Ann. Inst. Statist. Math.
Vol. 48, No. 2, 365–380 (1996)

THIRD ORDER EFFICIENCY IMPLIES FOURTH ORDER EFFICIENCY: A RESOLUTION OF THE CONJECTURE OF J. K. GHOSH

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(Received September 8, 1994; revised June 15, 1995)

Abstract. Under suitable regularity conditions, it is shown that a third order asymptotically efficient estimator is fourth order asymptotically efficient in some class of estimators in the sense that the estimator has the most concentration probability in any symmetric interval around the true parameter up to the fourth order in the class. This is a resolution of the conjecture by Ghosh (1994, *Higher Order Asymptotics*, Institute of Mathematical Statistics, Hayward, California). It is also shown that the bias-adjusted maximum likelihood estimator is fourth order asymptotically efficient in the class.

Key words and phrases: Asymptotically median unbiasedness, fourth order asymptotically symmetric efficiency, concentration probability, asymptotic cumulants, Edgeworth expansion, maximum likelihood estimator.

1. Introduction

In higher order asymptotics, the phenomenon "first order efficiency implies second order efficiency" is well known (see, e.g. Akahira and Takeuchi (1976, 1981), Takeuchi and Akahira (1976), Pfanzagl (1979) and Bickel *et al.* (1981)). Under the regularity conditions it is also shown that a second order asymptotically efficient estimator is third order asymptotically efficient in some class of estimators, and that a modified maximum likelihood estimator is third order asymptotically efficient in the class (see, e.g. Akahira and Takeuchi (1981), Akahira (1986), Pfanzagl and Wefelmeyer (1985), Ghosh (1994), Amari (1985)). Further, the asymptotic deficiency of asymptotically efficient estimators has been investigated by Ghosh and Subramanyam (1974), Akahira (1986) and others.

Recently, the other phenomenon "third order efficiency implies fourth order efficiency" has been stated by Ghosh ((1994), p. 64) as a conjecture. The purpose of this paper is to give a resolution of the conjecture. Under appropriate regularity conditions, it is shown that a third order asymptotically efficient estimator is fourth order asymptotically symmetrically efficient in some class of estimators in the sense that the estimator has the most concentration probability in any symmetric interval around the true parameter up to the fourth order in the class. In the above, replacing "any symmetric interval" with "any interval", the fact does not generally hold. It is also shown that a bias-adjusted maximum likelihood estimator is fourth order asymptotically symmetrically efficient in the class.

2. Notations and assumptions

Suppose that $X_1, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) real random variables according to a density function $f(x, \theta)$ with respect to a σ-finite measure μ, where θ is a parameter in an open interval Θ of $\boldsymbol{R}^1$. We denote by $P_{\theta,n}$ the n-fold direct product of probability measure P_θ with the density $f(x, \theta)$. An estimator $\hat{\theta}_n(X_1, \ldots, X_n)$ of θ based on $X_1, \ldots, X_n$ is simply denoted by $\hat{\theta}_n$.

DEFINITION 2.1. For each $k = 1, 2, \ldots$, a $\sqrt{n}$-consistent estimator $\hat{\theta}_n$ is called k-th order asymptotically median unbiased (or k-th order AMU for short) if, for any $\eta \in \Theta$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\eta|<\delta} n^{(k-1)/2} \left| P_{\theta,n}\{\hat{\theta}_n \leq \theta\} - \frac{1}{2} \right| = 0,$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\eta|<\delta} n^{(k-1)/2} \left| P_{\theta,n}\{\hat{\theta}_n \geq \theta\} - \frac{1}{2} \right| = 0.$$

For each $k = 1, 2, \ldots$, denote by $\boldsymbol{A}_k$ the class of all the k-th order AMU estimators, and let $\boldsymbol{S}_k$ be a subclass of $\boldsymbol{A}_k$. If an estimator $\hat{\theta}_n$ belongs to the class $\boldsymbol{S}_k$, then we call it $\boldsymbol{S}_k$-estimator.

DEFINITION 2.2. For each $k = 1, 2, \ldots$, a $\boldsymbol{S}_k$-estimator $\hat{\theta}_n^*$ is called k-th order asymptotically (as.) efficient in the class $\boldsymbol{S}_k$ if for any $\boldsymbol{S}_k$-estimator $\hat{\theta}_n$

$$P_{\theta,n}\{-a \leq \sqrt{n}(\hat{\theta}_n^* - \theta) \leq b\} \geq P_{\theta,n}\{-a \leq \sqrt{n}(\hat{\theta}_n - \theta) \leq b\} + o(n^{-(k-1)/2}) \tag{2.1}$$

for all $a \geq 0$, all $b \geq 0$ and all $\theta \in \Theta$.

This means that the estimator $\hat{\theta}_n^*$ has the most concentration probability in any interval around θ up to the k-th order, i.e., the order $n^{-(k-1)/2}$ in the class $\boldsymbol{S}_k$.

DEFINITION 2.3. For each $k = 1, 2, \ldots$, a $\boldsymbol{S}_k$-estimator $\hat{\theta}_n^*$ is called k-th order as. symmetrically efficient in the class $\boldsymbol{S}_k$ if for any $\boldsymbol{S}_k$-estimator $\hat{\theta}_n$

$$P_{\theta,n}\{\sqrt{n}|\hat{\theta}_n^* - \theta| \leq a\} \geq P_{\theta,n}\{\sqrt{n}|\hat{\theta}_n - \theta| \leq a\} + o(n^{-(k-1)/2}) \tag{2.2}$$

for all $a \geq 0$ and all $\theta \in \Theta$.

This means that the estimator $\hat{\theta}_n^*$ has the most concentration probability in any symmetric interval around θ up to the k-th order in the class $\boldsymbol{S}_k$. Note that the k-th order in the above definitions corresponds to $(k+1)/2$-th order in the terminology due to Rao (1961) and Fisher (1925) (see, e.g. Akahira (1986), Ghosh (1994)).

It is known that a bias-adjusted best asymptotically normal (BAN) estimator in the class $\boldsymbol{A}_2$ is second order as. efficient in $\boldsymbol{A}_2$, but it in the class $\boldsymbol{A}_3$ is not third order as. efficient in $\boldsymbol{A}_3$ (see, e.g. Akahira and Takeuchi (1981)). In order to consider the third order asymptotic efficiency, it may be necessary to confine the estimators to some class. Now, we discuss some restricted classes of estimators. We assume that, for almost all $x[\mu]$, $f(x,\theta)$ is twice continuously differentiable in θ. We put

$$Z_1(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i, \theta),$$

$$Z_2(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\theta^2} \log f(X_i, \theta) + I(\theta) \right\},$$

where $I(\theta)$ designates the amount of the Fisher information on f, i.e.

$$I(\theta) = E_\theta \left[\left\{ \frac{\partial}{\partial\theta} \log f(X, \theta) \right\}^2 \right].$$

Let $\boldsymbol{C}$ be the class of all the bias-adjusted BAN estimators $\hat{\theta}_n$ which are third order AMU and asymptotically expanded as

$$\sqrt{n}(\hat{\theta}_n - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{1}{\sqrt{n}} Q(\theta) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{2.3}$$

where $Q(\theta) = O_p(1)$, and the distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ admits the Edgeworth expansion up to the order $o(n^{-1})$. We call the class $\boldsymbol{D}$ the class of $\boldsymbol{C}$-estimators which satisfy $E_\theta[Z_1(\theta)Q^2(\theta)] = o(1)$. Note that the class $\boldsymbol{D}$ is a subclass of $\boldsymbol{C}$. Then it is shown that the bias-adjusted maximum likelihood estimator (MLE) is third order as. symmetrically efficient in the class $\boldsymbol{C}$ (see, e.g. Akahira (1986) and Akahira and Takeuchi (1981)). In order to discuss the fourth order asymptotic efficiency we assume the following conditions.

(A1) $\{x : f(x,\theta) > 0\}$ does not depend on θ.

(A2) For almost all $x[\mu]$, $f(x,\theta)$ is six times continuously differentiable in θ. For each $\eta \in \Theta$, there exist a compact neighborhood K_η of η and functions $g(x)$, $h(x)$ satisfying $|\ell^{(i)}(\theta,x)| \le g(x)$ $(i = 1,2,3,4,5)$, $|\ell^{(6)}(\theta,x)| \le h(x)$ for $\theta \in K_\eta$ and $\sup_{\theta\in K_\eta} E_\theta[g^5(X)] < \infty$, $\sup_{\theta\in K_\eta} E_\theta[h(X)] < \infty$, where $\ell^{(i)}(\theta,x) = (\partial^i/\partial\theta^i)\ell(\theta,x)$ with $\ell(\theta,x) = \log f(x,\theta)$.

(A3) For each $\theta \in \Theta$

$$0 < I(\theta) = E_\theta[\{\ell^{(1)}(\theta,X)\}^2] = -E_\theta[\ell^{(2)}(\theta,X)] < \infty.$$

(A4) There exist

$$J(\theta) = E_\theta[\ell^{(1)}(\theta, X)\ell^{(2)}(\theta, X)], \qquad K(\theta) = E_\theta[\{\ell^{(1)}(\theta, X)\}^3],$$
$$L(\theta) = E_\theta[\ell^{(1)}(\theta, X)\ell^{(3)}(\theta, X)], \qquad M(\theta) = E_\theta[\{\ell^{(2)}(\theta, X)\}^2] - I^2(\theta),$$
$$N(\theta) = E_\theta[\{\ell^{(1)}(\theta, X)\}^2\ell^{(2)}(\theta, X)] + I^2(\theta),$$
$$H(\theta) = E_\theta[\{\ell^{(1)}(\theta, X)\}^4] - 3I^2(\theta),$$

and $E_\theta[\ell^{(3)}(\theta, X)] = -3J(\theta) - K(\theta)$, $E_\theta[\ell^{(4)}(\theta, X)] = -4L(\theta) - 3M(\theta) - 6N(\theta) - H(\theta)$.

We put

$$Z_3(\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\ell^{(3)}(\theta, X_i) + 3J(\theta) + K(\theta)\}.$$

Let $\boldsymbol{F}$ be the class of all the bias-adjusted BAN estimators $\hat{\theta}_n$ which are fourth order AMU and asymptotically expanded as

$$\sqrt{n}(\hat{\theta}_n - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{1}{\sqrt{n}}Q(\theta) + \frac{1}{n}R(\theta) + O_p\left(\frac{1}{n\sqrt{n}}\right), \tag{2.4}$$

where $Q(\theta) = O_p(1)$, $E_\theta[Z_1 Q^2(\theta)] = O(1/\sqrt{n})$, $E_\theta[Z_2 Q(\theta)] = O(1/\sqrt{n})$, $E_\theta[Z_2 Q^2(\theta)] = o(1)$, $E_\theta[Z_3 Q(\theta)] = o(1)$, $R(\theta) = O_p(1)$, $\partial R(\theta)/\partial\theta = O_p(\sqrt{n})$, $\mathrm{Cov}_\theta(Q, R) = o(1)$, and the distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ admits the Edgeworth expansion up to the order $o(n^{-3/2})$, where $\mathrm{Cov}_\theta(Q, R)$ designates the asymptotic covariance of Q and R. Note that the class $\boldsymbol{F}$ is a subclass of the class $\boldsymbol{D}$. If the underlying distribution belongs to an exponential family of distributions, then it is enough to consider estimators based on a sufficient statistic as follows. Suppose that $X_1, \dots, X_n$ are i.i.d. random variables according to an exponential type distribution with the density (w.r.t. a σ-finite measure μ)

$$f(x, \theta) = h(x)c(\theta)\exp\left\{\sum_{i=1}^{m} s_i(\theta)t_i(x)\right\},$$

where θ is a real valued parameter which belongs to an open interval $\Theta(\subset \boldsymbol{R}^1)$, and $s_i(\theta)$'s are continuously differentiable real valued functions of θ and $t_i(x)$'s are real valued measurable functions of x. We assume that $s_1(\theta), \dots, s_m(\theta)$ are linearly independent as θ varies in Θ, and also that $t_1(x), \dots, t_m(x)$ are linearly independent functions of x. For the sample $(X_1, \dots, X_n)$, a sufficient statistic is given by $T = (T_1, \dots, T_m)$ with $T_i = (1/n)\sum_{j=1}^{n} t_i(X_j)$ $(i = 1, \dots, m)$. Then, we define an estimator called an extended regular (ER) estimator which can be expressed as

$$\hat{\theta} = g(T_1, \dots, T_m) + \frac{1}{n}h_1(T_1, \dots, T_m) + \frac{1}{n^2}h_2(T_1, \dots, T_m),$$

where g, h_1 and h_2 are smooth functions independent of n. If we consider a class of ER best asymptotically normal estimators, then it is seen in a similar way to Section 5.1 of Akahira and Takeuchi (1981) that the class satisfies the conditions of the class $\boldsymbol{F}$.

In the subsequent discussion, we consider the fourth order asymptotic efficiency of the estimators in the class $\boldsymbol{F}$ under the conditions (A1) to (A4).

3. Higher order asymptotic cumulants of F-estimators and the Edgeworth expansion

In this section, we obtain the asymptotic cumulants of $\boldsymbol{F}$-estimators and the Edgeworth expansion of the distribution of $\boldsymbol{F}$-estimators up to the order $o(n^{-3/2})$. In order to do so, we use the following convenient lemma.

LEMMA 3.1. *Suppose that Y_θ is a function of $X_1, \dots, X_n$ and θ and is differentiable in θ. Then*

$$E_\theta(Z_1 Y_\theta) = \frac{1}{\sqrt{n}} \frac{d}{d\theta} E_\theta(Y_\theta) - \frac{1}{\sqrt{n}} E_\theta\left(\frac{\partial Y_\theta}{\partial \theta}\right),$$
$$E_\theta(Z_1^2 Y_\theta) = \frac{1}{\sqrt{n}} \frac{d}{d\theta} E_\theta(Z_1 Y_\theta) - \frac{1}{\sqrt{n}} E_\theta\left(Y_\theta \frac{\partial Z_1}{\partial \theta}\right) - \frac{1}{\sqrt{n}} E_\theta\left(Z_1 \frac{\partial Y_\theta}{\partial \theta}\right),$$

provided that the differentiation under the integral sign of $E_\theta(Y_\theta)$ and $E_\theta(Z_1 Y_\theta)$ is allowed.

The proof is omitted since it is given in Akahira and Takeuchi ((1981), p. 140, p. 145) and Akahira ((1986), p. 41, p. 42). Using Lemma 3.1 we see that, for a $\boldsymbol{F}$-estimator $\hat{\theta}_n$ with (2.4),

$$E_\theta[Z_1(\theta) Q(\theta)] = O(1/\sqrt{n}), \tag{3.1}$$

if the differentiation under the integral sign of $E_\theta[Q(\theta)]$ is allowed. This can be proven in a similar way to Lemma 2.1.3 of Akahira (1986) and Lemma 5.2.2 of Akahira and Takeuchi (1981).

THEOREM 3.1. *Assume that the conditions* (A1) *to* (A4) *hold. Let $\hat{\theta}_n$ be a $\boldsymbol{F}$-estimator with* (2.4). *Then its asymptotic cumulants up to the order $o(n^{-3/2})$ are given as follows: For $T_n = \sqrt{n}(\hat{\theta}_n - \theta)$,*

$$\kappa_1 = E_\theta(T_n) = \frac{\mu_1(\theta)}{\sqrt{n}} + \frac{\mu_3(\theta)}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right),$$
$$\kappa_2 = V_\theta(T_n) = \frac{1}{I} + \frac{2\mu_1'(\theta)}{In} + \frac{\tau(\theta)}{n} + o\left(\frac{1}{n\sqrt{n}}\right),$$
$$\begin{aligned}\kappa_3 = \kappa_{3,\theta}(T_n) &= E_\theta[\{T_n - E_\theta(T_n)\}^3] \\ &= \frac{\beta_3(\theta)}{\sqrt{n}} + \frac{\gamma^*(\theta)}{n\sqrt{n}} + \frac{\eta^*(\theta)}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right),\end{aligned}$$
$$\begin{aligned}\kappa_4 = \kappa_{4,\theta}(T_n) &= E_\theta[\{T_n - E_\theta(T_n)\}^4] - 3\{V_\theta(T_n)\}^2 \\ &= \frac{\beta_4(\theta)}{n} + o\left(\frac{1}{n\sqrt{n}}\right),\end{aligned}$$
$$\begin{aligned}\kappa_5 = \kappa_{5,\theta}(T_n) &= E_\theta[\{T_n - E_\theta(T_n)\}^5] - 10 V_\theta(T_n)\kappa_{3,\theta}(T_n) \\ &= \frac{\beta_5(\theta)}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right),\end{aligned}$$
$$\kappa_i = \kappa_{i,\theta}(T_n) = o\left(\frac{1}{n\sqrt{n}}\right) \qquad (i \geq 6),$$

where

$$\mu_1(\theta) = \frac{1}{6}I(\theta)\beta_3(\theta) \quad \textit{with} \quad \beta_3(\theta) = -\frac{1}{I^3(\theta)}\{3J(\theta)+2K(\theta)\},$$

$$\begin{aligned}\mu_1'(\theta) = \frac{1}{6}\Big[&(2J(\theta)+K(\theta))\beta_3(\theta)\\ &-\frac{1}{I^2(\theta)}\{3(L(\theta)+M(\theta)+N(\theta))+2(H(\theta)+3N(\theta))\}\\ &+\frac{3}{I^3(\theta)}\{2J(\theta)+K(\theta)\}\{3J(\theta)+2K(\theta)\}\Big],\end{aligned}$$

$$\tau(\theta) = V_\theta(Q(\theta)), \qquad \gamma^*(\theta) = \frac{3\sqrt{n}}{2I(\theta)}E_\theta[Z_1(\theta)Q^2(\theta)] + o(1),$$

$$\begin{aligned}\eta^*(\theta) = &\frac{3}{I^3(\theta)}\{\mu_1''(\theta)I(\theta) - \mu_1'(\theta)(2J(\theta)+K(\theta))\} + \frac{3\tau'(\theta)}{2I(\theta)}\\ &+\frac{3}{I(\theta)}E_\theta[Z_1(\theta)(Q(\theta)-\mu_1(\theta))R(\theta)] + E_\theta[\{Q(\theta)-\mu_1(\theta)\}^3],\end{aligned}$$

$$\begin{aligned}\beta_4(\theta) = &\frac{12}{I^5(\theta)}\{2J(\theta)+K(\theta)\}\{J(\theta)+K(\theta)\}\\ &-\frac{1}{I^4(\theta)}\{3H(\theta)+4L(\theta)+12N(\theta)\},\end{aligned}$$

and $\beta_5(\theta)$ *and* $\mu_3(\theta)$ *are certain functions of* θ *and* $\tau'(\theta) = d\tau(\theta)/d\theta$, $\mu_1''(\theta) = (d^2/d\theta^2)\mu_1(\theta)$.

The proof is given in Section 5 along a line of those of Theorem 2.1.1 of Akahira (1986) and Theorem 3.1 of Akahira and Takeuchi (1989).

Remark 3.1. It is noted from Theorem 3.1 and its proof that the terms $\tau(\theta)$, $\gamma^*(\theta)$, $\eta^*(\theta)$ and $\mu_3(\theta)$ in the above cumulants depend on the estimator $\hat{\theta}_n$. It is not necessarily clear whether $\beta_5(\theta)$ depends on the estimator or not, but this does not affect the subsequent discussion. It is still an unsolved problem. It is seen from Theorem 2.1.1 of Akahira (1986) and Theorem 3.1 of Akahira and Takeuchi (1989) that the structure of a $\boldsymbol{F}$-estimator is quite similar to that of a $\boldsymbol{C}$-estimator in the sense that the term of n^{-1} in the third order cumulant of the $\boldsymbol{C}$-estimator is replaced with the term of $n^{-3/2}$ in that of the $\boldsymbol{F}$-estimator.

THEOREM 3.2. *Assume that the conditions* (A1) *to* (A4) *hold. Let* $\hat{\theta}_n$ *be a* $\boldsymbol{F}$*-estimator with* (2.4). *Then the Edgeworth expansion of the distribution of* $\hat{\theta}_n$ *up to the order* $o(n^{-3/2})$ *is given by*

$$\begin{aligned}&P_{\theta,n}\{\sqrt{nI}(\hat{\theta}_n-\theta)\le t\}\\ &\quad= \Phi(t) - \frac{I^{3/2}\beta_3}{6\sqrt{n}}t^2\phi(t) - \frac{I\tau+2\mu_1'}{2n}t\phi(t) - \frac{I\mu_1^2}{2n}t\phi(t) - \frac{I^2\beta_4}{24n}(t^3-3t)\phi(t)\\ &\qquad - \frac{I^3\beta_3^2}{72n}(t^5-10t^3+15t)\phi(t) - \frac{I^2\mu_1\beta_3}{6n}(t^3-3t)\phi(t)\end{aligned}$$

$$
\begin{aligned}
&-\frac{I^{3/2}(\gamma^*+\eta^*)}{6n\sqrt{n}}t^2\phi(t)\\
&-\frac{\sqrt{I}\mu_1(I\tau+2\mu_1')}{2n\sqrt{n}}t^2\phi(t)-\frac{I^{3/2}\mu_1^3}{6n\sqrt{n}}t^2\phi(t)-\frac{I^{5/2}(\tau+\mu_1^2)\beta_3}{12n\sqrt{n}}(t^4-6t^2)\phi(t)\\
&-\frac{I^{3/2}\mu_1'\beta_3}{6n\sqrt{n}}(t^4-6t^2)\phi(t)-\frac{I^{5/2}\mu_1\beta_4}{24n\sqrt{n}}(t^4-6t^2)\phi(t)\\
&-\frac{I^{5/2}\beta_5}{120n\sqrt{n}}(t^4-6t^2)\phi(t)-\frac{I^{7/2}\mu_1\beta_3^2}{72n\sqrt{n}}(t^6-15t^4+45t^2)\phi(t)\\
&-\frac{I^{7/2}\beta_3\beta_4}{144n\sqrt{n}}(t^6-15t^4+45t^2)\phi(t)\\
&-\frac{I^{9/2}\beta_3^3}{1296n\sqrt{n}}(t^8-28t^6+210t^4-420t^2)\phi(t)\\
&+o\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

where $\Phi(t)=\int_{-\infty}^t\phi(x)dx$ *with* $\phi(x)=(1/\sqrt{2\pi})e^{-x^2/2}$.

The proof is given in Section 5.

4. Fourth order asymptotic efficiency of F-estimators

Using Theorem 3.2 we can derive the phenomenon "third order efficiency implies fourth order efficiency".

THEOREM 4.1. *Assume that the conditions* (A1) *to* (A4) *hold. If a* $\boldsymbol{F}$*-estimator* $\hat\theta_n^*$ *is third order as. symmetrically efficient in the class* $\boldsymbol{D}$, *i.e. for any* $\boldsymbol{D}$*-estimator* $\hat\theta_n$

$$
P_{\theta,n}\{\sqrt{n}|\hat\theta_n^*-\theta|\le a\}\ge P_{\theta,n}\{\sqrt{n}|\hat\theta_n-\theta|\le a\}+o(n^{-1}) \tag{4.1}
$$

for all $a\ge 0$ *and all* $\theta\in\Theta$, *then* $\hat\theta_n^*$ *is fourth order as. symmetrically efficient in the class* $\boldsymbol{F}$, *i.e. for any* $\boldsymbol{F}$*-estimator* $\hat\theta_n$

$$
P_{\theta,n}\{\sqrt{n}|\hat\theta_n^*-\theta|\le a\}\ge P_{\theta,n}\{\sqrt{n}|\hat\theta_n-\theta|\le a\}+o(n^{-3/2})
$$

for all $a\ge 0$ *and all* $\theta\in\Theta$.

Remark 4.1. If a $\boldsymbol{F}$-estimator is third order as. efficient in the class $\boldsymbol{D}$ then it is third order as. symmetrically efficient in $\boldsymbol{D}$, hence (4.1) can be replaced with

$$
P_{\theta,n}\{-a\le\sqrt{n}(\hat\theta_n^*-\theta)\le b\}\ge P_{\theta,n}\{-a\le\sqrt{n}(\hat\theta_n-\theta)\le b\}+o(n^{-1})
$$

for all $a\ge 0$, all $b\ge 0$ and all $\theta\in\Theta$.

PROOF. From Theorem 3.1 we see that the only term $\tau(\theta)$ depends on the estimator in its asymptotic cumulants up to the order $o(n^{-1})$. From Theorem 3.2 it follows that, in the Edgeworth expansion of the distribution $P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n-\theta)\le t\}$ of any $\boldsymbol{F}$-estimator $\hat{\theta}_n$ up to the order $o(n^{-3/2})$, the only following depends on the estimator

$$
\begin{aligned}
(4.2)\qquad & -\frac{I\tau}{2n}t\phi(t)-\frac{I^{3/2}(\gamma^*+\eta^*)}{6n\sqrt{n}}t^2\phi(t)-\frac{I^{3/2}\mu_1\tau}{2n\sqrt{n}}t^2\phi(t)\\
&-\frac{I^{5/2}\tau\beta_3}{12n\sqrt{n}}(t^4-6t^2)\phi(t)-\frac{I^{5/2}\beta_5}{120n\sqrt{n}}(t^4-6t^2)\phi(t).
\end{aligned}
$$

Since all terms except for the first one in (4.2) are even functions of t, it follows that

$$
P_{\theta,n}\{\sqrt{In}|\hat{\theta}_n-\theta|\le a\}=\cdots-\frac{I\tau}{n}a\phi(a)+\cdots+o\left(\frac{1}{n\sqrt{n}}\right),
$$

where the terms "$\cdots$" are independent of the estimator. Since $\hat{\theta}_n^*$ minimizes $\tau=V_\theta(Q)$ in the class $\boldsymbol{D}$, it also does τ in the class $\boldsymbol{F}$, that is, $\hat{\theta}_n^*$ is fourth order as. symmetrically efficient in the class $\boldsymbol{F}$. Thus, we complete the proof.

Remark 4.2. The structure of the proof of Theorem 4.1 is similar to that of Pfanzagl (1979) showing first order efficiency implies second order efficiency. If there exists a third order as. symmetrically efficient estimator in the class $\boldsymbol{D}$, then, in order to prove its fourth order as. symmetric efficiency in the class $\boldsymbol{F}$, from Theorem 4.1 it is seen to be enough to show that the estimator belongs to the class $\boldsymbol{F}$. Since, in the Edgeworth expansion of the distribution $P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n-\theta)\le t\}$ of any $\boldsymbol{F}$-estimator $\hat{\theta}_n$ up to the order $o(n^{-3/2})$, (4.2) depends on the estimator, it is seen that a third order as. efficient $\boldsymbol{F}$-estimator is not generally fourth order as. efficient in the sense of Definition 2.2.

Remark 4.3. As is stated in Ghosh ((1994), p. 64), on the question whether anything like third order asymptotic efficiency holds when we go to the fifth order, Ghosh and Sinha (1982) give a counterexample so that this is impossible.

Remark 4.4. In the case of curved exponential distributions, it is shown by Kano (1994) that, for any third order as. efficient estimator, an asymptotically improved estimator up to the fourth order could be obtained by an appropriate bias-adjustment.

Next, we consider the fourth order asymptotic efficiency of the bias-adjusted MLE of θ.

THEOREM 4.2. *Assume that the conditions* (A1) *to* (A4) *hold. Then the stochastic expansion of the MLE* $\hat{\theta}_{ML}$ *is given by*

$$
(4.3)\quad \sqrt{n}(\hat{\theta}_{ML}-\theta)=\frac{Z_1}{I}+\frac{1}{I^2\sqrt{n}}\left(Z_1Z_2-\frac{3J+K}{2I}Z_1^2\right)
$$

$$+\frac{1}{I^3 n}\left\{Z_1 Z_2^2-\frac{3(3J+K)}{2I}Z_1^2 Z_2+\frac{1}{2}Z_1^2 Z_3\right.$$
$$\left.+\frac{(3J+K)^2}{2I^2}Z_1^3-\frac{4L+3M+6N+H}{6I}Z_1^3\right\}$$
$$+O_p\left(\frac{1}{n\sqrt{n}}\right).$$

The proof is omitted since it is given in Akahira and Takeuchi ((1981), p. 109) (see also Akahira (1992)). Let $\hat{\theta}^*_{ML}$ be the MLE bias-adjusted so that it is fourth order AMU. Then it follows from (4.3) that $\hat{\theta}^*_{ML}$ belongs to the class $\boldsymbol{F}$, since the term of order $n^{-1/2}$ is a quadratic form of Z_1 and Z_2, and that of order n^{-1} is a cubic form of Z_1, Z_2 and Z_3. It is known that the bias-adjusted MLE $\hat{\theta}^*_{ML}$ is third order as. efficient in the class $\boldsymbol{D}$ (see, e.g. Akahira and Takeuchi (1981), Akahira (1986)).

THEOREM 4.3. *Assume that the conditions* (A1) *to* (A4) *hold. Then the bias-adjusted MLE $\hat{\theta}^*_{ML}$ is fourth order as. symmetrically efficient in the class $\boldsymbol{F}$.*

The proof is straightforward from Theorem 4.1 since $\hat{\theta}^*_{ML}(\in \boldsymbol{F})$ is third order as. symmetrically efficient in the class $\boldsymbol{D}$.

5. Proofs and remark

Here, the proofs of Theorems in Section 3 and a remark are given.

PROOF OF THEOREM 3.1. Let $E_\theta(T_n)=\frac{\mu_1(\theta)}{\sqrt{n}}+\frac{\mu_2(\theta)}{n}+\frac{\mu_3(\theta)}{n\sqrt{n}}+o(\frac{1}{n\sqrt{n}})$. Putting $U_{\theta,n}=T_n-E_\theta(T_n)$, we have

$$\begin{aligned}(5.1)\qquad U_{\theta,n}&=\sqrt{n}(\hat{\theta}_n-E_\theta(\hat{\theta}_n))\\&=\frac{Z_1}{I}+\frac{1}{\sqrt{n}}(Q-\mu_1)+\frac{1}{n}(R-\mu_2)+O_p\left(\frac{1}{n\sqrt{n}}\right),\end{aligned}$$

where $\mu_1:=\mu_1(\theta)=E_\theta(Q)$, $\mu_2:=\mu_2(\theta)=E_\theta(R)$, and Z_1, I, Q and R denote $Z_1(\theta)$, $I(\theta)$, $Q(\theta)$ and $R(\theta)$, respectively. Henceforth such θ's are often omitted. Since, by (5.1),

$$(5.2)\qquad \frac{1}{\sqrt{n}}\frac{\partial U_{\theta,n}}{\partial\theta}=-1-\frac{1}{n}\mu_1'(\theta)-\frac{1}{n\sqrt{n}}\mu_2'(\theta)+o\left(\frac{1}{n\sqrt{n}}\right),$$

it follows from Lemma 3.1 that

$$\begin{aligned}(5.3)\qquad E_\theta[Z_1U_{\theta,n}]&=\frac{1}{\sqrt{n}}\frac{d}{d\theta}E_\theta(U_{\theta,n})-\frac{1}{\sqrt{n}}E_\theta\left[\frac{\partial U_{\theta,n}}{\partial\theta}\right]\\&=1+\frac{1}{n}\mu_1'(\theta)+\frac{1}{n\sqrt{n}}\mu_2'(\theta)+o\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

Since $\hat{\theta}_n$ is a $\boldsymbol{F}$-estimator, we have from (5.3)

$$\kappa_2 = E_\theta(U_{\theta,n}^2) = E_\theta\left[\left\{\left(U_{\theta,n} - \frac{Z_1}{I}\right) + \frac{Z_1}{I}\right\}^2\right] \tag{5.4}$$

$$= -\frac{1}{I} + \frac{2}{I}E_\theta(Z_1 U_{\theta,n}) + \frac{1}{n}E_\theta[(Q-\mu_1)^2]$$

$$+ \frac{2}{n\sqrt{n}}E_\theta[(Q-\mu_1)(R-\mu_2)] + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$= -\frac{1}{I} + \frac{2}{I}\left(1 + \frac{\mu_1'}{n} + \frac{\mu_2'}{n\sqrt{n}}\right) + \frac{\tau(\theta)}{n}$$

$$+ \frac{2}{n\sqrt{n}}\mathrm{Cov}_\theta(Q,R) + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$= \frac{1}{I} + \frac{2\mu_1'}{In} + \frac{\tau}{n} + \frac{2\mu_2'}{In\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right).$$

Putting $a = Z_1/I$ and $b = U_{\theta,n} - (Z_1/I)$ in the identity

$$(a+b)^3 = \frac{3}{2}a(a+b)^2 - \frac{1}{2}a^3 + \frac{3}{2}ab^2 + b^3,$$

we have

$$\kappa_3 = E_\theta(U_{\theta,n}^3) = \frac{3}{2I}E_\theta(Z_1 U_{\theta,n}^2) - \frac{1}{2I^3}E_\theta(Z_1^3) \tag{5.5}$$

$$+ \frac{3}{2I}E_\theta\left[Z_1\left(U_{\theta,n} - \frac{Z_1}{I}\right)^2\right] + E_\theta\left[\left(U_{\theta,n} - \frac{Z_1}{I}\right)^3\right].$$

By Lemma 3.1, (5.2) and (5.4) we obtain

$$E_\theta[Z_1 U_{\theta,n}^2] = \frac{1}{\sqrt{n}}\frac{d}{d\theta}E_\theta(U_{\theta,n}^2) - \frac{2}{\sqrt{n}}E_\theta\left[U_{\theta,n}\frac{\partial U_{\theta,n}}{\partial\theta}\right] \tag{5.6}$$

$$= \frac{1}{\sqrt{n}}\frac{d}{d\theta}\left(\frac{1}{I} + \frac{2\mu_1'}{In} + \frac{\tau}{n}\right) + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$= -\frac{2J+K}{I^2\sqrt{n}} + \frac{2}{I^2 n\sqrt{n}}\{\mu_1'' I - \mu_1'(2J+K)\} + \frac{\tau'}{n\sqrt{n}}$$

$$+ o\left(\frac{1}{n\sqrt{n}}\right).$$

Since $E_\theta(Z_1^3) = K/\sqrt{n}$ and, by (5.1),

$$E_\theta\left[Z_1\left(U_{\theta,n} - \frac{Z_1}{I}\right)^2\right]$$

$$= \frac{1}{n}E_\theta(Z_1 Q^2) + \frac{2}{n\sqrt{n}}E_\theta[Z_1(Q-\mu_1)(R-\mu_2)] + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$E_\theta\left[\left(U_{\theta,n} - \frac{Z_1}{I}\right)^3\right] = \frac{1}{n\sqrt{n}}E_\theta[(Q-\mu_1)^3] + o\left(\frac{1}{n\sqrt{n}}\right),$$

it follows from (5.5) and (5.6) that

$$(5.7)\quad \begin{aligned}\kappa_3 = &-\frac{3J+2K}{I^3\sqrt{n}} + \frac{3\{\mu_1''I - \mu_1'(2J+K)\}}{I^3 n\sqrt{n}} + \frac{3\tau'}{2In\sqrt{n}} + \frac{\gamma^*}{n\sqrt{n}} \\ &+ \frac{3}{In\sqrt{n}} E_\theta[Z_1(Q-\mu_1)(R-\mu_2)] \\ &+ \frac{1}{n\sqrt{n}} E_\theta[(Q-\mu_1)^3] + o\left(\frac{1}{n\sqrt{n}}\right) \\ = &\frac{\beta_3}{\sqrt{n}} + \frac{\beta_3^*}{n\sqrt{n}} + \frac{3\tau'}{2In\sqrt{n}} + \frac{\gamma^*}{n\sqrt{n}} + \frac{3\delta}{In\sqrt{n}} + \frac{\eta}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right) \\ = &\frac{\beta_3}{\sqrt{n}} + \frac{\gamma^*}{n\sqrt{n}} + \frac{\eta^*}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right) \qquad \text{(say)},\end{aligned}$$

where $\beta_3 = -(3J+2K)/I^3$, $\beta_3^* = 3\{\mu_1''I - \mu_1'(2J+K)\}/I^3$, $\gamma^* + o(1) =$ $3\sqrt{n}E_\theta(Z_1Q^2)/(2I)$, $\delta = E_\theta[Z_1(Q-\mu_1)(R-\mu_2)]$ and $\eta = E_\theta[(Q-\mu_1)^3]$.

Putting $a = Z_1/I$ and $b = U_{\theta,n} - (Z_1/I)$ in the identity

$$(a+b)^4 = \frac{8}{3}a(a+b)^3 - 2a^2(a+b)^2 + \frac{1}{3}a^4 + \frac{4}{3}ab^3 + b^4,$$

we obtain

$$(5.8)\quad \begin{aligned}E_\theta(U_{\theta,n}^4) = &\frac{8}{3I}E_\theta[Z_1U_{\theta,n}^3] - \frac{2}{I^2}E_\theta[Z_1^2U_{\theta,n}^2] + \frac{1}{3I^4}E_\theta(Z_1^4) \\ &+ \frac{4}{3I}E_\theta\left[Z_1\left(U_{\theta,n} - \frac{Z_1}{I}\right)^3\right] + E_\theta\left[\left(U_{\theta,n} - \frac{Z_1}{I}\right)^4\right].\end{aligned}$$

Since, by (5.7),

$$E_\theta(U_{\theta,n}^3) = \frac{\beta_3(\theta)}{\sqrt{n}} + o\left(\frac{1}{n}\right),$$

it follows from Lemma 3.1 and (5.2) that

$$(5.9)\quad \begin{aligned}E_\theta[Z_1U_{\theta,n}^3] &= \frac{1}{\sqrt{n}}\frac{d}{d\theta}E_\theta(U_{\theta,n}^3) - \frac{3}{\sqrt{n}}E_\theta\left[U_{\theta,n}^2\frac{\partial U_{\theta,n}}{\partial\theta}\right] \\ &= \frac{3}{I} + \frac{\beta_3'}{n} + \frac{3\tau}{n} + \frac{9\mu_1'}{In} + \frac{9\mu_2'}{In\sqrt{n}} \\ &\quad + \frac{6}{n\sqrt{n}}\mathrm{Cov}_\theta(Q,R) + o\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

Since $\partial Z_1/\partial\theta = Z_2 - \sqrt{n}I$, by Lemma 3.1, we have

$$(5.10)\quad \begin{aligned}I^2E_\theta\left[U_{\theta,n}^2\frac{\partial Z_1}{\partial\theta}\right] = &E_\theta\left[\frac{\partial Z_1}{\partial\theta}Z_1^2\right] + 2E_\theta\left[\frac{\partial Z_1}{\partial\theta}Z_1(IU_{\theta,n} - Z_1)\right] \\ &+ I^2E_\theta\left[\frac{\partial Z_1}{\partial\theta}\left(U_{\theta,n} - \frac{Z_1}{I}\right)^2\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{N}{\sqrt{n}} - \sqrt{n} I^2 \\
&\quad + 2\left\{ -\frac{J(2J+K)}{I\sqrt{n}} - \frac{I^2\mu_1'}{\sqrt{n}} + \frac{M}{\sqrt{n}} \right. \\
&\qquad \left. + \frac{I}{n}\frac{\partial}{\partial\theta} E_\theta(Z_2 Q) - \frac{I}{n} E_\theta(Z_3 Q) - \frac{I^2 \mu_2'}{n} \right\} \\
&\quad + I^2 \left[-\frac{I\tau}{\sqrt{n}} + \frac{1}{n}\{E_\theta(Z_2 Q^2) - 2\mu_1 E_\theta(Z_2 Q)\} \right. \\
&\qquad \left. - \frac{2I}{n}\mathrm{Cov}_\theta(Q, R) \right] + o\left(\frac{1}{n}\right) \\
&= -\sqrt{n} I^2 + \frac{N}{\sqrt{n}} - \frac{I^3}{\sqrt{n}}\tau - \frac{2J(2J+K)}{I\sqrt{n}} \\
&\quad - \frac{2I^2\mu_1'}{\sqrt{n}} + \frac{2M}{\sqrt{n}} - \frac{2I^2\mu_2'}{n} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

Since, by (5.2) and (5.3)

$$
\frac{1}{\sqrt{n}} E_\theta\left[Z_1 U_{\theta,n} \frac{\partial U_{\theta,n}}{\partial\theta} \right] = -\left(1 + \frac{2\mu_1'}{n} + \frac{2\mu_2'}{n\sqrt{n}} \right) + o\left(\frac{1}{n\sqrt{n}} \right),
$$

it follows from Lemma 3.1, (5.6), (5.9) and (5.10) that

$$
\begin{aligned}
(5.11) \quad E_\theta[Z_1^2 U_{\theta,n}^2] &= \frac{1}{\sqrt{n}}\frac{d}{d\theta} E_\theta[Z_1 U_{\theta,n}^2] - \frac{1}{\sqrt{n}} E_\theta\left[U_{\theta,n}^2 \frac{\partial Z_1}{\partial\theta} \right] \\
&\quad - \frac{2}{\sqrt{n}} E_\theta\left[Z_1 U_{\theta,n} \frac{\partial U_{\theta,n}}{\partial\theta} \right] \\
&= 3 + \frac{6\mu_1'}{n} - \frac{1}{n}\frac{d}{d\theta}\left(\frac{2J+K}{I^2} \right) - \frac{N}{I^2 n} + \frac{2J(2J+K)}{I^3 n} \\
&\quad - \frac{2M}{I^2 n} + \frac{I\tau}{n} + \frac{6}{n\sqrt{n}}\mu_2' + o\left(\frac{1}{n\sqrt{n}} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
(5.12) \qquad & E_\theta\left(Z_1^4\right) = 3I^2 + \frac{H}{n}, \\
& E_\theta\left[Z_1 \left(U_{\theta,n} - \frac{Z_1}{I} \right)^3 \right] = o\left(\frac{1}{n\sqrt{n}} \right), \\
& E_\theta\left[\left(U_{\theta,n} - \frac{Z_1}{I} \right)^4 \right] = o\left(\frac{1}{n\sqrt{n}} \right),
\end{aligned}
$$

it follows from (5.1), (5.8), (5.9) and (5.11) that

$$
\begin{aligned}
(5.13) \qquad E_\theta(U_{\theta,n}^4) &= \frac{3}{I^2} + \frac{1}{n}\left\{ \frac{8\beta_3'}{3I} + \frac{12\mu_1'}{I^2} + \frac{6\tau}{I} + \frac{2}{I^2}\frac{d}{d\theta}\left(\frac{2J+K}{I^2} \right) \right. \\
&\qquad \left. + \frac{2N}{I^4} - \frac{4J(2J+K)}{I^5} + \frac{4M}{I^4} + \frac{H}{3I^4} \right\} \\
&\quad + \frac{12\mu_2'}{I^2 n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}} \right).
\end{aligned}
$$

Since

$$J' = \frac{dJ(\theta)}{d\theta} = L(\theta) + M(\theta) + N(\theta), \qquad K' = \frac{dK(\theta)}{d\theta} = H(\theta) + 3N(\theta),$$

it follows from (5.4) and (5.13) that

$$\begin{aligned}(5.14) \qquad \kappa_4 &= E_\theta(U_{\theta,n}^4) - 3\{E_\theta(U_{\theta,n}^2)\}^2 \\ &= \frac{12(2J+K)(J+K)}{I^5 n} - \frac{3H+4L+12N}{I^4 n} + o\left(\frac{1}{n\sqrt{n}}\right) \\ &= \frac{\beta_4(\theta)}{n} + o\left(\frac{1}{n\sqrt{n}}\right) \qquad \text{(say)}.\end{aligned}$$

Since, by Lemma 3.1 and (5.2),

$$E_\theta[Z_1^4(Q-\mu_1)] = -6(J+K) + O\left(\frac{1}{\sqrt{n}}\right), \qquad E_\theta[Z_1^3(Q-\mu_1)^2] = O\left(\frac{1}{\sqrt{n}}\right),$$

$$E_\theta[Z_1^4(R-\mu_2)] = O\left(\frac{1}{\sqrt{n}}\right),$$

and

$$E_\theta(Z_1^5) = \frac{1}{n\sqrt{n}} E_\theta[\{\ell^{(1)}(\theta, X)\}^5] + 10\left(\frac{1}{\sqrt{n}} - \frac{1}{n\sqrt{n}}\right) IK,$$

it follows that

$$\begin{aligned} E_\theta(U_{\theta,n}^5) &= E_\theta\left[\left\{\frac{Z_1}{I} + \frac{1}{\sqrt{n}}(Q-\mu_1) + \frac{1}{n}(R-\mu_2) + o_p\left(\frac{1}{n}\right)\right\}^5\right] \\ &= \frac{1}{I^5}E_\theta(Z_1^5) + \frac{5}{I^4\sqrt{n}}E_\theta[Z_1^4(Q-\mu_1)] + \frac{10}{I^3 n}E_\theta[Z_1^3(Q-\mu_1)^2] \\ &\quad + \frac{5}{I^4 n}E_\theta[Z_1^4(R-\mu_2)] + O\left(\frac{1}{n\sqrt{n}}\right) \\ &= -\frac{10(3J+2K)}{I^4\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

From (5.4) and (5.7) we have

$$(5.15) \qquad \kappa_5 = E_\theta(U_{\theta,n}^5) - 10E_\theta(U_{\theta,n}^2)E_\theta(U_{\theta,n}^3) = \frac{\beta_5(\theta)}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right) \qquad \text{(say)}.$$

In a similar way to the above we can prove that $\kappa_i = o(n^{-3/2})$ for $i \geq 6$ (see also Bhattacharya and Ghosh (1978)). From (5.1), (5.4), (5.5), (5.14) and (5.15) it follows that the Edgeworth expansion of the distribution of $\hat{\theta}_n$ up to the order $o(n^{-3/2})$ is given by

$$(5.16) \quad P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n - \theta) \leq t\}$$

$$
\begin{aligned}
&= \Phi(t) - \frac{\sqrt{I}\mu_1}{\sqrt{n}}\phi(t) - \frac{I^{3/2}\beta_3}{6\sqrt{n}}(t^2-1)\phi(t) \\
&\quad - \frac{\sqrt{I}\mu_2}{n}\phi(t) - \frac{(I\tau + 2\mu_1')}{2n}t\phi(t) - \frac{I\mu_1^2}{2n}t\phi(t) \\
&\quad - \frac{I^2\beta_4}{24n}(t^3-3t)\phi(t) - \frac{I^3\beta_3^2}{72n}(t^5-10t^3+15t)\phi(t) \\
&\quad - \frac{I^2\mu_1\beta_3}{6n}(t^3-3t)\phi(t) \\
&\quad - \frac{\sqrt{I}\mu_3}{n\sqrt{n}}\phi(t) - \frac{I^{3/2}(\gamma^* + \eta^*)}{6n\sqrt{n}}(t^2-1)\phi(t) \\
&\quad - \frac{\mu_2'}{n\sqrt{n}}t\phi(t) - \frac{I\mu_1\mu_2}{n\sqrt{n}}t\phi(t) \\
&\quad - \frac{I^2\mu_2\beta_3}{6n\sqrt{n}}(t^3-3t)\phi(t) - \frac{I^{3/2}\mu_1^3}{6n\sqrt{n}}(t^2-1)\phi(t) \\
&\quad - \frac{\sqrt{I}\mu_1(I\tau + 2\mu_1')}{2n\sqrt{n}}(t^2-1)\phi(t) \\
&\quad - \frac{I^{5/2}\mu_1^2\beta_3}{12n\sqrt{n}}(t^4-6t^2+3)\phi(t) - \frac{I^{3/2}\mu_1'\beta_3}{6n\sqrt{n}}(t^4-6t^2+3)\phi(t) \\
&\quad - \frac{I^{5/2}\tau\beta_3}{12n\sqrt{n}}(t^4-6t^2+3)\phi(t) \\
&\quad - \frac{I^{5/2}\mu_1\beta_4}{24n\sqrt{n}}(t^4-6t^2+3)\phi(t) - \frac{I^{5/2}\beta_5}{120n\sqrt{n}}(t^4-6t^2+3)\phi(t) \\
&\quad - \frac{I^{7/2}\mu_1\beta_3^2}{72n\sqrt{n}}(t^6-15t^4+45t^2-15)\phi(t) \\
&\quad - \frac{I^{7/2}\beta_3\beta_4}{144n\sqrt{n}}(t^6-15t^4+45t^2-15)\phi(t) \\
&\quad - \frac{I^{9/2}\beta_3^3}{1296n\sqrt{n}}(t^8-28t^6+210t^4-420t^2+105)\phi(t) \\
&\quad + o\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

where $\Phi(t) = \int_{-\infty}^{t}\phi(x)dx$ with $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ (see also Kendall and Stuart (1969), p. 164). By the fourth order AMU condition of $\hat{\theta}_n$ we have from (5.16)

$$
\begin{aligned}
&P_{\theta,n}\{\sqrt{In}(\hat{\theta}_n - \theta) \le 0\} \\
&= \frac{1}{2} - \frac{\sqrt{I}\mu_1}{\sqrt{n}}\phi(0) + \frac{I^{3/2}\beta_3}{6\sqrt{n}}\phi(0) - \frac{\sqrt{I}\mu_2}{n}\phi(0) - \frac{\sqrt{I}\mu_3}{n\sqrt{n}}\phi(0) \\
&\quad + \frac{I^{3/2}(\gamma^* + \eta^*)}{6n\sqrt{n}}\phi(0) + \frac{I^{3/2}\mu_1^3}{6n\sqrt{n}}\phi(0) + \frac{\sqrt{I}\mu_1(I\tau + 2\mu_1')}{2n\sqrt{n}}\phi(0) \\
&\quad - \frac{I^{5/2}(\tau + \mu_1^2)\beta_3}{4n\sqrt{n}}\phi(0) - \frac{I^{3/2}\mu_1'\beta_3}{2n\sqrt{n}}\phi(0) - \frac{I^{5/2}\mu_1\beta_4}{8n\sqrt{n}}\phi(0) - \frac{I^{5/2}\beta_5}{40n\sqrt{n}}\phi(0)
\end{aligned}
$$

$$+\frac{5I^{7/2}\mu_1\beta_3^2}{24n\sqrt{n}}\phi(0)+\frac{5I^{7/2}\beta_3\beta_4}{48n\sqrt{n}}\phi(0)-\frac{35I^{9/2}\beta_3^3}{432n\sqrt{n}}\phi(0)+o\left(\frac{1}{n\sqrt{n}}\right)$$
$$=\frac{1}{2}+o\left(\frac{1}{n\sqrt{n}}\right),$$

hence

$$\mu_1=\frac{I\beta_3}{6},\qquad \mu_2=0, \tag{5.17}$$

and

$$\begin{aligned}\mu_3=&\frac{I}{6}(\gamma^*+\eta^*)+\frac{I}{6}\mu_1^3+\frac{1}{2}\mu_1(I\tau+2\mu_1')-\frac{1}{4}I^2(\tau+\mu_1^2)\beta_3-\frac{1}{2}I\mu_1'\beta_3\\&-\frac{1}{8}I^2\mu_1\beta_4-\frac{1}{40}I^2\beta_5+\frac{5}{24}I^3\mu_1\beta_3^2+\frac{5}{48}I^3\beta_3\beta_4-\frac{35}{432}I^4\beta_3^3.\end{aligned} \tag{5.18}$$

Thus, we complete the proof.

Remark 5.1. In order to obtain $\beta_5(\theta)$ exactly, it may be helpful to put $a=Z_1/I$ and $b=U_{\theta,n}-(Z_1/I)$ in the identity

$$(a+b)^5=\frac{15}{4}a(a+b)^4-5a^2(a+b)^3+\frac{5}{2}a^3(a+b)^2-\frac{1}{4}a^5+\frac{5}{4}ab^4+b^5$$

and calculate

$$\begin{aligned}E_\theta(U_{\theta,n}^5)=&\frac{15}{4I}E_\theta[Z_1U_{\theta,n}^4]-\frac{5}{I^2}E_\theta[Z_1^2U_{\theta,n}^3]+\frac{5}{2I^3}E_\theta[Z_1^3U_{\theta,n}^2]-\frac{1}{4I^5}E_\theta(Z_1^5)\\&+\frac{5}{4I}E_\theta\left[Z_1\left(U_{\theta,n}-\frac{Z_1}{I}\right)^4\right]+E_\theta\left[\left(U_{\theta,n}-\frac{Z_1}{I}\right)^5\right].\end{aligned}$$

PROOF OF THEOREM 3.2. The Edgeworth expansion of the distribution of the $\boldsymbol{F}$-estimator $\hat{\theta}_n$ straightforwardly follows from (5.16), (5.17) and (5.18).

Acknowledgements

The author wishes to thank the referee for his kind comments including a future problem on related topics.

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