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IMPROVED ESTIMATION UNDER PITMAN'S MEASURE OF CLOSENESS

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Abstract. Stein-type and Brown-type estimators are constructed for general families of distributions which improve in the sense of Pitman closeness on the closest (in a class) estimator of a parameter. The results concern mainly scale parameters but a brief discussion on improved estimation of location parameters is also included. The loss is a general continuous and strictly bowl shaped function, and the improved estimators presented do not depend on it, i.e., uniform domination is established with respect to the loss. The normal and inverse Gaussian distributions are used as illustrative examples. This work unifies and extends previous relevant results available in the literature.

Key words and phrases: Pitman's measure of closeness, closest estimator, Stein-type estimator, Brown-type estimator, equivariant estimator, loss function, testimator.

1. Introduction

Let X and Y be independent random variables, where X has a normal distribution $N(\mu, \sigma^2)$ and Y/σ^2 has a chi-squared distribution χ_n^2 with n degrees of freedom. Assume that μ and σ^2 are unknown and that based on (X, Y) we want to estimate σ^2 by an estimator δ using as criterion Pitman's measure of closeness (PMC) under the quadratic loss $(\delta/\sigma^2 - 1)^2$. Kubokawa (1991) showed that the closest estimator of σ^2 in the class of affine equivariant estimators cY, $c > 0$, is $\delta_0 = c_0 Y$, where $c_0 = \frac{1}{m_n}$ and m_n is the median of χ_n^2. He also showed that δ_0 can be improved on by considering a larger class of scale equivariant estimators

$$\delta = \phi(Z)Y, \qquad Z = \frac{X^2}{Y}$$

for ϕ a positive function. Specifically he found a Stein (1964)-type improved estimator

$$\delta_s = \phi_s(Z)Y, \qquad \phi_s(Z) = \min\left\{c_0, \frac{1+Z}{m_{n+1}}\right\}$$

and a Brown (1968)-type improved estimator

$$\delta_B = \phi_B(Z)Y, \qquad \phi_B(Z) = \begin{cases} a_0, & Z < r \\ c_0, & Z \geq r, \end{cases}$$

where r is any positive constant and a_0 is the reciprocal of the median of the conditional distribution of Y given that $Z < r$ when $\mu = 0$ and $\sigma^2 = 1$. Note that Z is a maximal invariant under the scale group and also a test statistic for testing $\mu = 0$.

Analogous results were obtained by Kourouklis (1995*a*, 1995*b*) for the exponential scale parameter and quantiles.

For $\omega = (\mu, \sigma^2)$, let $G(y \mid z; \omega)$ and $H(y \mid r; \omega)$ denote the conditional distribution functions of Y given that $Z = z$, $z > 0$, and given that $Z < r$ respectively. Let also $F_n(y)$ be the distribution function of Y when $\sigma^2 = 1$ (i.e., the χ^2_n distribution function). Then the following stochastic ordering-type conditions are satisfied (see Example 3.1 in Section 3).

A. $G(y \mid z; \omega) \leq G(y \mid z; \omega_0)$ for $y > 0$ and $\omega_0 = (0, \sigma^2)$.

B. $H(y \mid r; \omega) \leq H(y \mid r; \omega_0)$ for $y > 0$ and $\omega_0 = (0, \sigma^2)$.

C. $H(y \mid r; \omega_1) < F_n(y)$ for $y > 0$ and $\omega_1 = (0, 1)$.

Examination of the approach used by Kubokawa (1991) reveals that the first condition plays an important role in deriving δ_s, whereas the latter two in deriving δ_B.

The purpose of this paper is to exploit these conditions and to demonstrate that, when they hold, Stein-type and Brown-type improved estimators can be derived for general families of distributions. We present results mainly for scale parameters but we also discuss briefly estimation of location parameters. The loss we use is a general continuous and strictly bowl shaped function and the improved estimators presented turn out to be independent of it, i.e., we establish uniform domination with respect to the loss. Stein-type improvements also turn out to be testimators. We note that testimators with respect to PMC were first introduced by Keating and Czitrom (1989).

In Section 2 we describe our framework and conditions and obtain the main results. In Section 3 we present two illustrative examples from the normal and inverse Gaussian distributions. The latter is, in particular, interesting because it is not a pure location and scale model. Section 4 treats briefly the location parameter case.

2. Framework and main results

2.1 *Framework*

We consider the following framework. Data $\boldsymbol{X}$ comes from a distribution that depends on an unknown parameter $\omega = (\eta, \theta)$ where θ is the component of interest and η is a nuisance component. Further, there is a statistic Y, which is a function of $\boldsymbol{X}$, whose distribution is absolutely continuous with respect to Lebesque measure on $(0, \infty)$, does not depend upon η, and has $\tau = \tau(\theta) > 0$ as a scale parameter. Thus, Y has density $f(y; \tau) = \frac{1}{\tau} f(\frac{y}{\tau})$, $y > 0$. The problem is to estimate τ based on $\boldsymbol{X}$.

The criterion for selecting an estimator δ of τ is Pitman's measure of closeness (PMC) with respect to a general loss of the form $L(\delta/\tau)$, where $L(t)$ is continuous on $(0, \infty)$ and strictly bowl shaped assuming its minimum at $t = 1$, i.e., strictly decreasing for $0 < t < 1$ and strictly increasing for $t > 1$ with $L(1) = 0$. We refer to the book by Keating *et al.* (1993), to a special issue of *Communications in Statistics, Theory and Methods* ((1991), Vol. 20, No. 11), and to the discussion article by Robert *et al.* (1993) for a detailed account on PMC. Here, according to PMC an estimator δ_1 is called Pitman closer to τ than δ_2 if

$$(2.1) \qquad P_\omega(L(\delta_1/\tau) < L(\delta_2/\tau)) \geq P_\omega(L(\delta_2/\tau) < L(\delta_1/\tau)) \quad \text{for all} \quad \omega$$

with strict inequality for some ω. Setting

$$(2.2) \qquad PC(\delta_1, \delta_2; \omega) = P_\omega(L(\delta_1/\tau) < L(\delta_2/\tau)) + \frac{1}{2}P_\omega(L(\delta_1/\tau) = L(\delta_2/\tau))$$

(2.1) equivalently becomes

$$(2.3) \qquad PC(\delta_1, \delta_2; \omega) \geq \frac{1}{2} \quad \text{for all} \quad \omega.$$

In the sequel, it will be more convenient to work with (2.2) and (2.3) rather than (2.1).

Let C be the class of estimators given by

$$C = \{cY : c > 0\}.$$

For this class and under absolute error loss Keating (1985) established Rao's (1981) phenomenon, i.e., that shrinking the unbiased estimator to a minimum risk estimator does not improve the PMC property.

Let

$$(2.4) \qquad c_0 = \frac{1}{m} \quad \text{and} \quad \delta_0 = c_0 Y,$$

where m is the median of Y when $\tau = 1$. Nayak (1990) showed that δ_0 is the unique closest estimator of τ in the class C. Note that δ_0 does not depend on the loss and is median unbiased for τ (Ghosh and Sen (1989)). However, δ_0 utilizes information supplied by the data $\boldsymbol{X}$ only through Y. This observation sets the ground for improving on δ_0 (in the sense of PMC).

2.2 *Stein-type improvement*

Following Stein (1964), for improving on δ_0 we consider a larger class of estimators than C, namely,

$$D = \{\phi(Z)Y : \phi \text{ is positive (measurable) function}\}$$

where Z is a properly chosen statistic (function of $\boldsymbol{X}$). Invariance considerations are helpful in choosing Z, but not crucial, see Section 3. With Z granted, Theorem

2.1 below describes a general way for constructing an estimator δ in D which improves on δ_0. For the theorem we assume that the conditional distribution of Y given $Z = z$, when well defined, has a unique median $m(z;\omega)$.

THEOREM 2.1. *Suppose that there exist a (Borel) set B and a positive (measurable) function $\psi(z)$ defined on B such that $\frac{m(z;\omega)}{\tau} \geq$ (resp. $\leq$)$\psi(z)$ for all ω and all $z \in B$. Define*

$$\phi(z) = \begin{cases} \min\left\{c_0, \dfrac{1}{\psi(z)}\right\} \left(\text{resp. } \max\left\{c_0, \dfrac{1}{\psi(z)}\right\}\right), & z \in B \\ c_0, & z \notin B. \end{cases}$$

Then, $\delta = \phi(Z)Y$ is closer to τ than $\delta_0 = c_0 Y$ provided $P_\omega(\delta \neq \delta_0) > 0$ for some ω.

PROOF. We consider only the case $\frac{m(z;\omega)}{\tau} \geq \psi(z)$. Let z be such that $\phi(z) \neq c_0$. Then, $z \in B$ and $\phi(z) = \frac{1}{\psi(z)} < c_0$. By the properties of the loss, $L(\frac{Y}{\tau\psi(z)}) < L(\frac{c_0 Y}{\tau})$ holds iff $\frac{Y}{\tau} > u(z)$, where the point $u(z)$ satisfies $\frac{1}{\psi(z)}u(z) < 1 < c_0 u(z)$, see Keating *et al.* ((1993), p. 148) or Kourouklis (1995*a*). Hence $u(z) < \psi(z) \leq \frac{m(z;\omega)}{\tau}$ and $P_\omega(L(\delta/\tau) < L(\delta_0/\tau) \mid Z = z) = P_\omega(\frac{Y}{\tau} > u(z) \mid Z = z) > \frac{1}{2}$. Now, (2.2) entails $PC(\delta, \delta_0; \omega) \geq \frac{1}{2}P_\omega(\delta \neq \delta_0) + \frac{1}{2}P_\omega(\delta = \delta_0) = \frac{1}{2}$ with strict inequality whenever $P_\omega(\delta \neq \delta_0) > 0$. This completes the proof.

Implementation of Theorem 2.1 requires the derivation of an appropriate lower or upper bound for $\frac{m(z;\omega)}{\tau}$ as a function of ω. We next give simple conditions which guarantee such a bound. Let the conditional distribution of Y given that $Z = z$ be absolutely continuous with respect to Lebesque measure with density $g(y \mid z;\omega)$, $y \in S_\omega$, and set $G(y \mid z;\omega) = \int_0^y g(t \mid z;\omega)dt$. Thus, $m(z;\omega)$ is the unique median of $G(y \mid z;\omega)$.

Assume that there is a (Borel) set B and a value η_0 of η (the nuisance parameter) such that the following conditions hold.

(A.1) $G(y \mid z;\omega) \leq$ (resp. $\geq$)$G(y \mid z;\omega_0)$ for $y \in S_\omega$, $z \in B$, $\omega = (\eta, \theta)$, and $\omega_0 = (\eta_0, \theta)$.

(A.2) $\tau = \tau(\theta)$ is a scale parameter for $G(y \mid z;\omega_0)$, i.e.,

$$G(y \mid z;\omega_0) = G_1\left(\frac{y}{\tau};z\right), \qquad y \in S_{\omega_0}, \quad z \in B,$$

for some distribution function $G_1(\cdot;z)$.

Note that (A.1) is ensured when, for $z \in B$,

(A$'$.1) $\frac{g(y|z;\omega)}{g(y|z;\omega_0)}$ is increasing (resp. decreasing) in $y \in S_{\omega_0}$ and $S_\omega \subset S_{\omega_0}$,

cf. Lehmann ((1959), p. 74).

Let now

$$m(z) = \text{median of } G_1(\cdot;z), \qquad z \in B.$$

We then have the following result.

THEOREM 2.2. *Suppose that* (A.1) *and* (A.2) *hold. Define*

$$\phi_s(z) = \begin{cases} \min\left\{c_0, \dfrac{1}{m(z)}\right\} \left(\textit{resp. } \max\left\{c_0, \dfrac{1}{m(z)}\right\}\right), & z \in B \\ c_0, & z \notin B. \end{cases}$$

Then, $\delta_s = \phi_s(Z)Y$ *is closer to* τ *than* $\delta_0 = c_0 Y$ *provided* $P_\omega(\delta_s \neq \delta_0) > 0$ *for some* ω.

PROOF. (A.1) and (A.2) give $\frac{m(z;\omega)}{\tau} \geq$ (resp. $\leq$)$\frac{m(z;\omega_0)}{\tau} = m(z)$ for all ω and $z \in B$. Theorem 2.1 now applies with $\psi(z) = m(z)$, $z \in B$.

Remark 2.1. The estimator δ_s does not depend on the loss.

2.3 *Brown-type improvement*

Following Brown (1968), for improving on δ_0 we consider a subclass of D, namely, estimators of the form

$$\delta_1 = \phi(Z)Y, \qquad \phi(Z) = \begin{cases} a_0, & Z \in E \\ c_0, & Z \notin E. \end{cases} \tag{2.5}$$

Here, a_0 is a positive constant and E a (Borel) set to be chosen properly. The choice of a_0 and E is discussed later on. The next theorem describes a general way for constructing an estimator δ_1 of the form (2.5) which improves on δ_0. For the theorem we assume that the conditional distribution of Y given that $Z \in E$ is well defined and has a unique median $m(E;\omega)$.

THEOREM 2.3. *Suppose that* a_0 *and* E *in* (2.5) *are chosen so that*

$$0 < a_0 < c_0 \tag{2.6}$$

and

$$\frac{m(E;\omega)}{\tau} \geq \frac{1}{a_0} \quad \textit{for all} \quad \omega. \tag{2.7}$$

Then, the estimator δ_1 *is closer to* τ *than* δ_0.

PROOF. By conditioning on the event $(Z \in E)$ and using (2.6), (2.7) and a similar argument as in the proof of Theorem 2.1 we obtain $P_\omega(L(\delta_1/\tau) < L(\delta_0/\tau), Z \in E) > \frac{1}{2}P_\omega(Z \in E)$. Then, (2.2) gives $PC(\delta_1, \delta_0; \omega) > \frac{1}{2}P_\omega(Z \in E) + \frac{1}{2}P_\omega(Z \notin E) = \frac{1}{2}$, which completes the proof.

Implementation of Theorem 2.3 requires the derivation of an appropriate lower bound for $\frac{m(E;\omega)}{\tau}$ as a function of ω. Then as a_0 can be taken the reciprocal of this bound provided (2.6) is also satisfied. Regarding the choice of E, as in Brown (1968) and Brewster and Zidek (1974), we have found in applications that taking as E a suitable acceptance region of a test for the nuisance parameter η usually works. It is evident that to bound $\frac{m(E;\omega)}{\tau}$, a similar approach can be followed as in Subsection 2.2. This approach is described below.

With E granted let the conditional distribution of Y given that $Z \in E$ be absolutely continuous with respect to Lebesque measure with density $h(y \mid E;\omega)$, $y \in M_\omega$. Let also $H(y \mid E;\omega) = \int_0^y h(t \mid E;\omega)dt$, so that $m(E;\omega)$ is the median of $H(y \mid E;\omega)$. Finally, let $F(y) = \int_0^y f(t)dt$ so that m in (2.4) is the median of $F(y)$.

Assume that there is a value η_1 of η such that the following conditions hold.

(A.3) $H(y \mid E;\omega) \le H(y \mid E;\omega_1)$ for $y \in M_\omega$, $\omega = (\eta,\theta)$, and $\omega_1 = (\eta_1,\theta)$.

(A.4) $\tau = \tau(\theta)$ is a scale parameter for $H(y \mid E;\omega_1)$, i.e.,

$$H(y \mid E;\omega_1) = H_1\left(\frac{y}{\tau};E\right), \qquad y \in M_{\omega_1},$$

for some distribution function $H_1(\cdot;E)$.

(A.5) $H_1(y;E) < F(y)$ for all y such that $0 < H_1(y;E) < 1$. Note that (A.3) is satisfied when

(A$'$.3) $\frac{h(y|E;\omega)}{h(y|E;\omega_1)}$ is increasing in $y \in M_{\omega_1}$ and $M_\omega \subset M_{\omega_1}$ and (A.5) is satisfied when

(A$'$.5) $\frac{h_1(y;E)}{f(y)}$ is strictly increasing in $y > 0$, where $h_1(y;E)$ is the density of $H_1(y;E)$.

Let now

$$m(E) = \text{median of } H_1(\cdot;E). \tag{2.8}$$

We then have the following result.

THEOREM 2.4. *Suppose that* (A.3), (A.4), *and* (A.5) *hold. Define*

$$\delta_B = \phi_B(Z)Y, \qquad \phi_B(Z) = \begin{cases} \dfrac{1}{m(E)}, & Z \in E \\ c_0, & Z \notin E \end{cases}.$$

Then, δ_B is closer to τ than δ_0.

PROOF. (A.3), (A.4), and (A.5) ensure that $\frac{m(E;\omega)}{\tau} \ge \frac{m(E;\omega_1)}{\tau} = m(E) > m$. Hence, Theorem 2.3 applies with $a_0 = \frac{1}{m(E)}$.

Remark 2.2. As with δ_s, the estimator δ_B does not depend on the loss.

Remark 2.3. Kubokawa ((1991), Proposition 3.3) showed that, contrary to decision theory, in the case of normal variance a Brewster-Zidek (1974)-type estimator (which is the smooth version of the Brown-type estimator δ_B in the introduction) is inferior to δ_0 in the sense of PMC. His proof rests on general conditions as B and C rather than special properties of the normal distribution, and thus the same result holds for general families of distributions considered here.

3. Applications

Here we present two examples illustrating the conditions in Section 2. More examples are given in Kourouklis (1995*c*) and include the multivariate normal distribution for the estimation of the generalized variance (Kubokawa (1990)) and the exponential distribution for the estimation of the scale parameter (Kourouklis (1995*a*)).

For later use we denote by $f_k(y;\lambda)$ and $F_k(y;\lambda)$ the density and distribution functions of $\chi_k^2(\lambda)$. When $\lambda = 0$ we simply write $f_k(y)$ and $F_k(y)$. Also, recall that m_k denotes the median of χ_k^2.

Example 3.1. (Normal distribution) Let $\boldsymbol{U}$ and Y be independent, where $\boldsymbol{U} \sim N_p(\boldsymbol{\mu}, \tau I_p)$, $Y/\tau \sim \chi_n^2$, and $\boldsymbol{\mu}$ and τ are unknown. We want to estimate τ based on $\boldsymbol{X} = (\boldsymbol{U}, Y)$. This problem was studied by Kubokawa (1991) for squared error loss. Here $\delta_0 = \frac{Y}{m_n}$ and for improving on δ_0 we choose $Z = \frac{\|\boldsymbol{U}\|^2}{Y}$ by invariance considerations. Let $\omega = (\boldsymbol{\mu}, \tau)$, $\omega_0 = (\boldsymbol{0}, \tau) = \omega_1$, $B = (0, \infty)$, and $E = (0, r)$ for $r > 0$. Then, it is easy to check that

$$g(y \mid z;\omega) = \frac{y f_p\left(\frac{yz}{\tau};\lambda\right) f_n\left(\frac{y}{\tau}\right)}{\int_0^\infty t f_p\left(\frac{tz}{\tau};\lambda\right) f_n\left(\frac{t}{\tau}\right) dt}, \qquad y > 0, \ z \in B$$

and

$$h(y;E,\omega) = \frac{F_p\left(\frac{ry}{\tau};\lambda\right) f_n\left(\frac{y}{\tau}\right)}{\int_0^\infty F_p\left(\frac{rt}{\tau};\lambda\right) f_n\left(\frac{t}{\tau}\right) dt}, \qquad y > 0,$$

where $\lambda = \frac{\|\boldsymbol{\mu}\|^2}{\tau}$. Now (A$'$.1) holds since $\frac{f_p(t;\lambda)}{f_p(t)}$ is increasing in t, by the well-known monotone likelihood ratio property of $\chi_p^2(\lambda)$. (A$'$.3) is satisfied since $\frac{F_p F_0(t;\lambda)}{F_p(t)}$ is also increasing in t, by Lemma 4.2 in Cohen (1972). (A.2), (A.4), and (A$'$.5) can be verified by simple inspection. Thus, Theorems 2.2 and 2.4 apply. Here $m(z) = \frac{m_{n+p}}{1+z}$, $z > 0$, and the improved estimator of Theorem 2.2 is $\delta_s = \min\{\frac{Y}{m_n}, \frac{\|\boldsymbol{U}\|^2+Y}{m_{n+p}}\}$. Note that δ_s is a testimator which chooses among $\frac{Y}{m_n}$, the closest estimator of τ when $\boldsymbol{\mu}$ is unknown, and $\frac{\|\boldsymbol{U}\|^2+Y}{m_{n+p}}$, the closest estimator of τ when $\boldsymbol{\mu}$ is known to be $\boldsymbol{0}$.

Example 3.2. (Inverse Gaussian distribution) Let $\boldsymbol{X} = (X_1, \ldots, X_n)$, $n \geq 2$, be a random sample from an inverse Gaussian distribution $IG(\mu, \lambda)$ with density $(\frac{\lambda}{2\pi})^{1/2} x^{-3/2} \exp\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\}$, $x > 0$, where $\mu > 0$ and $\lambda > 0$ are unknown. We want to estimate $\tau = \frac{1}{\lambda}$ based on $\boldsymbol{X}$. In the literature, τ is usually referred to as a measure of dispersion and plays for the inverse Gaussian the analogous role of σ^2 for the Normal $N(v, \sigma^2)$. The decision theoretic estimation of τ was studied by Pal and Sinha (1989), MacGibbon and Shorrock (1995), and Kourouklis (1996). It is known that $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $Y = \sum_{i=1}^n (\frac{1}{X_i} - \frac{1}{\bar{X}})$ are complete and sufficient, and are independent with $IG(\mu, n\lambda)$, $\tau\chi^2_{n-1}$ distributions respectively (Tweedie (1957), or Chhikara and Folks (1989)). Besides, $\frac{n\lambda(\bar{X}-\mu)^2}{\mu^2\bar{X}}$ is distributed as χ^2_1 (Shuster (1968)). In this case $\delta_0 = \frac{Y}{m_{n-1}}$, and for improving on δ_0 we choose $Z = \frac{V}{Y}$, where $V = \frac{n(\bar{X}-1)^2}{\bar{X}}$. For motivating this choice, note that Z is a test statistic for testing that (the nuisance parameter) $\mu = 1$. We refer to Pal and Sinha (1989) for another motivation and point out that, unlike the previous example, here invariance does not seem to play a major role in choosing Z. Let $\omega = (\mu, \lambda)$, $\omega_0 = \omega_1 = (1, \lambda)$, $B = (0, \infty)$ and $E = (0, r)$ for $r > 0$. Also, denote by $k(t;\omega)$ and $K(t;\omega)$ the density and distribution functions of V respectively. Then, using the independence of V and Y, it is easy to check that

$$g(y \mid z; \omega) = \frac{yk(yz;\omega)f_{n-1}(y/\tau)}{\int_0^\infty tk(tz;\omega)f_{n-1}(t/\tau)dt}, \qquad y > 0, \ \ z \in B$$

and

$$h(y \mid E; \omega) = \frac{K(ry;\omega)f_{n-1}(y/\tau)}{\int_0^\infty K(rt;\omega)f_{n-1}(t/\tau)dt}, \qquad y > 0.$$

It is shown in Kourouklis (1996) that $\frac{k(t;\omega)}{k(t;\omega_0)}$ is increasing in t, and this in turn implies that $\frac{K(t;\omega)}{K(t;\omega_0)}$ is also increasing in t (Cohen (1972)). It now follows that (A$'$.1) and (A$'$.3) are satisfied. Besides, (A.2), (A.4), and (A$'$.5) can be directly verified, so that Theorems 2.2 and 2.4 apply. Here, it can be shown that $m(z) = \frac{m_n}{1+z}$, $z > 0$, and the improved estimator in Theorem 2.2 is $\delta_s = \min\{\frac{Y}{m_{n-1}}, \frac{Y+V}{m_n}\}$. Note that $Y + V = \sum_{i=1}^n \frac{(X_i-1)^2}{X_i}$ and δ_s is a testimator which chooses among $\frac{Y}{m_{n-1}}$, the closest estimator when μ is unknown, and $\frac{Y+V}{m_n}$, the closest estimator when μ is known to be 1.

4. Improved estimation of a location parameter

In this section we consider the framework of Section 2 except that τ is now a location parameter, i.e., Y has density $f(y;\tau) = f(y-\tau)$, $y \in R$, $\tau \in R$. The problem is to estimate τ by an estimator δ using as criterion Pitman's measure of closeness with respect to a general loss of the form $L(\delta - \tau)$, where $L(t)$ is continuous and strictly bowl shaped assuming its minimum at $t = 0$, i.e., strictly decreasing for $t < 0$ and strictly increasing for $t > 0$ with $L(0) = 0$.

Let C be the class of estimators given by $C = \{Y + c : c \in R\}$, $c_0 = -m$ and $\delta_0 = Y + c_0$, where m is the median of Y when $\tau = 0$. Nayak (1990) showed that δ_0 is the unique closest estimator of τ in the class C. For improving on δ_0, the techniques of Section 2 can be employed again. Below we discuss briefly only Stein-type improvements.

We consider a larger class of estimators than C, namely, $D = \{Y + \phi(Z) : \phi$ is a real valued (measurable) function$\}$, where Z is a properly chosen statistic (function of the full data $\boldsymbol{X}$). In analogy to Theorem 2.1 we have the following result, for which we assume that the conditional distribution of Y given that $Z = z$, when well defined, has a unique median $m(z;\omega)$.

THEOREM 4.1. *Suppose that there exist a (Borel) set B and a (measurable) function $\psi(z)$ defined on B such that $m(z;\omega) - \tau \geq$ (resp. $\leq$)$\psi(z)$ for all ω and all $z \in B$. Define*

$$\phi(z) = \begin{cases} \min\{c_0, -\psi(z)\}(\textit{resp. } \max\{c_0, -\psi(z)\}), & z \in B \\ c_0, & z \notin B. \end{cases}$$

Then, the estimator $\delta = Y + \phi(Z)$ is closer to τ than $\delta_0 = Y + c_0$ provided $P_\omega(\delta \neq \delta_0) > 0$ for some ω.

An application of Theorem 4.1 is given in Kourouklis (1995*c*).

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