1, k-2, \dots, p+1 \quad (\text{if} \quad k \geq p+2) \\
&\operatorname{col}(k) - \sum_{s=1}^{p} \phi_s \operatorname{col}(s) \to \operatorname{col}(k).
\end{aligned}$$

In virtue of (2.2), all $\rho(k)$'s in E_1, E_2, D, except in the diagonal elements, are eliminated, and all off-diagonal entries are linear functions of the $\epsilon(i)$'s. The submatrix A (unaffected) is the usual matrix of "early" sample autocorrelations, $R(s)$, $s = 0, 1, \dots, p-1$. The last entry on the diagonal of D is $H^2\epsilon(k)$, while the other $(k-p-1)$ diagonal entries are each

$$-\sum_{s=0}^{p} \phi_s \rho(s) - \sum_{s=0}^{p} \phi_s \epsilon(s) - \sum_{s=0}^{p} \phi_s H\epsilon(s).$$

Now consider, for a fixed k, the effect of letting $N \to \infty$. Since by assumption $\{N^{1/2}\epsilon(k)\} \xrightarrow{d} \{V(k)\}$, it follows $\epsilon(k) \xrightarrow{p} 0$, $k = 1, \ldots, p+q$, and so $R(k) \xrightarrow{p} \rho(k)$, $k = 0, 1, \ldots, p-1$. Hence by Slutsky's theorem, $E_1, E_2 \xrightarrow{p} 0$ and

$$\det(D) \xrightarrow{d} \left\{ -\sum_{s=0}^{p} \phi_s \rho(s) \right\}^{k-p-1} H^2 V(k)$$

and $\det(A)$ converges in probability to

$$\det\{\rho(i-j)\} \qquad |i-j| = 0, 1, \ldots, p-1.$$

But it is well-known (for example, Barndorff-Nielsen and Schou (1973), equation (5)) that this last is $\prod_{i=1}^{p-1}(1-\pi_i^2)^{p-i}$. Further, from equation (6) of the same reference,

$$-\sum_{s=0}^{p} \phi_s \rho(s) = \prod_{i=1}^{p}(1-\pi_i^2).$$

Thus

$$N^{1/2} \det(U_k) \xrightarrow{d} \prod_{i=1}^{p}(1-\pi_i^2)^{k-i-1} H^2 V(k). \tag{2.3}$$

In the fashion of $\det(A)$,

$$\det(L_k) \xrightarrow{p} \prod_{i=1}^{k-1}(1-\pi_i^2)^{k-i} = \prod_{i=1}^{p}(1-\pi_i^2)^{k-i} \tag{2.4}$$

since $k-1 \geq p$, and for $i > p$, $\pi_i = 0$. Hence again by Slutsky's theorem, for *fixed* $k \geq p+1$, as $N \to \infty$,

$$N^{1/2}\hat{\beta}_k = N^{1/2} \det(U_k)/\det(L_k) \xrightarrow{d} H^2 V(k) \Big/ \prod_{i=1}^{p}(1-\pi_i^2)$$

where $H^2 V(k) = \sum_{i=0}^{p} \sum_{j=0}^{p} \phi_i \phi_j V(k-i-j)$.

Let us now consider for arbitrary integer $q \geq 1$, *any* linear function

$$\begin{aligned} N^{1/2} \sum_{k=p+1}^{p+q} \alpha_k \hat{\beta}_k &= N^{1/2} \sum_{k=p+1}^{p+q} (\alpha_k / L_k) U_k \\ &\xrightarrow{d} \sum_{k=p+1}^{p+q} \alpha_k W_k \end{aligned}$$

by the above reasoning and Slutsky's theorem. Hence by the Cramér-Wold device, the vector convergence in distribution obtains.

3. Application to the classical setting

Let $\{X(t)\}$, $t = 0, \pm 1, \ldots$, be a linear process of the form

$$X(t) - \mu = \sum_{j=-\infty}^{\infty} \alpha_j Z(t-j) \tag{3.1}$$

where the $\{Z(t)\}$ are independently and identically distributed (i.i.d.), with zero mean and variance $\sigma_z^2 < \infty$, $\sum_{j=-\infty}^{\infty} |\alpha_j| < \infty$ and $\sum_{j=-\infty}^{\infty} |j|\alpha_j^2 < \infty$. A now-classical result of Anderson and Walker (1964), following on from Bartlett (1946), states that the joint distribution of $\{N^{1/2}\epsilon(k)\}$, $-G \leq k \leq G$ for arbitrary positive integer G converges to the multivariate normal distribution with zero mean and covariance matrix $\mathbf{\Sigma} = \{\sigma(i,j)\}$, where

$$\begin{aligned}\sigma(i,j) = \sum_{r=-\infty}^{\infty} \{&\rho(r+i)\rho(r+j) + \rho(r-i)\rho(r+j) - 2\rho(j)\rho(r)\rho(r+i) \\ &-2\rho(i)\rho(r)\rho(r+j) + 2\rho(i)\rho(j)\rho^2(r)\}.\end{aligned} \tag{3.2}$$

A modern presentation of this theory is given in Chapter 7 of Brockwell and Davis (1986). See also Cavazos-Cadena (1994).

Let us now suppose that our process $\{X(t)\}$ in (3.1) is in fact a stationary AR(p) process, so $\alpha_j = 0$, $j < 0$, the summability conditions on the α_j's are certainly satisfied; and the conditions of Theorem 1.1 are also satisfied since $\{X(t)\}$ is the same process as described by $\{Y(t)\}$ of (1.2).

Write

$$\begin{aligned}\boldsymbol{V} &= \{V(i)\} \quad i = p+q, \ldots, 1, 0, -1, \ldots, -p+1 \\ \boldsymbol{W} &= \{W(i)\} \quad i = p+1, \ldots, p+q.\end{aligned}$$

Then from (1.5), $\boldsymbol{W} = A\boldsymbol{V}$, where A is $q \times (2p+q)$ and has form (where summing from 1, for example, means for $i + j = 1$)

$$A = \frac{\gamma(0)}{\sigma_z^2}\begin{pmatrix} 0 & \cdots & 0 & 1 & \sum_1 \phi_i\phi_j & \cdots & \cdots & \sum_{2p} \phi_i\phi_j \\ 0 & \cdots & 1 & \sum_1 \phi_i\phi_j & \cdots & \cdots & \sum_{2p} \phi_i\phi_j & 0 \\ \vdots & & & \cdots & \cdots & & & \vdots \\ 0 & 1 & \sum_1 \phi_i\phi_j & & \cdots & & 0 & 0 \\ 1 & \sum_1 \phi_i\phi_j & \cdots & \cdots & \sum_{2p} \phi_i\phi_j & 0 & \cdots & 0 \end{pmatrix}$$

since by equation (8) of Barndorff-Nielsen and Schou (1973), $\prod_{i=1}^{p}(1 - \pi_i^2) = \sigma_z^2/\gamma(0)$, where σ_z^2 is var$(Z(t))$ and $\gamma(0) = $ var$(X(t))$ of the stationary AR(p) process $\{X(t)\}$. Then the covariance matrices of $\boldsymbol{W}$ and $\boldsymbol{V}$ are related by

$$\Sigma_{\boldsymbol{W}} = A\Sigma_{\boldsymbol{V}}A', \tag{3.3}$$

where $\Sigma_{\boldsymbol{V}}$ is $(2p+q)\times(2p+q)$ and its entries are given by (3.2), and $\Sigma_{\boldsymbol{W}}$ is $q\times q$. Define

$$h(k)=\sum_{r=0}^{p}\sum_{s=0}^{p}\phi_r\phi_s\rho(k-r-s)=\sum_{u=0}^{2p}\sum_{i+j=u}\phi_i\phi_j\rho(k-u). \tag{3.4}$$

We now compute the $(1,1)$ entry for $\Sigma_{\boldsymbol{W}}$ for purposes of illustration. Using (3.2), (3.3) and (3.4),

$$\begin{aligned}
\Sigma_{\boldsymbol{W}}(1,1) &= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2\sum_{u=0}^{2p}\sum_{m+n=u}\phi_m\phi_n\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\sigma(p+1-k,p+1-u)\\
&= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\left(\sum_{u=0}^{2p}\sum_{m+n=u}\phi_m\phi_n\right)\\
&\quad\times\sum_{r=-\infty}^{\infty}\{\rho(r+p+1-k)\rho(r+p+1-u)\\
&\qquad\qquad+\rho(p+1-k-r)\rho(r+p+1-u)\\
&\qquad\qquad-2\rho(p+1-u)\rho(r)\rho(r+p+1-k)\\
&\qquad\qquad-2\rho(p+1-k)\rho(r)\rho(r+p+1-u)\\
&\qquad\qquad+2\rho(p+1-k)\rho(p+1-u)\rho^2(r)\}\\
&= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2\sum_{r=-\infty}^{\infty}\left\{\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\rho(r+p+1-k)h(r+p-1)\right.\\
&\qquad\qquad+\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\rho(p+1-k-r)h(r+p+1)\\
&\qquad\qquad-2\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\rho(r)\rho(r+p+1-k)h(p+1)\\
&\qquad\qquad-2\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\rho(r)\rho(p+1-k)h(r+p+1)\\
&\qquad\qquad\left.+2\sum_{k=0}^{2p}\sum_{i+j=k}\phi_i\phi_j\rho^2(r)\rho(p+1-k)h(p+1)\right\}\\
&= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2\sum_{r=-\infty}^{\infty}\{h(r+p+1)h(r+p+1)\\
&\qquad\qquad+h(p+1-r)h(r+p+1)\\
&\qquad\qquad-2\rho(r)h(r+p+1)h(p+1)\\
&\qquad\qquad-2\rho(r)h(p+1)h(r+p+1)\\
&\qquad\qquad+2\rho^2(r)h(p+1)h(p+1)\}.
\end{aligned}$$

Now, in his Property 3, Choi (1990a) has established properties for the stationary AR(p) process using the Yule-Walker equations which imply that $h(p+1) = 0$ and that

$$\sum_{r=-\infty}^{\infty} h^2(r+p+1) = (\sigma_z^2/\gamma(0))^2$$

$$\sum_{r=-\infty}^{\infty} h(p+1-r)h(r+p+1) = 0.$$

Thus $\Sigma_{\boldsymbol{W}}(1,1) = 1$. Analogously for $s \geq 1$, $t \geq 1$,

$$\begin{aligned}\Sigma_{\boldsymbol{W}}(s,t) &= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2 \sum_{u=0}^{2p} \sum_{m+n=u} \phi_m\phi_n \sum_{k=0}^{2p} \sum_{i+j=k} \phi_i\phi_j\sigma(p+s-k,p+t-u) \\ &= \left(\frac{\gamma(0)}{\sigma_z^2}\right)^2 \sum_{r=-\infty}^{\infty} \{h(r+p+s)h(r+p+t) \\ &\qquad + h(p+s-r)h(r+p+t) \\ &\qquad - 2\rho(r)h(r+p+s)h(p+t) \\ &\qquad - 2\rho(r)h(p+s)h(r+p+t) \\ &\qquad + 2\rho^2(r)h(p+s)h(p+t)\} \\ &= \delta_{s,t}.\end{aligned}$$

Thus $\Sigma_{\boldsymbol{W}} = I$, hence $\{W(k)\}$, $k = p+1,\ldots,p+q$, are independent standard normal, that is, for a stationary AR(p) process $\{X(t)\}$ in the limit as $N \to \infty$, the random variables $\{N^{1/2}\hat{\beta}_k\}$, $k \geq p+1$, are i.i.d. $N(0,1)$ as is more or less well-known, albeit generally still with somewhat difficult and/or vague proofs, and is Quenouille's (1949) "second" result. Notice that this result, which implies that

$$N \sum_{k=p+1}^{p+T} \hat{\beta}_k^2 \sim \chi_T^2$$

for large N approximately, which is the usual basis for Quenouille's test, uses only the $\hat{\beta}_k$'s and hence does not require any preliminary estimation of the parameter ϕ_i's for its application, although (see for example Mills and Seneta (1989), Section 1), the least squares estimation of these parameters is implicit in the definition of $\hat{\beta}_k$'s. A direct rigorous but more difficult proof, involving asymptotic maximum likelihood ideas, of these results is given in Barndorff-Nielsen and Schou (1973).

Especially relevant is the relation of our general theory to Quenouille's (1947) "first" result. For a recent presentation and perception of these two results, see Hosking (1986). We note, in this connection, that for $k > p$ and by the Yule-Walker equations (1.3),

$$N^{1/2} \sum_i^p \sum_j^p \phi_i\phi_j R(k-i-j) = N^{1/2} \sum_i^p \sum_j^p \phi_i\phi_j (R(k-i-j) - \rho(k-i-j))$$

$$= N^{1/2}\sum_i^p\sum_j^p \phi_i\phi_j\epsilon(k-i-j)$$

$$\xrightarrow{d}\sum_i^p\sum_j^p \phi_i\phi_j V(k-i-j)$$

$$= (\sigma_z^2/\gamma(0))W(k)$$

from (1.5), since as noted, $\prod_{i=1}^p(1-\pi_i^2) = \sigma_z^2/\gamma(0)$. The $N(0,1)$ independence of the $W(k)$'s yields Quenouille's (1947) first result on the behaviour of an appropriate linear combination of sample autocorrelations.

4. Other settings

The essential features required for the applicability of our Theorem 1.1 to a stochastic process $\{X(t)\}$, $t = 0, \pm1, \ldots,$ are the Yule-Walker-type equations (1.3) for quantities $\{\rho(k)\}$ and the existence of the joint asymptotic distribution of $\{N^{1/2}\epsilon(k)\}$, where $\epsilon(k) = R(k) - \rho(k)$. The results of Mills and Seneta (1989) for a branching process setting (with $p = 1$) show that Theorem 1.1 has applicability beyond the classical AR(p) setting of Section 3. In fact, Theorem 1.1 was developed in order to apply it to a non-trivial variant of the Mills and Seneta (1989) branching process set-up, namely to inference for the sum process of two independent branching processes with immigration, a model considered by Suresh Chandra and Koteeswaran ((1986), p. 312). As in Mills and Seneta (1989), the process need not be stationary but is asymptotically stationary. This limit-mean adjusted process is an instance of the case $p = 2$ of our model, and ϕ_1, ϕ_2 of Theorem 1.1 are simple functions of the two offspring distribution means, so the assumption is easy to check. The special branching process structure then permits more specific consequences, in a manner resembling the classical AR(p) case, which has much simpler structure of residuals (1.2). On the whole, processes $\{X(t)\}$ satisfying the conditions of Theorem 1.1 might be expected to be at least asymptotically stationary.

On the other hand, we might consider extensions of Theorem 1.1 in directions suggested by the classical stationary ARMA(p,q) processes, where $q \geq 1$, with i.i.d. innovations. The "autocorrelations" $\rho(k)$ would then be assumed to satisfy a more complex set of equations than (1.3). The simplest extension in this direction is the case $p = q = 1$ where we assume $\rho(i)$, $i \geq 0$, are any real numbers such that $\rho(0) = 1$, $\rho(i) = \rho(-i)$, and $\{\rho(|i-j|)\}$, $i,j = 1,2,\ldots,k$, is a positive definite $(k \times k)$ matrix for any $k \geq 1$, and

$$\rho(k) - \phi_1\rho(k-1) = 0 \qquad k \geq 2$$

for some fixed real ϕ_1, $|\phi_1| < 1$. In this case, ARMA(1,1), one of us (Ku (1996)) has obtained a result paralleling Theorem 1.1, although the presence of an MA effect considerablely complicates matters. The extension does not parallel the linear combination of sample autocorrelations in Walker (1950) nor the treatments of

Choi (1990*b*) or Hosking (1980). Extensions of this kind (with $p \geq 1$) may be applied to bilinear processes $\{X(t)\}$ of the form BL$(p, 0, p, 1)$; that is satisfying

$$X(t) + \sum_{j=1}^{p} \phi_j X(t-j) = Z(t) + \sum_{i=1}^{p} b_i X(t-i) Z(t-1)$$

where $\{Z(t)\}$ are i.i.d. random variables with $EZ(t) = 0$, $EZ^3(t) = 0$, and $EZ^4(t) < \infty$, when a sufficient condition on the ϕ_j's and b_i's (Liu and Brockwell (1988), Theorem 2.1) for asymptotic stationarity is satisfied. This is because of the fact (see Priestley (1988), p. 63) that in the stationary regime of this process, the actual autocorrelations $\rho(k)$ satisfy

$$\sum_{j=0}^{p} \phi_j \rho(i-j) = 0 \qquad i \geq 2$$

$(\phi_0 = -1)$, analogously to the Yule-Walker equations for the autocorrelations of a stationary ARMA$(p, 1)$ process. Results of Liu ((1992), p. 491 ff.) indicate that the convergence of $\{N^{1/2}\epsilon(k)\}$ in this situation is likely.

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