Ann. Inst. Statist. Math.
Vol. 48, No. 3, 519–534 (1996)

RELATIONSHIPS FOR MOMENTS OF ORDER STATISTICS FROM THE RIGHT-TRUNCATED GENERALIZED HALF LOGISTIC DISTRIBUTION

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(Received February 13, 1995; revised June 2, 1995)

Abstract. In this paper, we establish several recurrence relations satisfied by the single and the product moments for order statistics from the right-truncated generalized half logistic distribution. These relationships may be used in a simple recursive manner in order to compute the single and the product moments of all order statistics for all sample sizes and for any choice of the truncation parameter P. These generalize the corresponding results for the generalized half logistic distribution derived recently by Balakrishnan and Sandhu (1995, *J. Statist. Comput. Simulation*, **52**, 385–398).

Key words and phrases: Order statistics, recurrence relations, single moments, product moments, generalized half logistic distribution, right-truncated distribution, IFR family.

1. Introduction

Balakrishnan and Sandhu (1995) considered the generalized half logistic distribution with cdf

$$(1.1) \qquad G(x) = \frac{1 - (1 - kx)^{1/k}}{1 + (1 - kx)^{1/k}}, \qquad 0 \le x \le \frac{1}{k}, \qquad k \ge 0$$

and pdf

$$(1.2) \qquad g(x) = \frac{2(1 - kx)^{(1/k)-1}}{\{1 + (1 - kx)^{1/k}\}^2}, \qquad 0 \le x \le \frac{1}{k}, \quad k \ge 0,$$

discussed its properties and showed that it is an IFR (Increasing Failure Rate) family. Hence, they pointed out that the above given generalized half logistic

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distribution will be quite useful as a life-span model; see Cohen and Whitten (1988) for an excellent treatment of many such life-span models.

If the shape parameter $k \to 0$, the cdf in (1.1) becomes $(1-e^{-x})/(1+e^{-x})$ and the pdf in (1.2) becomes $2e^{-x}/(1+e^{-x})^2$, $x \geq 0$. This is the standard half logistic distribution introduced and discussed in detail by Balakrishnan (1985); see also Balakrishnan and Cohen (1991). It should be mentioned here that Hosking (1986, 1990) proposed, discussed and applied many such generalizations of standard distributions.

In this paper, we consider the right-truncated generalized half logistic distribution with pdf

$$f(x) = \begin{cases} \dfrac{1}{P}\dfrac{2(1-kx)^{(1/k)-1}}{\{1+(1-kx)^{1/k}\}^2}, & 0 \leq x \leq P_1 \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

and cdf

$$F(x) = \frac{1}{P}\left\{\frac{1-(1-kx)^{1/k}}{1+(1-kx)^{1/k}}\right\}, \quad 0 \leq x \leq P_1; \tag{1.4}$$

here, $1-P$ $(0 < P \leq 1)$ is the proportion of truncation on the right of the standard generalized half logistic distribution in (1.1), and

$$P_1 = \frac{1}{k}\left\{1-\left(\frac{1-P}{1+P}\right)^k\right\}, \quad k \geq 0 \tag{1.5}$$

is the point of truncation on the right. The right-truncated form of life-span models are often of great interest in reliability studies; for example, see Cohen (1991).

It is easy to note that $P = G(P_1)$ $(k \geq 0,\ 0 < P_1 \leq 1/k)$ is monotonic increasing in P_1 and hence, for fixed values of k and P, P_1 is uniquely determined. Similarly, we also note that $P_1 = G^{-1}(P)$ $(k \geq 0,\ 0 < P \leq 1)$ is monotonic increasing in P so that, for fixed values of k and P_1, P is uniquely determined. Next, we consider P_1 as a function of k. If we can show that, for fixed P, P_1 as a function of k is monotonic, it will immediately imply that k is uniquely determined for fixed values of P and P_1. Thus, by denoting $\frac{1-P}{1+P}$ by a, we would like to show that

$$\frac{dP_1}{dk} = \frac{1-a^k+ka^k\log a}{-k^2}$$

is either positive or negative for all $k \geq 0$. Since $0 < a < 1$ for $0 < P < 1$ and $k > 0$, we may use the expansion

$$\begin{aligned} -\log a^k &= -\log\{1-(1-a^k)\} \\ &= (1-a^k) + \frac{1}{2}(1-a^k)^2 + \frac{1}{3}(1-a^k)^3 + \cdots \\ &< (1-a^k) + (1-a^k)^2 + (1-a^k)^3 + \cdots \\ &= \frac{1-a^k}{a^k} \end{aligned}$$

which simply implies that

$$1 - a^k + ka^k \log a > 0$$

and, hence, $\frac{dP_1}{dk} < 0$. Also, $\lim_{k\to 0} \frac{dP_1}{dk} = \frac{-(\log a)^2}{2} < 0$. Therefore, for $0 < P < 1$, we have established that P_1 is a monotonic decreasing function of k. Further, for the special case when $P = 1$, we have $P_1 = 1/k$ which is again a monotonic decreasing function of k. Consequently, there is a unique choice of k for fixed values of P and P_1.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the right-truncated generalized half logistic distribution in (1.4), and $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Let us denote the single moments $E(X_{r:n}^i)$ by $\mu_{r:n}^{(i)}$ for $1 \le r \le n$ and $i = 0, 1, 2, \dots$, and the product moments $E(X_{r:n}X_{s:n})$ by $\mu_{r,s:n}$ for $1 \le r < s \le n$. For convenience, let us also use $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$ and $\mu_{r,r:n}$ for $\mu_{r:n}^{(2)}$.

From (1.3) and (1.4), we observe that the characterizing differential equation for the right-truncated generalized half logistic distribution is

$$(1 - kx)f(x) = \frac{1}{2P}\{1 - P^2F^2(x)\} \tag{1.6}$$

$$= \frac{1}{2P}[1 - P^2 + 2P^2\{1 - F(x)\} - P^2\{1 - F(x)\}^2] \tag{1.7}$$

$$= \frac{1}{2P}[1 - P + P\{1 - F(x)\} + P(1 - P)F(x) + P^2F(x)\{1 - F(x)\}] \tag{1.8}$$

for $0 \le x \le P_1$. We shall use eqs. (1.6)–(1.8) in the following sections to establish several recurrence relations satisfied by the single and the product moments of order statistics. These relations will enable one to compute all the single and the product moments of all order statistics for all sample sizes in a simple recursive manner. For example, by starting with the values of $E(X) = \mu_{1:1}$ and $E(X^2) = \mu_{1:1}^{(2)}$, one can determine the means, variances and covariances of all order statistics for all sample sizes through this recursive computational procedure.

Since the values of $E(X)$ and $E(X^2)$ are needed as initial values for the recursive process, we next derive exact explicit expressions for these two quantities. Consider

$$\begin{aligned}
E(X) &= \int_0^1 F^{-1}(u)du = \frac{1}{k}\int_0^1 \left\{1 - \left(\frac{1-Pu}{1+Pu}\right)^k\right\} du \\
&= \frac{1}{k}\left\{1 - \frac{2}{P}\int_{(1-P)/(1+P)}^1 \frac{y^k}{(1+y)^2}dy\right\} \qquad \left(\text{setting } y = \frac{1-Pu}{1+Pu}\right) \\
&= \frac{1}{k} - \frac{2}{kP}\left[\int_0^1 \frac{y^k}{(1+y)^2}dy - \int_0^{(1-P)/(1+P)} \frac{y^k}{(1+y)^2}dy\right] \\
&= \frac{1}{k} - \frac{2}{kP}\left\{\frac{1}{4(1+k)}{}_2F_1\left[2, 1; 2+k; \frac{1}{2}\right]\right.
\end{aligned} \tag{1.9}$$

Table 1. Mean of right truncated generalized half logistic distribution for selected values of k and P.

	k					
P	0.05	0.10	0.20	0.30	0.40	0.50
.50	.51406922	.50513286	.48795723	.47166276	.45619501	.44150328
.55	.57037461	.55932774	.53819504	.51826939	.49946767	.48171312
.60	.62854366	.61506220	.58939987	.56536005	.54281886	.52166297
.65	.68899982	.67270410	.64184933	.61314529	.58641064	.56148109
.70	.75229522	.73272875	.69589451	.66188430	.63043563	.60131332
.75	.81918113	.79577715	.75199925	.71191078	.67513376	.64133358
.80	.89073828	.86276277	.81081098	.76367907	.72082146	.68176195
.85	.96864634	.93509069	.87330364	.81785536	.76794854	.72289887
.90	1.05584620	1.01519025	.94111768	.87553030	.81722758	.76520131
.95	1.15874725	1.10826237	1.01764090	.93887393	.87002068	.80950358
1.00	1.30833018	1.23811426	1.11686427	1.01602278	.93098366	.85840735

$$- \frac{1}{(1+k)} \left(\frac{1-P}{1+P} \right)^{1+k} {}_2F_1 \left[2, 1+k; 2+k; \frac{1-P}{1+P} \right] \Bigg\},$$

where

$$(1.10) \qquad {}_2F_1[a,b;c;x] = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i x^i}{(c)_i i!}, \qquad c \neq 0, -1, -2, \ldots$$

is the Gaussian hypergeometric function. Proceeding similarly, we obtain

$$\begin{aligned}(1.11)\ E(X^2) = {} & \frac{1}{k^2} - \frac{4}{k^2 P} \\ & \cdot \left\{ \frac{1}{4(1+k)} {}_2F_1 \left[2, 1; 2+k; \frac{1}{2} \right] \right. \\ & \left. - \frac{1}{(1+k)} \left(\frac{1-P}{1+P} \right)^{1+k} {}_2F_1 \left[2, 1+k; 2+k; -\frac{1-P}{1+P} \right] \right\} \\ & + \frac{2}{k^2 P} \\ & \cdot \left\{ \frac{1}{4(1+2k)} {}_2F_1 \left[2, 1; 2+2k; \frac{1}{2} \right] \right. \\ & \left. - \frac{1}{(1+2k)} \left(\frac{1-P}{1+P} \right)^{1+2k} {}_2F_1 \left[2, 1+2k; 2+2k; -\frac{1-P}{1+P} \right] \right\}.\end{aligned}$$

Higher order moments may also be similarly derived.

The Gaussian hypergeometric function in (1.10) may be calculated to any desired accuracy by using, for example, the *hypergeom* command in MAPLE V,

Table 2. Variance of right truncated generalized half logistic distribution for selected values of k and P.

	k					
P	0.05	0.10	0.20	0.30	0.40	0.50
.50	.09318739	.08835339	.07951988	.07168372	.06472247	.05852971
.55	.11654737	.10980655	.09619784	.08696054	.07762065	.06942212
.60	.14419122	.13494501	.11841794	.10417733	.09187922	.08123468
.65	.17710827	.16456461	.14241634	.12363652	.10766796	.09405152
.70	.21671050	.19979344	.17031961	.14576191	.12522814	.10799843
.75	.26510137	.24229675	.20315206	.17116372	.14490741	.12326064
.80	.32560576	.29467625	.24248077	.20076013	.16722389	.14011511
.85	.40395028	.36134645	.29089176	.23603012	.19299650	.15899518
.90	.51142677	.45084026	.35319853	.27963374	.22365499	.18064008
.95	.67669998	.58418383	.44060923	.33744876	.26221222	.20654458
1.00	1.09690889	.89207280	.61142036	.43573174	.32047569	.24194366

Release 3. In order to facilitate the easy usage of the recursive algorithm developed in this paper, we have computed the values of the mean and variance of the right truncated generalized half logistic distribution from eqs. (1.9) and (1.11) and have presented them in Tables 1 and 2, respectively, for some selected choices of k and P. These values were calculated to 20 digit accuracy and are correct to all 8 decimal places reported in the tables.

The results established in this paper generalize the corresponding results for the generalized half logistic distribution proved recently by Balakrishnan and Sandhu (1995). Similar recurrence relations for moments of order statistics from exponential and truncated exponential distributions were derived by Joshi (1978, 1979, 1982). Results of this nature are also available for a number of other distributions, and interested readers may refer to the monograph on this topic by Arnold and Balakrishnan (1989).

2. Relationships for single moments

The density function of $X_{r:n}$ is given by (David (1981), p. 9, Arnold *et al.* (1992), p. 10)

$$(2.1)\quad f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x), \qquad 0 \le x \le P_1,$$

where $f(x)$, $F(x)$ and P_1 are as given in eqs. (1.3), (1.4) and (1.5), respectively.

Then, by making use of the characterizing differential equations in (1.6)–(1.8), we establish in this section several recurrence relations for the single moments of order statistics.

THEOREM 2.1. *For* $i = 0, 1, 2, \dots,$

$$\mu_{1:2}^{(i+1)} = \frac{1}{P^2}[(1-P^2)P_1^{i+1} + 2P^2\mu_{1:1}^{(i+1)} - 2P(i+1)\{\mu_{1:1}^{(i)} - k\mu_{1:1}^{(i+1)}\}], \tag{2.2}$$

and for $n \geq 2$

$$\mu_{1:n+1}^{(i+1)} = \frac{1}{P^2}\left[(1-P^2)\mu_{1:n-1}^{(i+1)} + 2P^2\mu_{1:n}^{(i+1)} - \frac{2P(i+1)}{n}\{\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)}\}\right]. \tag{2.3}$$

PROOF. For $n \geq 1$ and $i = 0, 1, 2, \dots,$ let us consider

$$\begin{aligned} \mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} &= n\int_0^{P_1} x^i(1-kx)\{1-F(x)\}^{n-1}f(x)dx \\ &= \frac{n}{2P}\int_0^{P_1} x^i[1-P^2+2P^2\{1-F(x)\} - P^2\{1-F(x)\}^2] \\ &\quad \cdot\{1-F(x)\}^{n-1}dx \\ &= \frac{n}{2P}[(1-P^2)I_{1,n-1} + 2P^2I_{1,n} - P^2I_{1,n+1}], \end{aligned} \tag{2.4}$$

upon using (1.7). Integration by parts directly gives

$$I_{1,n-1} = \begin{cases} P_1^{i+1}/(i+1), & \text{when } n = 1 \\ \mu_{1:n-1}^{(i+1)}/(i+1), & \text{when } n \geq 2, \end{cases}$$

$$I_{1,n} = \mu_{1:n}^{(i+1)}/(i+1)$$

and

$$I_{1,n+1} = \mu_{1:n+1}^{(i+1)}/(i+1).$$

Substituting these expressions in (2.4) and simplifying the resulting equations, we derive the recurrence relations in (2.2) and (2.3). □

THEOREM 2.2. *For* $i = 0, 1, 2, \dots,$

$$\mu_{2:2}^{(i+1)} = \frac{1}{P^2}[-(1-P^2)P_1^{i+1} + 2P(i+1)\{\mu_{1:1}^{(i)} - k\mu_{1:1}^{(i+1)}\}], \tag{2.5}$$

and for $n \geq 2$

$$\mu_{n+1:n+1}^{(i+1)} = \frac{1}{P^2}\left[\mu_{n-1:n-1}^{(i+1)} - (1-P^2)P_1^{i+1} + \frac{2P(i+1)}{n}\{\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)}\}\right]. \tag{2.6}$$

PROOF. For $n \geq 1$ and $i = 0, 1, 2, \dots,$ let us consider

$$\begin{aligned} \mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)} &= n\int_0^{P_1} x^i(1-kx)\{F(x)\}^{n-1}f(x)dx \\ &= \frac{n}{2P}\int_0^{P_1} x^i\{F(x)\}^{n-1}\{1-P^2F^2(x)\}dx \\ &= \frac{n}{2P}[I_{2,n-1} - P^2I_{2,n+1}], \end{aligned} \tag{2.7}$$

upon using (1.6). Integration by parts now gives

$$I_{2,n-1} = \begin{cases} P_1^{i+1}/(i+1), & \text{when } n = 1 \\ \{P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)}\}/(i+1), & \text{when } n \geq 2 \end{cases}$$

and

$$I_{2,n+1} = \{P_1^{i+1} - \mu_{n+1:n+1}^{(i+1)}\}/(i+1).$$

Upon substituting these expressions in (2.7) and simplifying the resulting equa tions, we derive the recurrence relations in (2.5) and (2.6). □

THEOREM 2.3. *For $2 \leq r \leq n-1$ and $i = 0, 1, 2, \ldots,$*

$$\begin{aligned} \mu_{r:n+1}^{(i+1)} = \mu_{r-1:n+1}^{(i+1)} &+ \frac{n+1}{(n-r+1)(n-r+2)P^2} \\ &\cdot [n(1-P^2)\{\mu_{r:n-1}^{(i+1)} - \mu_{r-1:n-1}^{(i+1)}\} \\ &+ 2(n-r+1)P^2\{\mu_{r:n}^{(i+1)} - \mu_{r-1:n}^{(i+1)}\} \\ &- 2(i+1)P\{\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)}\}], \end{aligned} \tag{2.8}$$

and for $n \geq 2$

$$\begin{aligned} \mu_{n:n+1}^{(i+1)} = \mu_{n-1:n+1}^{(i+1)} + \frac{n+1}{2P^2}[n(1-P^2)\{P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)}\} \\ + 2P^2\{\mu_{n:n}^{(i+1)} - \mu_{n-1:n}^{(i+1)}\} \\ - 2(i+1)P\{\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)}\}]. \end{aligned} \tag{2.9}$$

PROOF. For $2 \leq r \leq n$ and $i = 0, 1, 2, \ldots,$ let us consider

$$\begin{aligned} \mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} &= \frac{n!}{(r-1)!(n-r)!} \\ &\cdot \int_0^{P_1} x^i(1-kx)\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x)dx \\ &= \frac{n!}{(r-1)!(n-r)!2P} \\ &\cdot \int_0^{P_1} x^i\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} \\ &\cdot [1 - P^2 + 2P^2\{1-F(x)\} - P^2\{1-F(x)\}^2]dx \\ &= \frac{n!}{(r-1)!(n-r)!2P} \\ &\cdot [(1-P^2)I_{3,n-r} + 2P^2 I_{3,n-r+1} - P^2 I_{3,n-r+2}], \end{aligned} \tag{2.10}$$

upon using (1.7). Integration by parts directly yields

$$I_{3,n-r} = \begin{cases} \dfrac{(r-1)!(n-r)!}{(n-1)!(i+1)}\{\mu_{r:n-1}^{(i+1)} - \mu_{r-1:n-1}^{(i+1)}\}, & \text{when } 2 \le r \le n-1 \\ \dfrac{1}{(i+1)}\{P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)}\}, & \text{when } r = n, \end{cases}$$

$$I_{3,n-r+1} = \frac{(r-1)!(n-r+1)!}{n!(i+1)}\{\mu_{r:n}^{(i+1)} - \mu_{r-1:n}^{(i+1)}\}$$

and

$$I_{3,n-r+2} = \frac{(r-1)!(n-r+2)!}{(n+1)!(i+1)}\{\mu_{r:n+1}^{(i+1)} - \mu_{r-1:n+1}^{(i+1)}\}.$$

Upon substituting these expressions in (2.10) and simplifying the resulting equations, we derive the recurrence relations in (2.8) and (2.9). □

Remark 2.1. By letting the proportion of truncation $1-P \to 0$ (and, hence, $P_1 \to 1/k$) in Theorems 2.1–2.3, we deduce the recurrence relations established by Balakrishnan and Sandhu (1995) for the single moments of order statistics from the standard generalized half logistic distribution.

Remark 2.2. The relationships proved in Theorems 2.1–2.3 will enable one to compute all the single moments of all order statistics for all sample sizes in a simple recursive manner. By starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$ (see Tables 1 and 2), for example, one can use these results to determine the first two single moments (or the means and variances) of all order statistics for all sample sizes n. This can be done for any choice of the truncation parameter P and the shape parameter k.

3. Relationships for product moments

The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by (David (1981), p. 10, Arnold *et al.* (1992), p. 16)

$$(3.1)\quad f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1} \cdot \{1-F(y)\}^{n-s} f(x) f(y), \qquad 0 \le x < y \le P_1,$$

where $f(x)$, $F(x)$ and P_1 are as given in eqs. (1.3), (1.4) and (1.5), respectively.

Then, by making use of the characterizing differential equations in (1.6)–(1.8), we establish in this section several recurrence relations for the product moments of order statistics.

THEOREM 3.1. *For* $1 \le r \le n-2$,

$$(3.2)\quad \mu_{r,r+1:n+1} = \mu_{r:n+1}^{(2)} + \frac{2(n+1)}{n-r+1}\Bigg[\mu_{r,r+1:n} - \mu_{r:n}^{(2)} + \frac{1}{P(n-r)}\{k\mu_{r,r+1:n} - \mu_{r:n}\} + \frac{n(1-P^2)}{2(n-r)P^2}\{\mu_{r,r+1:n-1} - \mu_{r:n-1}^{(2)}\}\Bigg],$$

and for $n \geq 2$

$$\begin{aligned}(3.3)\qquad \mu_{n-1,n:n+1} = \mu^{(2)}_{n-1:n+1} &\\ + (n+1)\Big[\mu_{n-1,n:n} - \mu^{(2)}_{n-1:n} &\\ + \frac{1}{P}\{k\mu_{n-1,n:n} - \mu_{n-1:n}\} &\\ + \frac{n(1-P^2)}{2P^2}\{P_1\mu_{n-1:n-1} - \mu^{(2)}_{n-1:n-1}\}\Big]&.\end{aligned}$$

PROOF. For $1 \leq r \leq n-1$, let us consider from (3.1)

$$\begin{aligned}(3.4)\quad \mu_{r:n} - k\mu_{r,r+1:n} &= E(X_{r:n}X^0_{r+1:n} - kX_{r:n}X_{r+1:n}) \\ &= \frac{n!}{(r-1)!(n-r-1)!} \\ &\quad \cdot \int_0^{P_1}\int_x^{P_1} x(1-ky)\{F(x)\}^{r-1}\{1-F(y)\}^{n-r-1} \\ &\quad\quad f(x)f(y)dydx \\ &= \frac{n!}{2P(r-1)!(n-r-1)!}\int_0^{P_1} x\{F(x)\}^{r-1}f(x)J_1(x)dx\end{aligned}$$

upon using (1.7), where

$$\begin{aligned}(3.5)\quad J_1(x) &= (1-P^2)\int_x^{P_1}\{1-F(y)\}^{n-r-1}dy \\ &\quad + 2P^2\int_x^{P_1}\{1-F(y)\}^{n-r}dy - P^2\int_x^{P_1}\{1-F(y)\}^{n-r+1}dy \\ &= (1-P^2)J_{1,n-r-1} + 2P^2J_{1,n-r} - P^2J_{1,n-r+1}.\end{aligned}$$

Integration by parts directly gives

$$J_{1,n-r-1} = \begin{cases} -x\{1-F(x)\}^{n-r-1} + (n-r-1) & \\ \quad\cdot\displaystyle\int_x^{P_1} y\{1-F(y)\}^{n-r-2}f(y)dy, & \text{when } 1 \leq r \leq n-2 \\ P_1 - x, & \text{when } r = n-1,\end{cases}$$

$$J_{1,n-r} = -x\{1-F(x)\}^{n-r} + (n-r)\int_x^{P_1} y\{1-F(y)\}^{n-r-1}f(y)dy,$$

and

$$J_{1,n-r+1} = -x\{1-F(x)\}^{n-r+1} + (n-r+1)\int_x^{P_1} y\{1-F(y)\}^{n-r}f(y)dy.$$

Substituting these expressions in (3.5) and the resulting expression of $J_1(x)$ in (3.4) and then simplifying the resulting equation, we derive the recurrence relations in (3.2) and (3.3). □

THEOREM 3.2. *For $1 \le r < s \le n-1$ and $s-r \ge 2$,*

$$\begin{aligned}(3.6)\quad \mu_{r,s:n+1} &= \mu_{r,s-1:n+1}\\ &+ \frac{2(n+1)}{n-s+2}\bigg[\mu_{r,s:n} - \mu_{r,s-1:n}\\ &\qquad + \frac{1}{P(n-s+1)}\{k\mu_{r,s:n} - \mu_{r:n}\}\\ &\qquad + \frac{n(1-P^2)}{2(n-s+1)P^2}\{\mu_{r,s:n-1} - \mu_{r,s-1:n-1}\}\bigg],\end{aligned}$$

and for $1 \le r \le n-2$

$$\begin{aligned}(3.7)\quad \mu_{r,n:n+1} &= \mu_{r,n-1:n+1}\\ &+ (n+1)\bigg[\mu_{r,n:n} - \mu_{r,n-1:n} + \frac{1}{P}\{k\mu_{r,n:n} - \mu_{r:n}\}\\ &\qquad + \frac{n(1-P^2)}{2P^2}\{P_1\mu_{r:n-1} - \mu_{r,n-1:n-1}\}\bigg].\end{aligned}$$

PROOF. For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (3.1)

$$\begin{aligned}(3.8)\quad \mu_{r:n} - k\mu_{r,s:n} &= E(X_{r:n} - kX_{r:n}X_{s:n})\\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\\ &\quad \cdot \int_0^{P_1}\int_x^{P_1} x(1-ky)\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}\\ &\qquad \cdot \{1-F(y)\}^{n-s} f(x)f(y)dydx\\ &= \frac{n!}{2P(r-1)!(s-r-1)!(n-s)!}\\ &\quad \cdot \int_0^{P_1} x\{F(x)\}^{r-1} f(x) J_2(x)dx\end{aligned}$$

upon using (1.7), where

$$\begin{aligned}(3.9)\quad J_2(x) &= (1-P^2)\int_x^{P_1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s}dy\\ &+ 2P^2\int_x^{P_1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+1}dy\\ &- P^2\int_x^{P_1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+2}dy\\ &= (1-P^2)J_{2,n-s} + 2P^2 J_{2,n-s+1} - P^2 J_{2,n-s+2}.\end{aligned}$$

Integration by parts yields

$$J_{2,n-s} = \begin{cases} -(s-r-1)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-2}\{1-F(y)\}^{n-s}f(y)dy \\ \quad +(n-s)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s-1}f(y)dy, \\ \qquad\qquad\qquad\qquad \text{when } s \le n-1 \\ P_1\{1-F(x)\}^{n-r-1} - (n-r-1) \\ \quad \cdot \int_x^{P_1} y\{F(y)-F(x)\}^{n-r-2}f(y)dy, \quad \text{when } s = n, \end{cases}$$

$$J_{2,n-s+1} = -(s-r-1)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-2}\{1-F(y)\}^{n-s+1}f(y)dy$$
$$+ (n-s+1)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s}f(y)dy,$$

and

$$J_{2,n-s+2} = -(s-r-1)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-2}\{1-F(y)\}^{n-s+2}f(y)dy$$
$$+ (n-s+2)\int_x^{P_1} y\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+1}f(y)dy.$$

Upon substituting these expressions in (3.9) and the resulting expression of $J_2(x)$ in (3.8) and then simplifying the resulting equation, we derive the recurrence relations in (3.6) and (3.7). □

THEOREM 3.3. *For $n \ge 2$,*

$$\mu_{2,3:n+1} = \mu_{3:n+1}^{(2)} + (n+1)\left[\frac{1}{P}\{\mu_{2:n} - k\mu_{1,2:n}\} - \frac{n}{2P^2}\mu_{1:n-1}^{(2)}\right], \tag{3.10}$$

and for $2 \le r \le n-1$

$$\mu_{r+1,r+2:n+1} = \mu_{r+2:n+1}^{(2)} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}\{\mu_{r+1:n} - k\mu_{r,r+1:n}\} - \frac{n}{2P^2}\{\mu_{r:n-1}^{(2)} - \mu_{r-1,r:n-1}\}\right]. \tag{3.11}$$

PROOF. For $1 \le r \le n-1$, let us consider from (3.1)

$$\begin{aligned} &\mu_{r+1:n} - k\mu_{r,r+1:n} \\ &= E(X_{r:n}^0 X_{r+1:n} - kX_{r:n}X_{r+1:n}) \\ &= \frac{n!}{(r-1)!(n-r-1)!} \\ &\quad \cdot \int_0^{P_1}\int_0^y y(1-kx)\{F(x)\}^{r-1}\{1-F(y)\}^{n-r-1}f(x)f(y)dxdy \\ &= \frac{n!}{2P(r-1)!(n-r-1)!}\int_0^{P_1} y\{1-F(y)\}^{n-r-1}f(y)K_1(y)dy \end{aligned} \tag{3.12}$$

upon using (1.6), where

$$K_1(y) = \int_0^y \{F(x)\}^{r-1} dx - P^2 \int_0^y \{F(x)\}^{r+1} dx \\ = K_{1,r-1} - P^2 K_{1,r+1}. \tag{3.13}$$

Integration by parts yields

$$K_{1,r-1} = \begin{cases} y, & \text{when } r = 1 \\ y\{F(y)\}^{r-1} - (r-1)\int_0^y x\{F(x)\}^{r-2} f(x)dx, & \text{when } r \geq 2, \end{cases}$$

and

$$K_{1,r+1} = y\{F(y)\}^{r+1} - (r+1)\int_0^y x\{F(x)\}^r f(x)dx.$$

Upon substituting these expressions in (3.13) and the resulting expression of $K_1(y)$ in (3.12) and then simplifying the resulting equation, we derive the recurrence relations in (3.10) and (3.11). □

COROLLARY 3.1. *By setting $r = n-1$ in (3.11), we obtain for $n \geq 3$*

$$\mu_{n,n+1:n+1} = \mu^{(2)}_{n+1:n+1} + \frac{2(n+1)}{(n-1)n}\left[\frac{1}{P}\{\mu_{n:n} - k\mu_{n-1,n:n}\} - \frac{n}{2P^2}\{\mu^{(2)}_{n-1:n-1} - \mu_{n-2,n-1:n-1}\}\right]. \tag{3.14}$$

THEOREM 3.4. *For $3 \leq s \leq n$,*

$$\mu_{2,s+1:n+1} = \mu_{3,s+1:n+1} + (n+1)\left[\frac{1}{P}\{\mu_{s:n} - k\mu_{1,s:n}\} - \frac{n}{2P^2}\mu_{1,s-1:n-1}\right], \tag{3.15}$$

and for $2 \leq r < s \leq n$ and $s - r \geq 2$

$$\mu_{r+1,s+1:n+1} = \mu_{r+2,s+1:n+1} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}\{\mu_{s:n} - k\mu_{r,s:n}\} - \frac{n}{2P^2}\{\mu_{r,s-1:n-1} - \mu_{r-1,s-1:n-1}\}\right]. \tag{3.16}$$

PROOF. For $1 \leq r < s \leq n$ and $s - r \geq 2$, let us consider from (3.1)

$$\mu_{s:n} - k\mu_{r,s:n} = E(X^0_{r:n}X_{s:n} - kX_{r:n}X_{s:n}) \tag{3.17}$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \cdot \int_0^{P_1} \int_0^y y(1-kx)\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1} \cdot \{1-F(y)\}^{n-s} f(x)f(y)dxdy$$

$$= \frac{n!}{2P(r-1)!(s-r-1)!(n-s)!} \cdot \int_0^{P_1} y\{1-F(y)\}^{n-s} f(y)K_2(y)dy$$

upon using (1.6), where

$$K_2(y) = \int_0^y \{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}dx - P^2 \int_0^y \{F(x)\}^{r+1}\{F(y)-F(x)\}^{s-r-1}dx = K_{2,r-1} - P^2 K_{2,r+1}. \tag{3.18}$$

Integration by parts yields

$$K_{2,r-1} = \begin{cases} (s-2)\displaystyle\int_0^y x\{F(y)-F(x)\}^{s-3}f(x)dx, & \text{when } r=1 \\ -(r-1)\displaystyle\int_0^y x\{F(x)\}^{r-2}\{F(y)-F(x)\}^{s-r-1}f(x)dx & \\ \quad +(s-r-1)\displaystyle\int_0^y x\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-2}f(x)dx, & \text{when } r \geq 2, \end{cases}$$

and

$$K_{2,r+1} = -(r+1)\int_0^y x\{F(x)\}^{r}\{F(y)-F(x)\}^{s-r-1}f(x)dx + (s-r-1)\int_0^y x\{F(x)\}^{r+1}\{F(y)-F(x)\}^{s-r-2}f(x)dx.$$

Upon substituting these expressions in (3.18) and the resulting expression of $K_2(y)$ in (3.17) and then simplifying the resulting equation, we derive the recurrence relations in (3.15) and (3.16). □

COROLLARY 3.2. *Upon setting $s = n$ in (3.16), we obtain for $2 \leq r \leq n-2$*

$$\mu_{r+1,n+1:n+1} = \mu_{r+2,n+1:n+1} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}\{\mu_{n:n} - k\mu_{r,n:n}\} - \frac{n}{2P^2}\{\mu_{r,n-1:n-1} - \mu_{r-1,n-1:n-1}\}\right]. \tag{3.19}$$

THEOREM 3.5. *We have*

$$\mu_{1,3:3} = 3\left[\left(1+\frac{k}{P}\right)\mu_{1,2:2} + \mu_{1:2}^{(2)} - \frac{1}{3}\mu_{1,2:3} - \frac{1}{3}\mu_{1:3}^{(2)} - \frac{1}{P}\mu_{2:2} + \frac{1-P^2}{P^2}\mu_{1:1}^{(2)}\right], \tag{3.20}$$

and for $n \geq 3$

$$\mu_{1,n+1:n+1} = \frac{2(n+1)}{n(n-1)}\left[\left(n-1+\frac{k}{P}\right)\mu_{1,n:n} + \mu_{1,n-1:n} - \frac{n-1}{n+1}\mu_{1,n:n+1} - \frac{1}{n+1}\mu_{1,n-1:n+1} - \frac{1}{P}\mu_{n:n} + \frac{n(1-P^2)}{2P^2}\mu_{1,n-1:n-1}\right]. \tag{3.21}$$

PROOF. For $n \geq 2$, let us consider from (3.1)

$$\begin{aligned} \mu_{n:n} - k\mu_{1,n:n} &= E(X_{1:n}^0 X_{n:n} - kX_{1:n}X_{n:n}) \\ &= n(n-1)\int_0^{P_1}\int_0^y y(1-kx)\{F(y)-F(x)\}^{n-2} \cdot f(x)f(y)dxdy \\ &= \frac{n(n-1)}{2P}\int_0^{P_1} yf(y)L(y)dy \end{aligned} \tag{3.22}$$

upon using (1.7), where

$$\begin{aligned} L(y) &= (1-P^2)\int_0^y \{F(y)-F(x)\}^{n-2}dx \\ &\quad + 2P^2\int_0^y \{F(y)-F(x)\}^{n-2}\{1-F(x)\}dx \\ &\quad - P^2\int_0^y \{F(y)-F(x)\}^{n-2}\{1-F(x)\}^2dx \\ &= (1-P^2)L_{n-2} + 2P^2[L_{n-1} + \{1-F(y)\}L_{n-2}] \\ &\quad - P^2[L_n + 2\{1-F(y)\}L_{n-1} + \{1-F(y)\}^2L_{n-2}] \\ &= [1-P^2+2P^2\{1-F(y)\} - P^2\{1-F(y)\}^2]L_{n-2} \\ &\quad + [2P^2 - 2P^2\{1-F(y)\}]L_{n-1} - P^2L_n. \end{aligned} \tag{3.23}$$

Integration by parts yields

$$L_{n-2} = \begin{cases} y, & \text{when } n=2 \\ (n-2)\displaystyle\int_0^y x\{F(y)-F(x)\}^{n-3}f(x)dx, & \text{when } n \geq 3. \end{cases}$$

$$L_{n-1} = (n-1)\int_0^y x\{F(y)-F(x)\}^{n-2}f(x)dx$$

and

$$L_n = n \int_0^y x\{F(y) - F(x)\}^{n-1} f(x)dx.$$

Upon substituting these expressions in (3.23) and the resulting expression of $L(y)$ in (3.22) and then simplifying the resulting equation, we derive the recurrence relations in (3.20) and (3.21). $\square$

Remark 3.1. As mentioned earlier in Remark 2.1, if we let the proportion of truncation $1 - P \to 0$ in Theorems 3.1–3.5, we deduce the recurrence relations established by Balakrishnan and Sandhu (1995) for the product moments of order statistics from the standard generalized half logistic distribution.

Remark 3.2. The relationships established in Theorems 3.1–3.5 will enable one to compute all the product moments of all order statistics for all sample sizes in a simple recursive manner. This can be done for any choice of the shape parameter k and truncation parameter P.

4. Recursive computational algorithm

By starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$ (see Tables 1 and 2), $\mu_{1:2}$ and $\mu_{1:2}^{(2)}$ can be determined from (2.2) while $\mu_{2:2}$ and $\mu_{2:2}^{(2)}$ can be computed from (2.5). For the sample of size 3, $\mu_{1:3}$ and $\mu_{1:3}^{(2)}$ can be determined from (2.3), $\mu_{2:3}$ and $\mu_{2:3}^{(2)}$ from (2.9), and finally $\mu_{3:3}$ and $\mu_{3:3}^{(2)}$ from (2.6). Similarly, for the sample of size 4, $\mu_{1:4}$ and $\mu_{1:4}^{(2)}$ can be determined from (2.3), $\mu_{2:4}$ and $\mu_{2:4}^{(2)}$ from (2.8), $\mu_{3:4}$ and $\mu_{3:4}^{(2)}$ from (2.9), and finally $\mu_{4:4}$ and $\mu_{4:4}^{(2)}$ from (2.6). This process may be followed similarly to determine $\mu_{r:n}$ and $\mu_{r:n}^{(2)}$ for $1 \le r \le n$ and for $n = 5, 6, \ldots$. From these values, variances of order statistics can be readily computed.

By starting with the fact that $\mu_{1,2:2} = \mu_{1:1}^2$ (see David (1981), Arnold and Balakrishnan (1989)), $\mu_{1,2:3}$ can be determined from (3.3), $\mu_{2,3:3}$ from (3.10), and then $\mu_{1,3:3}$ from (3.20). For the sample of size 4, $\mu_{1,2:4}$ can be determined from (3.2), $\mu_{2,3:4}$ from (3.3), $\mu_{3,4:4}$ from (3.14), $\mu_{1,3:4}$ from (3.7), $\mu_{2,4:4}$ from (3.15), and finally $\mu_{1,4:4}$ from (3.21). For the sample of size 5, $\mu_{1,2:5}$ and $\mu_{2,3:5}$ can be determined from (3.2), $\mu_{3,4:5}$ from (3.3), $\mu_{4,5:5}$ from (3.14), $\mu_{1,3:5}$ from (3.6), $\mu_{2,4:5}$ from (3.15), $\mu_{3,5:5}$ from (3.19), $\mu_{1,4:5}$ from (3.7), $\mu_{2,5:5}$ from (3.15), and finally $\mu_{1,5:5}$ from (3.21). This process may be followed similarly to determine $\mu_{r,s:n}$ for $1 \le r < s \le n$ and for $n = 6, 7, \ldots$. From these values, covariances of order statistics can be readily computed.

Acknowledgements

Thanks are due to Sheree Cox for the excellent typing of the manuscript, and to the Natural Sciences and Engineering Research Council of Canada for funding this research. Thanks are also offered to two referees who made some suggestions which led to an improvement in the paper.

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