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A CHARACTERIZATION OF CERTAIN DISCRETE EXPONENTIAL FAMILIES

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Abstract. For independent random variables X and Y, define $S \equiv X + Y$. When the conditional expectations $E[g(X) \mid S] \equiv a(S)$ and $E[h(X) \mid S] \equiv b(S)$ are given, then under certain assumptions, the density function of X has the form of $u(x)k(\alpha)e^{\alpha x}$, where $u(x)$ is uniquely determined by the functions $a(\cdot)$ and $b(\cdot)$.

Key words and phrases: Characterization, conditional expectation.

1. Introduction

Let X, Y be two independent random variables and $g(\cdot)$, $h(\cdot)$ be two functions. Suppose that

$$E[g(X) \mid S] = a(S) \quad \text{and} \quad E[h(X) \mid S] = b(S),$$

where $S = X + Y$, then for several special forms of $g(\cdot)$, $h(\cdot)$, $a(\cdot)$ and $b(\cdot)$, the distributions of X and Y have been proved to be members of the one-parameter exponential family. Studies on this topic can be found in Kagan (1993), Li *et al.* (1994), Wesolowski (1989) and the references given therein. Some related results are given in Wang *et al.* (1995).

In this paper, we give a general result for discrete distributions which is a complement to some results given in Kagan (1993). Our result can also be considered a generalization of Theorem 1 given in Patil and Seshadri (1964) for the discrete case.

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2. The main results

THEOREM 2.1. *Let X and Y be two independent discrete random variables whose sum, denoted by S, has positive probabilities on and only on the set $\boldsymbol{N} \equiv \{0,1,2,\ldots,N\}$, where $1 < N \leq +\infty$. We further assume that X and Y have positive probabilities on the set $\{0,1\}$. Suppose that*

$$E[g(X) \mid S] = a(S) \quad \text{and} \quad E[h(X) \mid S] = b(S), \tag{2.1}$$

where the functions $g(\cdot)$, $h(\cdot)$, $a(\cdot)$ and $b(\cdot)$ satisfy

$$[g(0) - a(s)][h(s) - b(s)] - [h(0) - b(s)][g(s) - a(s)] \neq 0, \tag{2.2}$$

for $1 < s \leq N$, and

$$g(0) - a(1) \neq 0. \tag{2.3}$$

Then, for every $x \in \boldsymbol{N}$, $\Pr(X = x) = u(x)k(\alpha)e^{\alpha x}$, where $\alpha = \ln[\Pr(X = 1)/\Pr(X = 0)]$, and $u(x)$ is determined by consistent values of functions $g(\cdot)$, $h(\cdot)$, $a(\cdot)$ and $b(\cdot)$.

PROOF. Let $P(x) \equiv \Pr(X = x)$ and $Q(y) \equiv \Pr(Y = y)$. From the assumptions in the theorem, we can easily conclude that X and Y have positive probabilities on and only on two subsets of $\boldsymbol{N}$. From the definition of conditional expectation, for $s \leq N$, we have the following linear system

$$\begin{cases} \displaystyle\sum_{x=0}^{s} [g(x) - a(s)]P(x)Q(s-x) = 0 \\ \displaystyle\sum_{x=0}^{s} [h(x) - b(s)]P(x)Q(s-x) = 0. \end{cases} \tag{2.4}$$

Choose $s = 1$, and let $t \equiv P(1)/P(0)$. We get $P(1) = c_1 t P(0)$, and $Q(1) = d_1 t Q(0)$, where

$$c_1 = 1, \quad \text{and} \quad d_1 = -\frac{g(1) - a(1)}{g(0) - a(1)}.$$

Now we complete the proof by induction. Suppose that for $i \leq n - 1 < N$, we have $P(i) = c_i t^i P(0)$ and $Q(i) = d_i t^i Q(0)$, where c_i and d_i are uniquely determined. Next, by choosing $s = n$, we can rewrite (2.4) as

$$\begin{cases} [g(0) - a(n)]P(0)Q(n) + [g(n) - a(n)]P(n)Q(0) \\ \qquad + \displaystyle\sum_{i=1}^{n-1} [g(i) - a(n)]P(0)Q(0)d_{n-i}c_i t^n = 0 \\ [h(0) - b(n)]P(0)Q(n) + [h(n) - b(n)]P(n)Q(0) \\ \qquad + \displaystyle\sum_{i=1}^{n-1} [h(i) - b(n)]P(0)Q(0)d_{n-i}c_i t^n = 0. \end{cases} \tag{2.5}$$

Divide both equations by $P(0)Q(0)$. Then, by assumption (2.2), this system has a unique solution for $P(n)/P(0)$ and $Q(n)/Q(0)$. Therefore,

$$P(n) = c_n t^n P(0) \quad \text{and} \quad Q(n) = d_n t^n Q(0), \tag{2.6}$$

where c_n and d_n are uniquely determined. By induction, (2.6) holds for every integer n satisfying $1 \le n \le N$. Let $\alpha \equiv \ln t$. Since $\sum_{x=0}^{N} P(x) = P(0)(1 + \sum_{x=1}^{N} c_x e^{\alpha x}) = 1$, then $P(0) = 1/(1 + \sum_{x=1}^{N} c_x e^{\alpha x})$. Finally, by defining $k(\alpha) \equiv 1/(1 + \sum_{x=1}^{N} c_x e^{\alpha x})$, we get the desired result.

Remark 2.1. The proof provides a method to compute $P(x)$ and $Q(y)$ when the functions $g(\cdot)$, $h(\cdot)$, $a(\cdot)$ and $b(\cdot)$ are given. Note that c_n and d_n are uniquely determined by the functions $g(\cdot)$, $h(\cdot)$, $a(\cdot)$ and $b(\cdot)$. Hence if $N < +\infty$, then for some $n \le N$, we may have $c_n = 0$ and $d_n = 0$.

Multivariate extension of this result is given in a technical report available from the authors.

As an application of the above result, we give the following corollary.

COROLLARY 2.1. *Let X, Y, S and $\boldsymbol{N}$ be defined as in Theorem* 2.1. *Suppose that*

$$E(X \mid S) = a(S) \quad \textit{and} \quad E(X^2 \mid S) = b(S). \tag{2.7}$$

Then, for every $x \in \boldsymbol{N}$, $\Pr(X = x) = u(x)k(\alpha)e^{\alpha x}$, where $\alpha = \ln[\Pr(X = 1)/\Pr(X = 0)]$, and $u(x)$ is uniquely determined by consistent values of functions $a(\cdot)$ and $b(\cdot)$.

PROOF. This is a special case of Theorem 2.1 with $g(x) = x$ and $h(x) = x^2$. Therefore, we only need to verify that conditions (2.2) and (2.3) are satisfied. In fact,

$$\begin{aligned} g(0) - a(1) &= -a(1) = -E(X \mid S = 1) \\ &= -\Pr(X = 1)\Pr(Y = 0)/\Pr(S = 1) < 0, \end{aligned}$$

and hence (2.3) is satisfied. Next,

$$[g(0) - a(s)][h(s) - b(s)] - [h(0) - b(s)][g(s) - a(s)] = s^2 a(s) - s b(s).$$

When $s > 1$, it can be easily verified that $s^2 E(X \mid S = s) - sE(X^2 \mid S = s) > 0$.

Remark 2.2. It can be easily verified that condition (2.7) can be replaced by many other conditions. For example, it can be replaced by

$$E(X \mid S) = a(S) \quad \text{and} \quad E(X^3 \mid S) = b(S). \tag{2.8}$$

Corollary 2.1 can be used to characterize any X and Y whose probability functions are positive on the set $\boldsymbol{N}$. Particular cases are binomial, negative binomial and Poisson distributions.

Example 2.1. Let X, Y, S and $\boldsymbol{N}$ be defined as in Theorem 2.1 with $N = 2k > 1$. Then

$$E(X \mid S) = \frac{S}{2} \quad \text{and} \quad \mathrm{Var}(X \mid S) = \frac{S(2k - S)}{4(2k - 1)}$$

hold if and only if

$$\Pr(X = i) = \Pr(Y = i) = \binom{k}{i} p^i (1 - p)^{k-i},$$

where $0 \le i \le k$, $0 < p < 1$.

Example 2.2. Let X, Y, S and $\boldsymbol{N}$ be defined as in Theorem 2.1 with $N = +\infty$. Then formulas

$$E(X \mid S) = a + bS \quad \text{and} \quad \mathrm{Var}(X \mid S) = c + dS \tag{2.9}$$

are true if and only if X and Y have Poisson distributions. In (2.9), a, b, c, d are some constants satisfying $0 < b < 1$, $d \neq 0$.

This example is also studied in Kagan (1993).

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References

Kagan, A. M. (1993). Linearity of posterior mean for location parameter families when sample size equals one, *Mathematical Methods of Statistics*, **1**(1), 28–38.

Li, S., Huang, W. and Huang, M. L. (1994). Characterizations of the Poisson process as a renewal process via two conditional moments, *Ann. Inst. Statist. Math.*, **46**(2), 351–360.

Patil, G. P. and Seshadri, V. (1964). Characterization theorems for some univariate probability distributions, *J. Roy. Statist. Soc. Ser. B*, **26**, 286–292.

Wang, Y., Gupta, A. K. and Nguyen, T. T. (1995). Characterization theorems for some discrete distributions based on conditional structure, *Canad. J. Statist.* (to appear).

Wesolowski, J. (1989). A characterization of the gamma process by conditional moments, *Metrika*, **36**, 299–309.