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THE EXACT DISTRIBUTION OF INDEFINITE QUADRATIC FORMS IN NONCENTRAL NORMAL VECTORS

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Abstract. The exact density of the difference of two linear combinations of independent noncentral chi-square variables is obtained in terms of Whittaker's function and expressed in closed forms. Two distinct representations are required in order to cover all the possible cases. The corresponding expressions for the exact distribution function are also given.

Key words and phrases: Exact distribution function, exact density function, indefinite quadratic forms, noncentral chi-square variables, singular normal vectors, Whittaker's function.

1. Introduction

The distribution of linear combinations of chi-square variables or equivalently that of quadratic forms in normal vectors has been studied by several authors over the last four decades. Various representations of the distribution function of quadratic forms have been derived, and several procedures have been proposed for computing percentage points and preparing tables. Box (1954) considered a linear combination of chi-square variables having even degrees of freedom. Gurland (1948, 1953, 1956), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961), and Kotz *et al.* (1967*a*, 1967*b*) among others, have given representations of the distribution function of quadratic forms in terms of the distribution functions of chi-square variables, MacLaurin series and Laguerre polynomials. Gurland (1956) and Shah (1963) considered respectively central and noncentral indefinite quadratic forms but as pointed by Shah (1963), the expansions obtained are not practical. Numerical methods have been suggested by Imhof (1961), Davis (1973) and Rice (1980) for the evaluation of the distribution function of indefinite quadratic forms. Some representations of the density function of linear combinations of chi-square variables are available in Mathai and Saxena (1978) and Mathai and Provost (1992).

Linear combinations of noncentral chi-square random variables are involved in the asymptotic distribution of quadratic forms in order statistics from a uniform distribution, see for example Guttorp and Lockhart (1988). Furthermore if we let Q_1 and Q_2 denote two quadratic forms in normal variables, then in view of the relationship $\text{Prob}(Q_1/Q_2 \le t) = \text{Prob}(Q_1 - tQ_2 \le 0)$, it can be seen that the distribution of ratios of quadratic forms is available from that of the differences of linear combinations of chi-square variables. Such ratios arise for example in regression and analysis of variance problems associated with linear models. The sample serial correlation coefficient as defined in Anderson (1990) and discussed in Provost and Rudiuk (1995) also has this structure.

Series representations for the exact density function of an indefinite quadratic form in noncentral normal vectors are given in Section 2. Closed form expressions for the corresponding distribution functions are derived in Section 3.

2. The exact density function

First it is shown that an indefinite quadratic form in normal vectors can be expressed as a linear combination of independent noncentral chi-square variables involving positive and negative coefficients. Representations of the density function of such a structure are then obtained in terms of Whittaker's function and expressed in closed forms.

Let $Z = \boldsymbol{Y}'A\boldsymbol{Y}$ be an indefinite quadratic form in noncentral normal variables where $A = A'$, A' denoting the transpose of the matrix A, $\boldsymbol{Y} \sim N_r(\boldsymbol{\mu}, \Sigma)$ and $\Sigma > 0$. If we let $\Sigma = LL'$, $\boldsymbol{Y} = L\boldsymbol{S}$ so that $\boldsymbol{S} = L^{-1}\boldsymbol{Y}$, and $\lambda_1, \ldots, \lambda_r$ denote the eigenvalues of $L'AL$, (or equivalently those of ΣA), then $Z = \boldsymbol{S}'L'AL\boldsymbol{S} = \boldsymbol{S}'P\,\text{diag}(\lambda_1, \ldots, \lambda_r)P'\boldsymbol{S} = \boldsymbol{Y}'(L^{-1})'P\,\text{diag}(\lambda_1, \ldots, \lambda_r)P'L^{-1}\boldsymbol{Y} = \boldsymbol{W}'\,\text{diag}(\lambda_1, \ldots, \lambda_r)\,\boldsymbol{W}$ where $\boldsymbol{W} = P'L^{-1}\boldsymbol{Y}$ and P is an orthogonal matrix of the eigenvectors of $L'AL$. It follows that $\boldsymbol{W} = N_r(P'L^{-1}\boldsymbol{\mu}, I)$ which shows that the components of $\boldsymbol{W}$ are independently distributed. Let $\boldsymbol{W} = \boldsymbol{X} + \boldsymbol{d}$ where $\boldsymbol{X} \sim N_r(\boldsymbol{0}, I)$, $\boldsymbol{X} = (X_1, \ldots, X_r)'$, and $\boldsymbol{d} = (\delta_1, \ldots, \delta_r)' = P'L^{-1}\boldsymbol{\mu}$, and let $\lambda_i > 0$ for $i = 1, \ldots, \rho$, $\lambda_i < 0$ for $i = \rho + 1, \ldots, \rho + \xi$, and $\lambda_i = 0$ for $i = \rho + \xi + 1, \ldots, r$. Then Z can be expressed as follows:

$$Z = \sum_{i=1}^{r} \lambda_i (X_i + \delta_i)^2 = U - V \tag{2.1}$$

where

$$U = \sum_{i=1}^{\rho} \lambda_i (X_i + \delta_i)^2, \qquad V = \sum_{i=\rho+1}^{\rho+\xi} (-\lambda_i)(X_i + \delta_i)^2 \tag{2.2}$$

and

$$(X_i + \delta_i)^2 \overset{\text{ind}}{\sim} \chi_1^2({\delta_i}^2), \qquad i = 1, \ldots, \rho + \xi, \tag{2.3}$$

where $\chi_1^2({\delta_i}^2)$ denotes a chi-square distribution with one degree of freedom and noncentrality parameter δ_i^2. Clearly Z is distributed as the difference of two linear combinations of independent chi-square variables.

Ruben (1962) obtained the following representation of the density function of U (see also Kotz *et al.* (1967*b*) and Johnson and Kotz (1970)):

$$g_U(u) = \sum_{k=0}^{\infty} a_k u^{\alpha+k-1} e^{-u/2\beta} / (\Gamma(\alpha+k)(2\beta)^{\alpha+k}), \qquad u > 0, \tag{2.4}$$

where

$$a_0 = e^{-\delta/2} \prod_{j=1}^{\rho} \left(\frac{\beta}{\lambda_j}\right)^{1/2}$$

$$a_k = (2k)^{-1} \sum_{r=0}^{k-1} b_{k-r} a_r, \qquad k \geq 1,$$

$$b_k = k\beta \sum_{j=1}^{\rho} (\delta_j^2/\lambda_j) c_j^{k-1} + \sum_{j=1}^{\rho} c_j^k, \qquad k \geq 1,$$

where $\delta = \sum_{j=1}^{\rho} \delta_j^2$, $\alpha = \rho/2$, $c_j = 1 - \beta/\lambda_j$ and β is such that

$$|c_j| = |1 - \beta/\lambda_j| < 1, \qquad j = 1, \ldots, \rho. \tag{2.5}$$

The parameter β is chosen so as to accelerate the convergence of the series; β can be taken as some average of the λ_j's, $j = 1, \ldots, \rho$ such as the geometric mean or the harmonic mean as suggested by Ruben (1962).

Consider the positive linear combination U. Note that some of the λ_i's in U may be equal; in this case, some of the chi-square variables in the linear combination will have more than one degree of freedom. Let l_j, $j = 1, \ldots, t$, denote the t distinct positive eigenvalues among the λ_i's and let α_j denote the multiplicity of l_j (that is, the number of λ_i's in U which are equal to l_j). Then

$$U = \sum_{j=1}^{t} l_j \sum_{i=1}^{\alpha_j} (X_{j_i} + \delta_{j_i})^2 = \sum_{j=1}^{t} l_j T_j$$

is distributed as a linear combination of independent chi-square variables T_j each having α_j degrees of freedom and noncentrality parameter

$$d_j = \sum_{i=1}^{\alpha_j} \delta_{j_i}^2, \qquad j = 1, \ldots, t.$$

Similarly, one has

$$V = \sum_{j=t+1}^{t+w} l_j \sum_{i=1}^{\alpha_j} (X_{j_i} + \delta_{j_i})^2 = \sum_{j=t+1}^{t+w} l_j T_j\,.$$

where l_j denotes the w distinct values of $-\lambda_i$, $i = \rho+1, \ldots, \rho+\xi$, α_j denotes the multiplicity of l_j and the T_j's are independent chi-square variables each having α_j degrees of freedom and noncentrality parameter

$$d_j = \sum_{i=1}^{\alpha_j} \delta_{ji}^2, \qquad j = t+1, \ldots, t+w.$$

We now derive the density function of the difference of two linear combinations of independent noncentral chi-square random variables. Let

$$Z = U - V$$

where

$$U = \sum_{j=1}^{t} l_j T_j, \qquad V = \sum_{j=t+1}^{t+w} l_j T_j, \qquad T_j \overset{\text{ind}}{\sim} \chi^2_{\alpha_j}(d_j), \tag{2.6}$$

and $l_j > 0$, $j = 1, \ldots, t+w$. The probability density functions of U and V are respectively

$$g_U(u) = \sum_{k=0}^{\infty} \theta_k u^{\alpha+k-1} e^{-u/2\beta} / \Gamma(\alpha+k), \qquad u > 0$$

where

$$\theta_k = a_k/(2\beta)^{\alpha+k}, \qquad a_0 = \mathrm{e}^{-d/2} \prod_{j=1}^{t} \left(\frac{\beta}{l_j}\right)^{\alpha_j/2}, \tag{2.7}$$

$$a_k = (2k)^{-1} \sum_{r=0}^{k-1} b_{k-r} a_r,$$

$$b_k = k\beta \sum_{j=1}^{t} (d_j/l_j) c_j^{k-1} + \sum_{j=1}^{t} \alpha_j c_j^k, \qquad k = 1, 2, \ldots,$$

with $d = \sum_{j=1}^{t} d_j$, $\alpha = \sum_{j=1}^{t} \alpha_j/2$, $c_j = 1 - \beta/l_j$, and β such that $|1 - \beta/l_j| < 1$, $j = 1, \ldots, t$, and

$$g_V(v) = \sum_{\nu=0}^{\infty} \theta'_\nu v^{\alpha'+\nu-1} e^{-v/2\beta'} / \Gamma(\alpha'+\nu), \qquad v > 0 \tag{2.8}$$

where

$$\theta'_\nu = a'_\nu/(2\beta')^{\alpha'+\nu}, \qquad a'_0 = \mathrm{e}^{-\gamma/2} \prod_{j=t+1}^{t+w} \left(\frac{\beta'}{l_j}\right)^{\alpha_j/2}, \tag{2.9}$$

$$a'_\nu = (2\nu)^{-1} \sum_{r=0}^{\nu-1} b'_{\nu-r} a'_r,$$

$$b'_\nu = \nu\beta' \sum_{j=t+1}^{t+w} (d_j/l_j) c'^{\nu-1}_j + \sum_{j=t+1}^{t+w} \alpha_j c'^{\nu}_j, \qquad \nu = 1, 2, \ldots,$$

with $\gamma = \sum_{j=t+1}^{t+w} d_j$, $\alpha' = \sum_{i=t+1}^{t+w} \alpha_j/2$, $c'_j = 1 - \beta'/l_j$, and β' such that $|1 - \beta'/l_j| < 1$, $j = t+1, \ldots, t+w$. The recursive relationships follow directly from that given in (2.4).

Since U and V are independently distributed, the joint density of $Y = U$ and $Z = U - V$ is given by

$$\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k \theta'_\nu y^{\alpha+k-1} e^{-y/2\beta} (y-z)^{\alpha'+\nu-1} e^{-(y-z)/2\beta'}}{\Gamma(\alpha+k)\Gamma(\alpha'+\nu)} \tag{2.10}$$

for $z > 0$ and $y > z$ or for $z \leq 0$ and $y > 0$.

The marginal density of Z is therefore

$$\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k \theta'_\nu e^{z/2\beta'}}{\Gamma(\alpha+k)\Gamma(\alpha'+\nu)} \int_z^\infty y^{\alpha+k-1} (y-z)^{\alpha'+\nu-1} e^{-by} dy \tag{2.11}$$

for $z > 0$ and

$$\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k \theta'_\nu e^{z/2\beta'}}{\Gamma(\alpha+k)\Gamma(\alpha'+\nu)} \int_0^\infty y^{\alpha+k-1} (y-z)^{\alpha'+\nu-1} e^{-by} dy \tag{2.12}$$

for $z \leq 0$ where $b = (\beta^{-1} + (\beta')^{-1})/2$.

Using eq. 4, p. 319 of Gradshteyn and Ryzhik (1980) and noting that $\alpha'+\nu > 0$ and $bz > 0$ in (2.11), one can express the integral in (2.11) as follows

$$\begin{aligned} &b^{-(\alpha+k+\alpha'+\nu)/2} z^{(\alpha+k+\alpha'+\nu-2)/2} \Gamma(\alpha'+\nu) e^{-bz/2} \\ &\quad \times W_{(\alpha+k-\alpha'-\nu)/2, (1-\alpha-k-\alpha'-\nu)/2}(bz) \end{aligned}$$

for $z > 0$ where W$(\cdot)$ denotes Whittaker's function defined in (2.13).

Now letting $\rho = y + \xi$ where $\xi = -z$, the integral in (2.12) becomes

$$\begin{aligned} &\int_\xi^\infty (\rho-\xi)^{\alpha+k-1} \rho^{\alpha'+\nu-1} e^{-b(\rho-\xi)} d\rho \\ &\quad = e^{b\xi} \int_\xi^\infty (\rho-\xi)^{\alpha+k-1} \rho^{\alpha'+\nu-1} e^{-b\rho} d\rho \\ &\quad = e^{-bz} b^{-(\alpha+k+\alpha'+\nu)/2} (-z)^{(\alpha+k+\alpha'+\nu-2)/2} \Gamma(\alpha+k) e^{bz/2} \\ &\qquad \times W_{(-\alpha-k+\alpha'+\nu)/2, (1-\alpha-k-\alpha'-\nu)/2}(-bz) \end{aligned}$$

for $z \leq 0$.

When $z = 0$, the integral in (2.12) is equal to

$$b^{-(\alpha+k+\alpha'+\nu-1)} \Gamma(\alpha+k+\alpha'+\nu-1).$$

Making use of some identities from Sections 9.220 and 9.210.1 of Gradshteyn and Ryzhik (1980), one can express Whittaker's function as follows:

$$W_{l,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-l\right)} z^{\mu+(1/2)} e^{-z/2} {}_1F_1\left(\mu-l+\frac{1}{2}, 2\mu+1; z\right) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-l\right)} z^{-\mu+(1/2)} e^{-z/2} \times {}_1F_1\left(-\mu-l+\frac{1}{2}, -2\mu+1; z\right) \tag{2.13}$$

whenever the various quantities are defined, where for example

$$ {}_1F_1(d,\gamma,z) = \sum_{i=0}^{\infty} \frac{(d)_i z^i}{(\gamma)_i i!}$$

and $(a)_i = \Gamma(a+i)/\Gamma(a)$.

The exact density of Z is given in terms of Whittaker's function in the following theorem.

THEOREM 2.1. *Let $Z = U - V = \sum_{j=1}^{t} l_j T_j - \sum_{j=t+1}^{t+w} l_j T_j$ where the l_j's are positive real numbers and the T_j's are independent noncentral chi-square variables with α_i degrees of freedom and noncentrality parameter d_j, $j=1,\dots,t+w$. Let $\alpha = (\alpha_1+\cdots+\alpha_t)/2$, $\alpha' = (\alpha_{t+1}+\cdots+\alpha_{t+w})/2$, and $b = (\beta^{-1}+(\beta')^{-1})/2$, where β and β' are such that $|1-\beta/l_j|<1$, $j=1,\dots,t$, and $|1-\beta'/l_j|<1$, $j=t+1,\dots,t+w$. Then, provided α and α' are not both nonnegative integers plus $1/2$, the density of Z is given by*

$$f(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)} b^{-(\alpha+k+\alpha'+\nu)/2} z^{(\alpha+k+\alpha'+\nu-2)/2} \\ \qquad \times e^{z(\beta'^{-1}-\beta^{-1})/4} W_{(\alpha+k-\alpha'-\nu)/2,(1-\alpha'-v-\alpha-k)/2}(bz) & \text{for } z>0, \\ \displaystyle\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k\theta'_\nu}{\Gamma(\alpha'+\nu)} b^{-(\alpha+k+\alpha'+\nu)/2} (-z)^{(\alpha+k+\alpha'+\nu-2)/2} \\ \qquad \times e^{z(\beta'^{-1}-\beta^{-1})/4} W_{(\alpha'+\nu-\alpha-k)/2,(1-\alpha-k-\alpha'-\nu)/2}(-bz) & \text{for } z\le 0, \end{cases}$$

where $W_{l,\mu}(\cdot)$ denotes Whittaker's function defined in (2.13), and θ_k and θ'_ν are given in (2.7) and (2.9).

When α and α' are both nonnegative integers plus $1/2$, then $\mu-l+1/2$ is a negative integer plus $1/2$, $2\mu+1$ is a negative integer, and the above expressions for $f(z)$ both diverge since in that case the hypergeometric function ${}_1F_1(\mu-l+1/2, 2\mu+1; z)$ in the representation (2.13) of Whittaker's function diverges as $(\mu-l+1/2)_i/(2\mu+1)_i$ is then infinite for any $i \ge -2\mu$.

Theorem 2.1 also applies to linear combinations of central chi-square variables. In that case, the noncentrality parameters $d_j = 0$, $j = 1, \ldots, t+w$,

$$\text{(2.14)} \qquad \theta_k = a_k/(2\beta)^{\alpha+k}, \qquad a_0 = \prod_{j=1}^{t} \left(\frac{\beta}{l_j}\right)^{\alpha_j/2},$$

$$a_k = (2k)^{-1} \sum_{r=0}^{k-1} b_{k-r} a_r, \qquad b_k = \sum_{j=1}^{t} \alpha_j c_j^k, \qquad k = 1, 2, \ldots,$$

and

$$\text{(2.15)} \qquad \theta'_\nu = a'_\nu/(2\beta')^{\alpha'+\nu}, \qquad a'_0 = \prod_{j=t+1}^{t+w} \left(\frac{\beta'}{l_j}\right)^{\alpha_j/2}$$

$$a'_\nu = (2\nu)^{-1} \sum_{r=0}^{\nu-1} b'_{\nu-r} a'_r, \qquad b'_\nu = \sum_{j=t+1}^{t+w} \alpha_j c'^{\nu}_j, \qquad \nu = 1, 2, \ldots.$$

When α and α' are both nonnegative integers plus $1/2$, the following representation of Whittaker's function may be used:

$$\text{(2.16)} \quad W_{l,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+1/2} e^{-z/2}}{\Gamma\left(\frac{1}{2} - \mu - l\right) \Gamma\left(\frac{1}{2} + \mu - l\right)} \times \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - l + \frac{1}{2}\right)}{k!(2\mu+k)!} \Big[\psi(k+1) + \psi(2\mu+k+1) - \psi\left(\mu + k - l + \frac{1}{2}\right) - \ln(z)\Big] z^k + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu-k)\Gamma\left(k - \mu - l + \frac{1}{2}\right)}{k!} (-z)^k,$$

where the psi function can be evaluated by means of the following identity

$$\text{(2.17)} \qquad \psi(x) = \ln(x) - \sum_{\ell=0}^{\infty} \left[\frac{1}{x+\ell} - \ln\left(1 + \frac{1}{x+\ell}\right)\right],$$

see Gradshteyn and Ryzhik ((1980), p. 943 and p. 1063).

On substituting the representation (2.16) of Whittaker's function in the density of Z given in Theorem 2.1, one obtains the series representation given in the next theorem.

THEOREM 2.2. *When α and α' are both nonnegative integers plus $1/2$, the* *density function of Z as defined in Theorem 2.1 is given by*

$$f(z)=\begin{cases}
\displaystyle\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha'+v)}\frac{(-1)^{2\mu}e^{z/2\beta'}}{\Gamma\left(\frac{1}{2}-\mu-l\right)\Gamma\left(\frac{1}{2}+\mu-l\right)} \\
\displaystyle\times\left\{\sum_{i=0}^{\infty}\frac{\Gamma\left(-\mu+i-l+\frac{1}{2}\right)}{i!(-2\mu+i)!}b^i\left[\Bigg(\psi(i+1)+\psi(-2\mu+i+1)\right.\right. \\
\displaystyle-\psi\left(-\mu+i-l+\frac{1}{2}\right)-\ln(b)\Bigg)(-z)^{-2\mu+i} \\
\displaystyle\left.-\sum_{j=1}^{\infty}\sum_{n=0}^{j}\frac{(-1)^{n-1}}{n!(j-n)!}(-z)^{-2\mu+i+n}\right] \\
\displaystyle\left.+\sum_{i=0}^{-2\mu-1}\frac{\Gamma(-2\mu-i)\Gamma\left(i+\mu+l+\frac{1}{2}\right)}{i!}b^{2\mu+i}(-1)^{2\mu+i}(-z)^k\right\} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } z\le 0 \\
\displaystyle\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)}\frac{(-1)^{2\mu}e^{z/2\beta}}{\Gamma\left(\frac{1}{2}-\mu-l_2\right)\Gamma\left(\frac{1}{2}+\mu-l_2\right)} \\
\displaystyle\times\left\{\sum_{i=0}^{\infty}\frac{\Gamma\left(-\mu+i-l_2+\frac{1}{2}\right)}{i!(-2\mu+i)!}b^i\left[\Bigg(\psi(i+1)+\psi(-2\mu+i+1)\right.\right. \\
\displaystyle-\psi\left(-\mu+i-l_2+\frac{1}{2}\right)-\ln(b)\Bigg)z^{-2\mu+i} \\
\displaystyle\left.-\sum_{j=1}^{\infty}\sum_{n=0}^{j}\frac{(-1)^{n-1}}{n!(j-n)!}z^{-2\mu+i+n}\right] \\
\displaystyle\left.+\sum_{i=0}^{-2\mu-1}\frac{\Gamma(-2\mu-i)\Gamma\left(i+\mu+l_2+\frac{1}{2}\right)}{i!}b^{2\mu+i}(-1)^{2\mu+i}z^k\right\} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } z> 0
\end{cases}$$

where $\psi(\cdot)$ denotes Euler's psi function, θ_k and θ'_ν are respectively given in (2.7) *and* (2.9) *for the noncentral case and in* (2.14) *and* (2.15) *for the central case,* $l=-(\alpha+k-\alpha'-\nu)/2$, $\mu=(1-\alpha-k-\alpha'-\nu)/2$ *and* $l_2=(\alpha+k-\alpha'-\nu)/2$.

3. The exact distribution function

As explained in Section 2, indefinite quadratic forms in nonsingular normal vectors are distributed as the difference of two linear combinations of chi-square variables. Again we let

$$Z = U - V = \sum_{j=1}^{t} l_j T_j - \sum_{j=t+1}^{t+w} l_j T_j \tag{3.1}$$

where the l_j's are positive real numbers and the T_j's are independent noncentral chi-square variables with α_i degrees of freedom and noncentrality parameter d_j, $j = 1, \ldots, t+w$. Let $\alpha = \sum_{j=1}^{t} \alpha_j/2$ and $\alpha' = \sum_{j=t+1}^{t+w} \alpha_j/2$. It is first assumed that α and α' are not both negative integers plus $1/2$. For $z \leq 0$, the distribution function of Z denoted by $F(z)$ can be evaluated by integrating the representation of the density of Z given in Theorem 2.1 for negative values of z from $-\infty$ to z. Considering only the terms involving z and letting $l = -(\alpha + k - \alpha' - \nu)/2$ and $\mu = (1 - \alpha' - \nu - \alpha - k)/2$, we have

$$\begin{aligned}
&\int_{-\infty}^{z} (-s)^{-\mu-1/2} e^{s/2\beta'} \Bigg[(-bs)^{\mu+1/2} \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-l)} \sum_{i=0}^{\infty} \frac{(\mu - l + 1/2)_i}{(2\mu+1)_i} \frac{(-bs)^i}{i!} \\
&\qquad + (-bs)^{-\mu+1/2} \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-l)} \\
&\qquad \times \sum_{i=0}^{\infty} \frac{(-\mu - l + 1/2)_i}{(-2\mu+1)_i} \frac{(-bs)^i}{i!} \Bigg] ds \\
&= \sum_{i=0}^{\infty} \Bigg[\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - l\right)} \frac{\left(\mu - l + \frac{1}{2}\right)_i}{(2\mu+1)_i} \left(\frac{1}{i!}\right) \int_{-\infty}^{z} b^{\mu+1/2+i} e^{s/2\beta'} (-s)^i ds \\
&\qquad + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - l\right)} \frac{\left(-\mu - l + \frac{1}{2}\right)_i}{(-2\mu+1)_i} \left(\frac{1}{i!}\right) \\
&\qquad \times \int_{-\infty}^{z} b^{-\mu+1/2+i} e^{s/2\beta'} (-s)^{-2\mu+i} ds \Bigg].
\end{aligned}$$

Now letting

$$\begin{aligned}
I_{1i} &= \Gamma\left(i+1, -\frac{z}{2\beta'}\right) (2\beta')^{(i+1)} \quad \text{and} \\
I_{2i} &= \Gamma\left(-2\mu + i + 1, -\frac{z}{2\beta'}\right) (2\beta')^{(-2\mu+i+1)},
\end{aligned} \tag{3.2}$$

where

$$\Gamma(a, x) = \int_{x}^{\infty} e^{-t} t^{a-1} dt = \Gamma(a) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{n!(a+n)},$$

the distribution function of Z can be represented as follows:

$$
\begin{aligned}
(3.3)\qquad F(z) = {} & \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_k \theta'_\nu}{\Gamma(\alpha' + v)} b^{\mu-1/2} \\
& \times \left[\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - l\right)} \frac{\left(\mu - l + \frac{1}{2}\right)_i}{(2\mu+1)_i} \frac{b^{\mu+1/2+i}}{i!} I_{1i} \right. \\
& \left. + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - l\right)} \frac{\left(-\mu - l + \frac{1}{2}\right)_i}{(-2\mu+1)_i} \frac{b^{-\mu+1/2+i}}{i!} I_{2i} \right].
\end{aligned}
$$

For $z > 0$, the distribution function of Z denoted by $F(z)$ can be evaluated by adding $F(0)$ to the integral from 0 to z of the representation of the density of Z given in Theorem 2.1 for positive values of z. Considering only the terms involving z and letting $l_2 = (\alpha + k - \alpha' - \nu)/2$ and $\mu = (1 - \alpha' - \nu - \alpha - k)/2$, we have

$$
\begin{aligned}
& \int_0^z (s)^{-\mu-1/2} e^{-s/2\beta} \left[(bs)^{\mu+1/2} \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - l_2)} \sum_{i=0}^{\infty} \frac{(\mu - l_2 + 1/2)_i}{(2\mu+1)_i} \frac{(bs)^i}{i!} \right. \\
& \qquad + (bs)^{-\mu+1/2} \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - l_2)} \\
& \qquad \left. \times \sum_{i=0}^{\infty} \frac{(-\mu - l_2 + 1/2)_i}{(-2\mu+1)_i} \frac{(bs)^i}{i!} \right] ds \\
& = \sum_{i=0}^{\infty} \left[\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - l_2\right)} \frac{\left(\mu - l_2 + \frac{1}{2}\right)_i}{(2\mu+1)_i} \left(\frac{1}{i!}\right) \int_0^z b^{\mu+1/2+i} e^{-s/2\beta} (s)^i ds \right. \\
& \qquad + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - l_2\right)} \frac{\left(-\mu - l_2 + \frac{1}{2}\right)_i}{(-2\mu+1)_i} \left(\frac{1}{i!}\right) \\
& \qquad \left. \times \int_0^z b^{-\mu+1/2+i} e^{-s/2\beta} (s)^{-2\mu+i} ds \right].
\end{aligned}
$$

Now letting

$$
(3.4)\qquad
\begin{aligned}
I_{3i} &= \gamma\left(i+1, \frac{z}{2\beta}\right)(2\beta)^{(i+1)} \qquad \text{and} \\
I_{4i} &= \gamma\left(-2\mu + i + 1, \frac{z}{2\beta}\right)(2\beta)^{(-2\mu+i+1)}
\end{aligned}
$$

where the incomplete gamma function,

$$\gamma(a,x)=\int_0^x e^{-t}t^{a-1}dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{a+n}}{n!(a+n)}, \tag{3.5}$$

the distribution function of Z can be represented as follows:

$$\begin{aligned}F(z)=F(0)+\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{i=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)}b^{\mu-1/2}&\\ \times\Bigg[\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-l_2\right)}\frac{\left(\mu-l_2+\frac{1}{2}\right)_i}{(2\mu+1)_i}\frac{b^{\mu+1/2+i}}{i!}I_{3i}&\\ +\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-l_2\right)}\frac{\left(-\mu-l_2+\frac{1}{2}\right)_i}{(-2\mu+1)_i}\frac{b^{-\mu+1/2+i}}{i!}I_{4i}\Bigg],&\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}F(0)=\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{i=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha'+v)}b^{\mu-1/2}&\\ \times\Bigg[\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-l\right)}\frac{\left(\mu-l+\frac{1}{2}\right)_i}{(2\mu+1)_i}b^{\mu+1/2+i}(2\beta')^{(i+1)}&\\ +\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-l\right)}\frac{\left(-\mu-l+\frac{1}{2}\right)_i}{(-2\mu+1)_i}\frac{b^{-\mu+1/2+i}}{i!}&\\ \times\Gamma(-2\mu+i+1)(2\beta')^{(-2\mu+i+1)}\Bigg].&\end{aligned} \tag{3.7}$$

THEOREM 3.1. *Provided α and α' are not both nonnegative integers plus 1/2, the distribution function of Z as defined in Theorem 2.1 is given by (3.3) when $z<0$ and by (3.6) when $z\geq 0$.*

In order to obtain the distribution function for the case where α and α' are both nonnegative integers plus $1/2$, we integrate the representations of the density function given in Theorem 2.2. The resulting expressions are given in the next theorem.

THEOREM 3.2. *When α and α' are both nonnegative integers plus 1/2, the distribution function of Z as defined in Theorem 2.1 is given by*

$$F(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha'+v)}\frac{(-1)^{2\mu}}{\Gamma\left(\frac{1}{2}-\mu-l\right)\Gamma\left(\frac{1}{2}+\mu-l\right)} \\ \displaystyle\times\left\{\sum_{i=0}^{\infty}\frac{\Gamma\left(-\mu+i-l+\frac{1}{2}\right)}{i!(-2\mu+i)!}b^i\left[\left(\psi(i+1)+\psi(-2\mu+i+1)\right.\right.\right. \\ \displaystyle\left.-\psi\left(-\mu+i-l+\frac{1}{2}\right)-\ln(b)\right)\Gamma\left(-2\mu+i+1,-\frac{z}{2\beta'}\right)(2\beta')^{(-2\mu+i+1)} \\ \displaystyle\left.-\sum_{j=1}^{\infty}\sum_{n=0}^{j}\frac{(-1)^{n-1}}{n!(j-n)!}\Gamma\left(-2\mu+i+n+1,-\frac{z}{2\beta'}\right)(2\beta')^{(-2\mu+i+n+1)}\right] \\ \displaystyle+\sum_{i=0}^{-2\mu-1}\frac{\Gamma(-2\mu-i)\Gamma\left(i+\mu+l+\frac{1}{2}\right)}{i!}b^{2\mu+i}(-1)^{2\mu+i} \\ \displaystyle\left.\times\,\Gamma\left(i+1,-\frac{z}{2\beta'}\right)(2\beta')^{(i+1)}\right\} & \textit{for } z \le 0 \\ \displaystyle F(0)+\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)}\frac{(-1)^{2\mu}}{\Gamma\left(\frac{1}{2}-\mu-l_2\right)\Gamma\left(\frac{1}{2}+\mu-l_2\right)} \\ \displaystyle\times\left\{\sum_{i=0}^{\infty}\frac{\Gamma\left(-\mu+i-l_2+\frac{1}{2}\right)}{i!(-2\mu+i)!}b^i\left[\left(\psi(i+1)+\psi(-2\mu+i+1)\right.\right.\right. \\ \displaystyle\left.-\psi\left(-\mu+i-l_2+\frac{1}{2}\right)-\ln(b)\right)\gamma\left(-2\mu+i+1,\frac{z}{2\beta}\right)(2\beta)^{(-2\mu+i+1)} \\ \displaystyle\left.-\sum_{j=1}^{\infty}\sum_{n=0}^{j}\frac{(-1)^{n-1}}{n!(j-n)!}\gamma\left(-2\mu+i+n+1,\frac{z}{2\beta}\right)(2\beta)^{(-2\mu+i+n+1)}\right] \\ \displaystyle+\sum_{i=0}^{-2\mu-1}\frac{\Gamma(-2\mu-i)\Gamma\left(i+\mu+l_2+\frac{1}{2}\right)}{i!}b^{2\mu+i}(-1)^{2\mu+i} \\ \displaystyle\left.\times\,\gamma\left(i+1,\frac{z}{2\beta}\right)(2\beta)^{(i+1)}\right\} & \textit{for } z > 0, \end{cases}$$

where $\psi(\cdot)$ and $\gamma(\cdot,\cdot)$ respectively denote the psi function and the incomplete *gamma function*,

$$F(0) = \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\theta_k \theta'_\nu}{\Gamma(\alpha' + v)} \frac{(-1)^{2\mu}}{\Gamma\left(\frac{1}{2} - \mu - l\right)\Gamma\left(\frac{1}{2} + \mu - l\right)}$$

$$\times \left\{ \sum_{i=0}^{\infty} \frac{\Gamma\left(-\mu + i - l + \frac{1}{2}\right)}{i!(-2\mu + i)!} b^i \right.$$

$$\times \left[\left(\psi(i+1) + \psi(-2\mu + i + 1) \right.\right.$$

$$\left. - \psi\left(-\mu + i - l + \frac{1}{2}\right) - \ln(b) \right) \Gamma(-2\mu + i + 1)(2\beta')^{(-2\mu+i+1)}$$

$$\left. - \sum_{j=1}^{\infty}\sum_{n=0}^{j} \frac{(-1)^{n-1}}{n!(j-n)!} \Gamma(-2\mu + i + n + 1)(2\beta')^{(-2\mu+i+n+1)} \right]$$

$$+ \sum_{i=0}^{-2\mu-1} \frac{\Gamma(-2\mu - i)\Gamma\left(i + \mu + l + \frac{1}{2}\right)}{i!} b^{2\mu+i}(-1)^{2\mu+i}$$

$$\left. \times \Gamma(i+1)(2\beta')^{(i+1)} \right\},$$

$l = -(\alpha + k - \alpha' - \nu)/2$, $l_2 = (\alpha + k - \alpha' - \nu)/2$ *and* $\mu = (1 - \alpha' - \nu - \alpha - k)/2$.

The distribution function of Z for the central case is obtained by taking θ_k and θ'_ν as defined in (2.14) and (2.15) respectively.

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