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LOSS OF INFORMATION OF A STATISTIC FOR A FAMILY OF NON-REGULAR DISTRIBUTIONS

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Abstract. In the non-regular case, the asymptotic loss of amount of information (extended to as Rényi measure) associated with a statistic is discussed. It is shown that the second order asymptotic loss of information in reducing to a statistic consisting of extreme values and an asymptotically ancillary statistic vanishes. This result corresponds to the fact that the statistic is second order asymptotically sufficient in the sense of Akahira (1991, *Metron*, **49**, 133–143). Some examples on truncated distributions are also given.

Key words and phrases: Amount of Rényi type information, loss of information, non-regular case, extreme statistics, asymptotically ancillary statistic, truncated distributions.

1. Introduction

In the regular cases, the amounts of information like Fisher, Kullback-Leibler etc. play an important part in calculation of the loss of information associated with a statistic. However, in non-regular cases, they are not always useful in the calculation. So an amount of information is proposed and investigated by Akahira and Takeuchi (1991) from the viewpoint of non-regular estimation, and also by LeCam (1990) in relation to the concept of affinity introduced by Matusita (1955). The order of consistency of estimators in terms of the information is also discussed by Akahira (1995) in non-regular cases. Related results can be found in Papaioannou and Kempthorne (1971) and Akahira and Takeuchi (1995). In this paper, in a more general non-regular framework, we define the second order asymptotic loss of amount of information associated with a statistic using an amount of information extended to as Rényi measure, and consider the loss for a statistic consisting of extreme values and an asymptotically ancillary statistic. The main result is that the second order asymptotic loss of information of the statistic vanishes. Further, some examples on truncated distributions are given. In a similar situation, it is shown in Akahira (1991*b*) that the above statistic is second order asymptotically sufficient in some sense. The related results in the estimation problem on a location parameter of the double exponential distribution are obtained by Akahira and Takeuchi (1990).

2. The amount of information

Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) real random variables with a density function $f(x,\theta)$ with respect to a σ-finite measure μ, where θ belongs to a parameter space Θ. Then, we consider the quantity which was defined by Akahira and Takeuchi (1991) as an amount of information on X_1 between $f(\cdot,\theta_1)$ and $f(\cdot,\theta_2)$ for any points θ_1 and θ_2 in Θ as follows.

$$I_{X_1}(\theta_1,\theta_2) = -8\log\int\{f(x,\theta_1)f(x,\theta_2)\}^{1/2}d\mu(x). \tag{2.1}$$

Here, the integral in the above is called affinity between $f(\cdot,\theta_1)$ and $f(\cdot,\theta_2)$ (see, e.g. Matusita (1955) and also LeCam (1990)). This amount (2.1) of information was introduced in consideration of an application to non-regular cases, such as truncated densities, and a connection with the amount of Fisher information. The amount (2.1) of information is also extended to as Rényi measure:

$$I_{X_1}^{(\alpha)}(\theta_1,\theta_2) = -\frac{8}{1-\alpha^2}\log\int f(x,\theta_1)^{(1-\alpha)/2}f(x,\theta_2)^{(1+\alpha)/2}d\mu(x) \tag{2.2}$$

for $-1 < \alpha < 1$. When $\alpha = 0$, the amount (2.2) of information coincides with (2.1). Let $T_1 = T_1(\boldsymbol{X})$ and $T_2 = T_2(\boldsymbol{X})$ be statistics based on a sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ of size n. Let $f_\theta(t_1,t_2)$ be a joint density of T_1 and T_2 with respect to a direct product measure $\mu_{T_1}\otimes\mu_{T_2}$, $f_\theta(t_1 \mid t_2)$ be a conditional density of T_1, given T_2, with respect to the measure μ_{T_1}, and $g_\theta(t_2)$ be a marginal density with respect to the measure μ_{T_2}. Then, we define an amount $I_{T_1|T_2}(\theta_1,\theta_2)$ of information on T_1, given T_2, between $f_{\theta_1}(t_1 \mid t_2)$ and $f_{\theta_2}(t_1|t_2)$ for any disjoint points θ_1 and θ_2 in Θ as follows.

$$\begin{aligned} I_{T_1|T_2}^{(\alpha)}&(\theta_1,\theta_2) \\ &= -\frac{8}{1-\alpha^2}\log\int f_{\theta_1}(t_1 \mid t_2)^{(1-\alpha)/2}f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2}d\mu_{T_1}(t_1). \end{aligned} \tag{2.3}$$

LEMMA 2.1. *For any θ_1, θ_2 in Θ and $-1<\alpha<1$,*

$$\begin{aligned} I_{T_1,T_2}^{(\alpha)}(\theta_1,\theta_2) &= -\frac{8}{1-\alpha^2}\log\int\left[\exp\left\{-\frac{1-\alpha^2}{8}I_{T_1|T_2}^{(\alpha)}(\theta_1,\theta_2)\right\}\right] \\ &\qquad\cdot g_{\theta_1}(t_2)^{(1-\alpha)/2}g_{\theta_2}(t_2)^{(1+\alpha)/2}d\mu_{T_2}(t_2) \\ &= -\frac{8}{1-\alpha^2}\log E\left[\exp\left\{-\frac{1-\alpha^2}{8}I_{T_1|T_2}^{(\alpha)}(\theta_1,\theta_2)\right\}\right] \\ &\qquad + I_{T_2}^{(\alpha)}(\theta_1,\theta_2), \end{aligned} \tag{2.4}$$

where the expectation $E[\ ]$ is taken under the density

$$g_{\theta_1}(t_2)^{(1-\alpha)/2}g_{\theta_2}(t_2)^{(1+\alpha)/2} \Big/ \int g_{\theta_1}(t_2)^{(1-\alpha)/2}g_{\theta_2}(t_2)^{(1+\alpha)/2}d\mu_{T_2}(t_2).$$

PROOF. Let α be any fixed in $(-1,1)$. We have

$$
\begin{aligned}
(2.5)\quad & I^{(\alpha)}_{T_1,T_2}(\theta_1,\theta_2) \\
&= -\frac{8}{1-\alpha^2}\log\iint f_{\theta_1}(t_1,t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1,t_2)^{(1+\alpha)/2} \\
&\qquad\qquad \cdot d\mu_{T_1}(t_1)d\mu_{T_2}(t_2) \\
&= -\frac{8}{1-\alpha^2}\log\iint \{f_{\theta_1}(t_1 \mid t_2)g_{\theta_1}(t_2)\}^{(1-\alpha)/2} \\
&\qquad\qquad \cdot \{f_{\theta_2}(t_1 \mid t_2)g_{\theta_2}(t_2)\}^{(1+\alpha)/2} d\mu_{T_1}(t_1)d\mu_{T_2}(t_2) \\
&= -\frac{8}{1-\alpha^2} \\
&\quad \cdot \log\int\left\{\int f_{\theta_1}(t_1 \mid t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2} d\mu_{T_1}(t_1)\right\} \\
&\qquad\qquad \cdot g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2) \\
&= -\frac{8}{1-\alpha^2}\log\int\left[\exp\left\{-\frac{1-\alpha^2}{8} I^{(\alpha)}_{T_1|T_2}(\theta_1,\theta_2)\right\}\right] \\
&\qquad\qquad \cdot g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int\left[\int f_{\theta_1}(t_1|t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2} d\mu_{T_1}(t_1)\right] \\
&\qquad \cdot g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2) \\
&\quad = \int\left[\int f_{\theta_1}(t_1 \mid t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2} d\mu_{T_1}(t_1)\right] \\
&\qquad \cdot \frac{g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2}}{\int g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2)} \\
&\qquad \cdot \left[\int g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2)\right] d\mu_{T_2}(t_2),
\end{aligned}
$$

it follows from (2.5) that

$$
\begin{aligned}
& I^{(\alpha)}_{T_1,T_2}(\theta_1,\theta_2) \\
&\quad = -\frac{8}{1-\alpha^2}\log\int\left[\int f_{\theta_1}(t_1 \mid t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2} d\mu_{T_1}(t_1)\right] \\
&\qquad\qquad \cdot \frac{g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2}}{\int g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2)} d\mu_{T_2}(t_2) \\
&\qquad - \frac{8}{1-\alpha^2}\log\int g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} d\mu_{T_2}(t_2) \\
&\quad = -\frac{8}{1-\alpha^2}\log E\left[\int f_{\theta_1}(t_1 \mid t_2)^{(1-\alpha)/2} f_{\theta_2}(t_1 \mid t_2)^{(1+\alpha)/2} d\mu_{T_1}(t_1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ I_{T_2}^{(\alpha)}(\theta_1, \theta_2) \\
&= -\frac{8}{1-\alpha^2} \log E \left[\exp\left\{ -\frac{1-\alpha^2}{8} I_{T_1|T_2}^{(\alpha)}(\theta_1, \theta_2) \right\} \right] + I_{T_2}^{(\alpha)}(\theta_1, \theta_2).
\end{aligned}
$$

Hence (2.4) follows as required.

Since $X_1, \dots, X_n$ are i.i.d., it follows that the amount of information (extended to as Rényi measure) on $\boldsymbol{X}$ between $f(\cdot, \theta_1)$ and $f(\cdot, \theta_2)$ for any disjoint points θ_1 and θ_2 is given by

$$
I_{\boldsymbol{X}}^{(\alpha)}(\theta_1, \theta_2) = n I_{X_1}^{(\alpha)}(\theta_1, \theta_2) \tag{2.6}
$$

for $-1 < \alpha < 1$. It can be also shown that

$$
I_{T_1, T_2}^{(\alpha)}(\theta_1, \theta_2) \leq I_{\boldsymbol{X}}^{(\alpha)}(\theta_1, \theta_2) \tag{2.7}
$$

for $-1 < \alpha < 1$ (see Akahira and Takeuchi (1991) for $\alpha = 0$). Further, for each α with $-1 < \alpha < 1$ we can consider the loss of information of any statistic $T_n = T_n(\boldsymbol{X})$ as $I_{\boldsymbol{X}}^{(\alpha)}(\theta_1, \theta_2) - I_{T_n}^{(\alpha)}(\theta_1, \theta_2)$, and, in the next section, discuss the asymptotic loss up to the second order, i.e. the order $o(n^{-1})$ when $|\theta_1 - \theta_2| = O(n^{-1})$.

The relationship between the amount $I_{X_1}^{(\alpha)}$ of information and that of Fisher information is stated as follows. Under suitable regularity conditions on $f(x, \theta)$, we have for any fixed α and sufficiently small $\Delta\theta$

$$
\begin{aligned}
&I_{X_1}^{(\alpha)}(\theta, \theta + \Delta\theta) \\
&= -\frac{8}{1-\alpha^2} \log \int f(x, \theta)^{(1-\alpha)/2} f(x, \theta + \Delta\theta)^{(1+\alpha)/2} d\mu(x) \\
&= -\frac{8}{1-\alpha^2} \log \int \exp\left\{ \frac{1-\alpha}{2} \log f(x, \theta) \right. \\
&\qquad\qquad \left. + \frac{1+\alpha}{2} \log f(x, \theta + \Delta\theta) \right\} d\mu(x) \\
&= -\frac{8}{1-\alpha^2} \log \int \exp\left\{ \log f(x, \theta) + \frac{(1+\alpha)\Delta\theta}{2} \frac{\partial \log f(x, \theta)}{\partial \theta} \right. \\
&\qquad\qquad \left. + \frac{(1+\alpha)(\Delta\theta)^2}{4} \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2} + o((\Delta\theta)^2) \right\} d\mu(x) \\
&= -\frac{8}{1-\alpha^2} \log \left[1 + \frac{(1+\alpha)(\Delta\theta)^2}{4} \int \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2} f(x, \theta) d\mu(x) \right. \\
&\qquad\qquad + \frac{(1+\alpha)^2(\Delta\theta)^2}{8} \int \left\{ \frac{\partial \log f(x, \theta)}{\partial \theta} \right\}^2 f(x, \theta) d\mu(x) \\
&\qquad\qquad \left. + o((\Delta\theta)^2) \right] \\
&= -\frac{8}{1-\alpha^2} \log \left[1 - \frac{(1+\alpha)(\Delta\theta)^2}{4} I_{X_1}(\theta) \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{(1+\alpha)^2(\Delta\theta)^2}{8} I_{X_1}(\theta) + o((\Delta\theta)^2)\Bigg] \\
&= -\frac{8}{1-\alpha^2}\log\left[1 - \frac{(1-\alpha^2)(\Delta\theta)^2}{8} I_{X_1}(\theta) + o((\Delta\theta)^2)\right] \\
&= I_{X_1}(\theta)(\Delta\theta)^2 + o((\Delta\theta)^2),
\end{aligned}
$$

where $I_{X_1}(\theta) = E_\theta[\{(\partial/\partial\theta)\log f(X_1,\theta)\}^2]$ which is called the amount of Fisher information. Hence, in the regular case, the investigation on the loss of information associated with statistics is reduced to the works by Fisher (1925), Rao (1961), Ghosh and Subramanyam (1974) and others.

3. The loss of information

Suppose that $X_1,\dots,X_n$ are i.i.d. real random variables with the density $f(x,\theta)$ with respect to the Lebesgue measure and consider the location parameter family $f(x,\theta)$, $\theta \in \boldsymbol{R}^1$, defined by $f(x,\theta) = f_0(x-\theta)$ for $x \in \boldsymbol{R}^1$. We assume the following conditions:

$$
\text{(A.1)} \qquad
\begin{aligned}
&f_0(x) > 0 \quad \text{for} \quad a < x < b, \\
&f_0(x) = 0 \quad \text{for} \quad x \le a, \ x \ge b,
\end{aligned}
$$

where both a and b are finite.

(A.2) $f_0(x)$ is twice continuously differentiable in the open interval (a,b) and

$$
\begin{aligned}
&\lim_{x\to a+0} f_0(x) = \lim_{x\to b-0} f_0(x) = c, \\
&\lim_{x\to b-0} f_0'(x) = -\lim_{x\to a+0} f_0'(x) = h,
\end{aligned}
$$

where c is a positive constant and h is a constant.

$$
\text{(A.3)} \qquad 0 < I_0 = \int_a^b \{f_0'(x)\}^2/f_0(x)dx < \infty.
$$

Putting

$$
\text{(3.1)} \qquad I = -\int_a^b \frac{d^2 \log f_0(x)}{dx^2} f_0(x)dx,
$$

we have from (A.1) to (A.3)

$$
\text{(3.2)} \qquad I - I_0 = -2h.
$$

Indeed, since

$$
\begin{aligned}
I &= -\int_a^b \frac{d^2\log f_0(x)}{dx^2} f_0(x)dx \\
&= -\int_a^b f_0''(x)dx + \int_a^b \frac{\{f_0'(x)\}^2}{f_0(x)}dx \\
&= f_0'(a+0) - f_0'(b-0) + I_0 \\
&= -2h + I_0,
\end{aligned}
$$

we get (3.2). We also obtain, from (A.2) and (A.3), $0 < I < \infty$. In the situation, it is known that the order of consistency is equal to n. Then, we have the following.

THEOREM 3.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Then, for* $-1 < \alpha < 1$ *and a small* Δ

$$I_{X_1}^{(\alpha)}(\theta, \theta+\Delta) = \frac{1}{1-\alpha^2}[8c|\Delta| + \{4c^2 - 2h + I - \alpha^2(2h+I)\}\Delta^2] + o(\Delta^2)$$

and

$$I_{\mathbf{X}}^{(\alpha)}(\theta, \theta+\Delta) = \frac{1}{1-\alpha^2}[8cn|\Delta| + \{4c^2 - 2h + I - \alpha^2(2h+I)\}n\Delta^2] + o(n\Delta^2).$$

PROOF. Without loss of generality, we assume $\theta = 0$. First, we consider the case when $\Delta > 0$. Put $l(x) = \log f_0(x)$ and let α be any fixed in $(-1,1)$. Since

$$\begin{aligned}
&f_0(x)^{(1-\alpha)/2} f_0(x-\Delta)^{(1+\alpha)/2} \\
&= \exp\left\{\frac{1-\alpha}{2}\log f_0(x) + \frac{1+\alpha}{2}\log f_0(x-\Delta)\right\} \\
&= \exp\left\{\log f_0(x) - \frac{1+\alpha}{2}\Delta l'(x) + \frac{1+\alpha}{4}\Delta^2 l''(x) + o(\Delta^2)\right\} \\
&= f_0(x)\left\{1 - \frac{1+\alpha}{2}\Delta l'(x) + \frac{1+\alpha}{4}\Delta^2 l''(x) \right. \\
&\qquad \left. + \frac{(1+\alpha)^2}{8}\Delta^2\{l'(x)\}^2 + o(\Delta^2)\right\},
\end{aligned}$$

it follows that

$$\begin{aligned}
(3.3)\qquad &\int_{a+\Delta}^{b} f_0(x)^{(1-\alpha)/2} f_0(x-\Delta)^{(1+\alpha)/2} dx \\
&= \int_{a+\Delta}^{b} f_0(x)dx - \frac{1+\alpha}{2}\Delta\int_{a+\Delta}^{b} l'(x) f_0(x)dx \\
&\quad + \frac{1+\alpha}{4}\Delta^2\int_{a+\Delta}^{b} l''(x) f_0(x)dx \\
&\quad + \frac{(1+\alpha)^2}{8}\Delta^2\int_{a+\Delta}^{b}\{l'(x)\}^2 f_0(x)dx + o(\Delta^2).
\end{aligned}$$

From (A.2) we have

$$\begin{aligned}
(3.4)\qquad \int_{a+\Delta}^{b} f_0(x)dx &= 1 - \int_{a}^{a+\Delta} f_0(x)dx \\
&= 1 - \left\{\Delta f_0(a+0) + \frac{\Delta^2}{2} f_0'(a+0)\right\} + o(\Delta^2) \\
&= 1 - c\Delta + \frac{h}{2}\Delta^2 + o(\Delta^2).
\end{aligned}$$

From (A.2), (A.3) and (3.1) we obtain

$$
\begin{aligned}
(3.5)\qquad \int_{a+\Delta}^{b} l'(x)f_0(x)dx &= \int_a^b l'(x)f_0(x)dx - \int_a^{a+\Delta} l'(x)f_0(x)dx \\
&= -\int_a^{a+\Delta} f_0'(x)dx \\
&= h\Delta + O(\Delta^2),
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\qquad \int_{a+\Delta}^{b} l''(x)f_0(x)dx &= \int_a^b l''(x)f_0(x)dx - \int_a^{a+\Delta} l''(x)f_0(x)dx \\
&= -I + O(\Delta),
\end{aligned}
$$

$$
(3.7)\qquad \int_{a+\Delta}^{b} \{l'(x)\}^2 f_0(x)dx = I_0 + O(\Delta).
$$

Substituting (3.4), (3.5), (3.6) and (3.7) into (3.3), we have

$$
\begin{aligned}
&\int_{a+\Delta}^{b} f_0(x)^{(1-\alpha)/2} f_0(x-\Delta)^{(1+\alpha)/2}dx \\
&= 1 - c\Delta - \frac{\alpha}{2}h\Delta^2 - \frac{1+\alpha}{4}I\Delta^2 + \frac{(1+\alpha)^2}{8}I_0\Delta^2 + o(\Delta^2) \\
&= 1 - c\Delta - \frac{\alpha}{2}h\Delta^2 - \frac{1+\alpha}{4}I\Delta^2 + \frac{(1+\alpha)^2}{8}(I+2h)\Delta^2 + o(\Delta^2),
\end{aligned}
$$

hence, from (A.1)

$$
\begin{aligned}
I_{X_1}^{(\alpha)}(0,\Delta) &= -\frac{8}{1-\alpha^2}\log\int_{a+\Delta}^{b} f_0(x)^{(1-\alpha)/2} f_0(x-\Delta)^{(1+\alpha)/2}dx \\
&= -\frac{8}{1-\alpha^2}\log\Bigg(1 - c\Delta - \frac{\alpha}{2}h\Delta^2 - \frac{1+\alpha}{4}I\Delta^2 \\
&\qquad\qquad + \frac{(1+\alpha)^2}{8}(I+2h)\Delta^2 + o(\Delta^2)\Bigg) \\
&= \frac{8}{1-\alpha^2}c\Delta + \frac{1}{1-\alpha^2}\{4c^2 - 2h + I - \alpha^2(2h+I)\}\Delta^2 + o(\Delta^2).
\end{aligned}
$$

In a similar way to the case $\Delta > 0$, we have for $\Delta < 0$

$$
I_{X_1}^{(\alpha)}(0,\Delta) = -\frac{8}{1-\alpha^2}c\Delta + \frac{1}{1-\alpha^2}\{4c^2 - 2h + I - \alpha^2(2h+I)\}\Delta^2 + o(\Delta^2).
$$

It is easily seen that the asymptotic value of $I_{\boldsymbol{X}}(0,\Delta)$ follows from (2.6). This completes the proof.

We define the extreme statistics $\overline{\theta}$ and $\underline{\theta}$ by $\overline{\theta} = \min_{1\le i\le n} X_i - a$ and $\underline{\theta} = \max_{1\le i\le n} X_i - b$, and put

$$
Z_1(\theta) = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{f_0'(X_i-\theta)}{f_0(X_i-\theta)} \qquad \text{for} \quad \underline{\theta} < \theta < \overline{\theta}.
$$

Let θ_0 be a true parameter and put $\hat{\theta}^* = (\overline{\theta} + \underline{\theta})/2$. Then it is seen that $\hat{\theta}^*$ is a consistent estimator of θ_0. Putting $U = n(\overline{\theta} - \theta_0)$ and $V = n(\underline{\theta} - \theta_0)$, we see that the second order asymptotic joint density $g_n(u,v)$ of U and V is given by

$$g_n(u,v) = \begin{cases} c^2 e^{-c(u-v)} \left[1 + \dfrac{1}{n}\left\{-1 + 2c(u-v) \right.\right. \\ \qquad + \dfrac{h}{4}((u+v)^2 + (u-v)^2) \\ \qquad \left.\left. -\dfrac{c^2}{2}(u-v)^2 - \dfrac{h}{c}(u-v)\right\}\right] + o\left(\dfrac{1}{n}\right) \\ \qquad\qquad \text{for} \quad v < 0 < u, \\ 0 \qquad\qquad \text{otherwise,} \end{cases} \tag{3.8}$$

(see Akahira (1991a, 1993)). It is seen from (3.8) that U and V are asymptotically independent but not so up to the second order.

LEMMA 3.1. *Assume that the conditions* (A.1) *to* (A.3) *hold. Then, for* $-1 < \alpha < 1$ *and any* $\Delta \in \boldsymbol{R}^1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_n(u-\theta_0, v-\theta_0)^{(1-\alpha)/2} g_n(u-\theta_0-\Delta, v-\theta_0-\Delta)^{(1+\alpha)/2} dudv$$
$$= e^{-c|\Delta|}\left\{1 + \frac{1}{2n}(h-c^2)\Delta^2 + o\left(\frac{1}{n}\right)\right\}$$

and

$$I^{(\alpha)}_{n\overline{\theta}, n\underline{\theta}}(\theta_0, \theta_0+\Delta) = \frac{8}{1-\alpha^2}c|\Delta| + \frac{4}{(1-\alpha^2)n}(c^2-h)\Delta^2 + o\left(\frac{1}{n}\right).$$

PROOF. Without loss of generality, we assume that $\theta_0 = 0$. Let α be any fixed in $(-1,1)$. From (3.8) we have for any $\Delta > 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_n(u,v)^{(1-\alpha)/2} g_n(u-\Delta, v-\Delta)^{(1+\alpha)/2} dudv$$
$$= \int_{-\infty}^{0}\int_{\Delta}^{\infty} c^2 e^{-c(u-v)}\left[1 + \frac{1-\alpha}{2n}\left\{-1 + 2c(u-v)\right.\right.$$
$$+ \frac{h}{4}((u+v)^2 + (u-v)^2)$$
$$\left.\left. - \frac{c^2}{2}(u-v)^2 - \frac{h}{c}(u-v)\right\} + o\left(\frac{1}{n}\right)\right]$$
$$\cdot\left[1 + \frac{1+\alpha}{2n}\left\{-1 + 2c(u-v)\right.\right.$$
$$+ \frac{h}{4}((u+v-2\Delta)^2 + (u-v)^2)$$

$$- \frac{c^2}{2}(u-v)^2 - \frac{h}{c}(u-v)\Bigg\} + o\left(\frac{1}{n}\right)\Bigg] dudv$$

$$= \int_{-\infty}^{0}\int_{\Delta}^{\infty} c^2 e^{-c(u-v)}\Bigg[1 + \frac{1}{2n}\Bigg\{-2 + 4c(u-v)$$

$$+ \frac{h}{4}((u+v)^2 + 2(u-v)^2 + (u+v-2\Delta)^2)$$

$$- c^2(u-v)^2 - \frac{2h}{c}(u-v)\Bigg\}$$

$$+ \frac{\alpha}{2n}\left\{\frac{h}{4}((u+v-2\Delta)^2 - (u+v)^2)\right\}\Bigg] dudv$$

$$+ o\left(\frac{1}{n}\right)$$

$$= \int_{-\infty}^{0}\int_{\Delta}^{\infty} c^2 e^{-c(u-v)}\Bigg[1 + \frac{1}{n}\Bigg\{-1 + 2c(u-v) + \frac{h}{4}(u+v)^2 + \frac{h}{4}(u-v)^2$$

$$- \frac{c^2}{2}(u-v)^2 - \frac{h}{c}(u-v) - \frac{h}{2}(u+v)\Delta + \frac{h}{2}\Delta^2\Bigg\}$$

$$+ \frac{\alpha h}{2n}(-\Delta(u+v) + \Delta^2)\Bigg] dudv + o\left(\frac{1}{n}\right)$$

$$= e^{-c\Delta}\left\{1 + \frac{1}{2n}(h - c^2)\Delta^2 + o\left(\frac{1}{n}\right)\right\}.$$

Then we have

$$I^{(\alpha)}_{n\overline{\theta}, n\underline{\theta}}(0, \Delta)$$

$$= -\frac{8}{1-\alpha^2}\log\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_n(u,v)^{(1-\alpha)/2} g_n(u-\Delta, v-\Delta)^{(1+\alpha)/2} dudv$$

$$= -\frac{8}{1-\alpha^2}\log\left[e^{-c\Delta}\left\{1 + \frac{1}{2n}(h-c^2)\Delta^2 + o\left(\frac{1}{n}\right)\right\}\right]$$

$$= \frac{8}{1-\alpha^2}c\Delta + \frac{4}{(1-\alpha^2)n}(c^2 - h)\Delta^2 + o\left(\frac{1}{n}\right).$$

Similarly we have, for any $\Delta < 0$, the integral value and

$$I^{(\alpha)}_{n\overline{\theta}, n\underline{\theta}}(0, \Delta) = -\frac{8}{1-\alpha^2}c\Delta + \frac{4}{(1-\alpha^2)n}(c^2 - h)\Delta^2 + o\left(\frac{1}{n}\right).$$

Thus we complete the proof.

Put $Z_1^* = Z_1(\hat{\theta}^*)$. Then it is noted that Z_1^* is an asymptotically ancillary statistic.

LEMMA 3.2. *Assume that the conditions* (A.1) *to* (A.3) *hold and* $\Delta = O(1/n)$. *Then the conditional information of* $Z_1^*/(I_0\sqrt{n})$ *given* $\overline{\theta}$ *and* $\underline{\theta}$ *is obtained by*

$$I^{(\alpha)}_{Z_1^*/(I_0\sqrt{n})|\overline{\theta},\underline{\theta}}(\theta_0, \theta_0 + \Delta) = nI_0\Delta^2 + o\left(\frac{1}{n}\right)$$

for $-1 < \alpha < 1$.

PROOF. Without loss of generality we assume that $\theta_0 = 0$. Put $Z_1^0 = Z_1(0)$. Then the asymptotic conditional cumulants of Z_1^0 given $U = u$ and $V = v$, under $\theta_0 = 0$, are obtained by

$$E_0[Z_1^0 \mid u, v] = -\frac{h(u+v)}{\sqrt{n}} + O_p\left(\frac{1}{n\sqrt{n}}\right),$$
$$V_0(Z_1^0 \mid u, v) = I_0 + \frac{1}{n}\left\{-2I_0 + \left(cI_0 - \frac{h^2}{c}\right)(u-v)\right\} + O_p\left(\frac{1}{n^2}\right),$$
$$\kappa_3(Z_1^0 \mid u, v) = -\frac{K}{\sqrt{n}} + O_p\left(\frac{1}{n\sqrt{n}}\right),$$
$$\kappa_4(Z_1^0 \mid u, v) = \frac{H}{n} + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$K = -\int_a^b \{l'(x)\}^3 f_0(x)dx$$

and

$$H = \int_a^b \{l'(x)\}^4 f_0(x)dx - 3I_0^2.$$

Hence the Edgeworth expansion of the conditional distribution of $Z_1^0/\sqrt{I_0}$ given $U = u$ and $V = v$ is obtained by

$$\begin{aligned} F(z \mid u, v) &= P\{Z_1^0/\sqrt{I_0} \le z \mid U = u, V = v\} \\ &= \Phi(z) + \frac{h(u+v)}{\sqrt{nI_0}}\phi(z) - \frac{K}{6\sqrt{n}I_0^{3/2}}(z^2-1)\phi(z) \\ &\quad - \frac{H}{24nI_0^2}(z^3 - 3z)\phi(z) \\ &\quad + \frac{1}{2n}\left\{2 - \left(c - \frac{h^2}{cI_0}\right)(u-v) - \frac{h^2(u+v)^2}{I_0}\right\} z\phi(z) + o\left(\frac{1}{n}\right), \end{aligned}$$

which yields the asymptotic conditional density of $Z_1^0/\sqrt{I_0}$ given $U = u$ and $V = v$ as follows.

$$\begin{aligned} f_{Z_1^0/\sqrt{I_0}}(z \mid u, v) = \phi(z) &- \frac{h(u+v)}{\sqrt{nI_0}} z\phi(z) - \frac{K}{6\sqrt{n}I_0^{3/2}}(z^3 - 3z)\phi(z) \\ &+ \frac{H}{24nI_0^2}(z^4 - 6z^2 + 3)\phi(z) + \frac{1}{2n}W(1 - z^2)\phi(z) + o\left(\frac{1}{n}\right), \end{aligned}$$

where $\Phi(z) = \int_{-\infty}^z \phi(u)du$ with $\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$, and $W = 2 - (c - \frac{h^2}{cI_0})(u-v) - \frac{h^2(u+v)^2}{I_0}$. Let α be any fixed in $(-1, 1)$. Since $Z_1^* = Z_1^0 + o_p(1)$, it follows that

$$f_{Z_1^*/\sqrt{I_0}}(z \mid u, v)^{(1-\alpha)/2} f_{Z_1^*/\sqrt{I_0}}(z - \sqrt{nI_0}\Delta \mid u, v)^{(1+\alpha)/2}$$

$$
\begin{aligned}
&= \left\{1 - \frac{(1-\alpha)h(u+v)}{2\sqrt{nI_0}}z - \frac{(1-\alpha)K}{12\sqrt{n}I_0^{3/2}}(z^3-3z)\right. \\
&\quad + \frac{(1-\alpha)H}{48nI_0^2}(z^4-6z^2+3) + \frac{(1-\alpha)W}{4n}(1-z^2) \\
&\quad + \frac{(\alpha^2-1)h^2(u+v)^2}{8nI_0}z^2 + \frac{(\alpha^2-1)Kh(u+v)}{24nI_0^2}(z^4-3z^2) \\
&\quad \left. + \frac{(\alpha^2-1)K^2}{288nI_0^3}(z^3-3z)^2 + o_p\left(\frac{1}{n}\right)\right\} \\
&\quad \cdot \left\{1 - \frac{(1+\alpha)h(u+v)}{2\sqrt{nI_0}}(z-\sqrt{nI_0}\Delta)\right. \\
&\quad - \frac{(1+\alpha)K}{12\sqrt{n}I_0^{3/2}}(z^3-3z-3(z^2-1)\sqrt{nI_0}\Delta) \\
&\quad + \frac{(1+\alpha)H}{48nI_0^2}(z^4-6z^2+3) + \frac{(1+\alpha)W}{4n}(1-z^2) \\
&\quad + \frac{(\alpha^2-1)h^2(u+v)^2}{8nI_0}z^2 + \frac{(\alpha^2-1)Kh(u+v)}{24nI_0^2}(z^4-3z^2) \\
&\quad \left. + \frac{(\alpha^2-1)K^2}{288nI_0^3}(z^3-3z)^2 + o_p\left(\frac{1}{n}\right)\right\} \\
&\quad \cdot \left\{1 + \frac{1+\alpha}{2}\sqrt{nI_0}\Delta z + \frac{1+\alpha}{4}nI_0\Delta^2(z^2-1)\right. \\
&\quad \left. + \frac{\alpha^2-1}{8}nI_0\Delta^2 z^2 + o\left(\frac{1}{n}\right)\right\}\phi(z) \\
&= \left\{1 - \frac{h(u+v)}{\sqrt{nI_0}}z - \frac{K}{6\sqrt{n}I_0^{3/2}}(z^3-3z) + \frac{H}{24nI_0^2}(z^4-6z^2+3)\right. \\
&\quad \left. + \frac{W}{2n}(1-z^2) + \frac{1+\alpha}{2}h(u+v)\Delta + \frac{(1+\alpha)K}{4I_0}\Delta(z^2-1) + o_p\left(\frac{1}{n}\right)\right\} \\
&\quad \cdot \left\{1 + \frac{1+\alpha}{2}\sqrt{nI_0}\Delta z + \frac{1+\alpha}{4}nI_0\Delta^2(z^2-1)\right. \\
&\quad \left. + \frac{\alpha^2-1}{8}nI_0\Delta^2 z^2 + o\left(\frac{1}{n}\right)\right\}\phi(z) \\
&= \left\{1 + \frac{\alpha^2-1}{8}nI_0\Delta^2 z^2 + \frac{1+\alpha}{2}\sqrt{nI_0}\Delta z - \frac{h(u+v)}{\sqrt{nI_0}}z\right. \\
&\quad - \frac{K}{6\sqrt{n}I_0^{3/2}}(z^3-3z) + \frac{H}{24nI_0^2}(z^4-6z^2+3) + \frac{W}{2n}(1-z^2) \\
&\quad + \frac{1+\alpha}{4}\left(\frac{K}{I_0} - 2h(u+v)\right)\Delta(z^2-1) + \frac{1+\alpha}{4}nI_0\Delta^2(z^2-1) \\
&\quad \left. - \frac{(1+\alpha)K}{12I_0}\Delta z(z^3-3z) + o_p\left(\frac{1}{n}\right)\right\}\phi(z).
\end{aligned}
$$

Hence the conditional information of $Z_1^*/(\sqrt{n}I_0)$ given $\overline{\theta}$ and $\underline{\theta}$ is given by

$$
\begin{aligned}
&I^{(\alpha)}_{Z_1^*/\sqrt{I_0}|u,v}(0, \sqrt{nI_0}\Delta)\\
&= -\frac{8}{1-\alpha^2}\log\int_{-\infty}^{\infty} f_{Z_1^*/\sqrt{I_0}}(z \mid u,v)^{(1-\alpha)/2}\\
&\qquad\qquad \cdot f_{Z_1^*/\sqrt{I_0}}(z-\sqrt{nI_0}\Delta \mid u,v)^{(1+\alpha)/2}dz\\
&= -\frac{8}{1-\alpha^2}\log\Bigg[\int_{-\infty}^{\infty}\Bigg\{1+\frac{\alpha^2-1}{8}nI_0\Delta^2z^2+\frac{1+\alpha}{2}\sqrt{nI_0}\Delta z\\
&\qquad -\frac{h(u+v)}{\sqrt{nI_0}}z-\frac{K}{6\sqrt{n}I_0^{3/2}}(z^3-3z)\\
&\qquad +\frac{H}{24nI_0^2}(z^4-6z^2+3)+\frac{W}{2n}(1-z^2)\\
&\qquad +\frac{1+\alpha}{4}\left(\frac{K}{I_0}-2h(u+v)\right)\Delta(z^2-1)\\
&\qquad +\frac{1+\alpha}{4}nI_0\Delta^2(z^2-1)\\
&\qquad -\frac{(1+\alpha)K}{12I_0}\Delta z(z^3-3z)\Bigg\}\phi(z)dz+o\left(\frac{1}{n}\right)\Bigg]\\
&= -\frac{8}{1-\alpha^2}\log\left(1+\frac{\alpha^2-1}{8}nI_0\Delta^2+o\left(\frac{1}{n}\right)\right)\\
&= nI_0\Delta^2+o\left(\frac{1}{n}\right).
\end{aligned}
$$

This completes the proof.

From Lemmas 3.1 and 3.2 we have the following.

THEOREM 3.2. *Assume that the conditions* (A.1) *to* (A.3) *hold and* $\Delta = O(1/n)$. *Then the information of the statistic* $(Z_1^*/(\sqrt{n}I_0), \overline{\theta}, \underline{\theta})$ *is given by*

$$
\begin{aligned}
I^{(\alpha)}_{Z_1^*/(\sqrt{n}I_0),\overline{\theta},\underline{\theta}}(\theta_0,\theta_0+\Delta) &= \frac{8}{1-\alpha^2}cn|\Delta|\\
&\quad+\left\{\frac{4}{1-\alpha^2}(c^2-h)+I_0\right\}n\Delta^2+o(n\Delta^2)
\end{aligned}
$$

for $-1<\alpha<1$.

PROOF. Without loss of generality we assume that $\theta_0 = 0$. Let α be any fixed in $(-1,1)$. From Lemma 3.1 we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_n(u,v)^{(1-\alpha)/2}g_n(u-n\Delta, v-n\Delta)^{(1+\alpha)/2}dudv\\
&= e^{-2cn|\Delta|}\{1+(h-c^2)n\Delta^2+o(n\Delta^2)\}\\
&= C(n\Delta) \quad \text{(say)},
\end{aligned}
$$

and

$$I_{\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta) = I_{n\bar{\theta},n\underline{\theta}}^{(\alpha)}(0,n\Delta) \tag{3.9}$$
$$= \frac{8}{1-\alpha^2}cn|\Delta| + \frac{4}{1-\alpha^2}(c^2-h)n\Delta^2 + o(n\Delta^2).$$

From Lemma 3.2 we obtain

$$I_{Z_1^*/(\sqrt{n}I_0)|\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta) = I_{Z_1^*/\sqrt{I_0}|u,v}^{(\alpha)}(0,\sqrt{nI_0}\Delta) = I_0 n\Delta^2 + o(n\Delta^2),$$

hence, by Lemma 2.1,

$$I_{Z_1^*/(\sqrt{n}I_0),\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta) \tag{3.10}$$
$$= -\frac{8}{1-\alpha^2}\log E\left[\exp\left\{-\frac{1-\alpha^2}{8}I_{Z_1^*/(\sqrt{n}I_0)|\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta)\right\}\right] + I_{\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta)$$
$$= -\frac{8}{1-\alpha^2}\log E\left[\exp\left\{-\frac{1-\alpha^2}{8}I_0 n\Delta^2 + o(n\Delta^2)\right\}\right] + I_{\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta).$$

Since, by Lemma 2.1,

$$E\left[\exp\left\{-\frac{1-\alpha^2}{8}I_0 n\Delta^2 + o(n\Delta^2)\right\}\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\exp\left\{-\frac{1-\alpha^2}{8}I_0 n\Delta^2 + o(n\Delta^2)\right\}\right]\frac{1}{C(n\Delta)}$$
$$\cdot g_n(u+n\Delta, v+n\Delta)^{(1-\alpha)/2} g_n(u-n\Delta, v-n\Delta)^{(1+\alpha)/2}dudv$$
$$= e^{-(1-\alpha^2)I_0 n\Delta^2/8}\{1+o(n\Delta^2)\},$$

it follows from (3.9) and (3.10) that

$$I_{Z_1^*/(\sqrt{n}I_0),\bar{\theta},\underline{\theta}}^{(\alpha)}(0,\Delta) = I_0 n\Delta^2 + \frac{8}{1-\alpha^2}cn|\Delta| + \frac{4}{1-\alpha^2}(c^2-h)n\Delta^2 + o(n\Delta^2)$$
$$= \frac{8}{1-\alpha^2}cn|\Delta| + \left\{\frac{4}{1-\alpha^2}(c^2-h) + I_0\right\}n\Delta^2 + o(n\Delta^2).$$

This completes the proof.

Next we define the second order asymptotic loss of information (extended to as Rényi measure) of any statistic $T_n = T_n(\boldsymbol{X})$ as

$$L_n^{(\alpha)}(T_n) = \frac{1}{n\Delta^2}\{I_{\boldsymbol{X}}^{(\alpha)}(\theta,\theta+\Delta) - I_{T_n}^{(\alpha)}(\theta,\theta+\Delta)\} + o(1),$$

for $-1 < \alpha < 1$, where $\Delta = O(1/n)$. Then we have the following.

THEOREM 3.3. *Assume that the conditions* (A.1) *to* (A.3) *hold and* $\Delta = O(1/n)$. *Then the second order asymptotic loss of information of the statistic* $T_n^* = (Z_1^*/(\sqrt{n}I_0), \overline{\theta}, \underline{\theta})$ *vanishes, that is,*

$$L_n^{(\alpha)}(T_n^*) = o(1)$$

for $-1 < \alpha < 1$.

The proof is straightforward from Theorems 3.1 and 3.2 and (3.2), since

$$\begin{aligned}
&\frac{1}{n\Delta^2}\{I_{\boldsymbol{X}}^{(\alpha)}(0,\Delta) - I_{T_n^*}^{(\alpha)}(0,\Delta)\} \\
&\quad = \frac{1}{1-\alpha^2}\{-2h + I - \alpha^2(2h+I) + 4h\} - I_0 + o(1) \\
&\quad = I - I_0 + 2h + o(1) \\
&\quad = o(1)
\end{aligned}$$

for $-1 < \alpha < 1$.

Remark 3.1. Note that the above result does not depend on α, that is, the second order asymptotic loss $L_n^{(\alpha)}(T_n^*)$ is independent of α up to the order $o(1)$. This shows that the result becomes invariable for such types of amount of information. The result of Theorem 3.3 also corresponds to the fact that the statistic T_n^* is second order asymptotically sufficient in Akahira (1991*b*), and is also closely related to the fact that, for any fixed $t \in \boldsymbol{R}^1$, the maximum probability estimator $\hat{\theta}_{MP}^t$ constructed from T_n^* has the second order two-sided asymptotic efficiency in some class of estimators for the case $h = 0$ in Akahira (1991*a*). It means that, in T_n^*, the statistic $Z_1^*/(\sqrt{n}I_0)$ has the asymptotic full information on the inside of the interval $(a+\theta, b+\theta)$, i.e. the support of the density $f_0(x-\theta)$ and the remainders $\underline{\theta}$ and $\overline{\theta}$ have the one on the end points $a+\theta$ and $b+\theta$, up to the second order.

Remark 3.2. In (A.2) we assume that $f_0(x)$ and $f_0'(x)$ at each endpoint of the support have the two same limits, but the above may be similarly extended to the case when they have the two different limits.

4. Examples

In the previous framework, we now give some examples on truncated distributions.

Example 4.1. (Truncated normal distribution) Let $X_1, \dots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} ce^{-(x-\theta)^2/2} & \text{for} \quad |x-\theta| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where θ is a real-valued unknown parameter and c is some positive constant. Then it is easily seen that $\lim_{x\to -1+0} f_0(x) = \lim_{x\to 1-0} f_0(x) = ce^{-1/2}$ and that $h = \lim_{x\to 1-0} f_0'(x) = -\lim_{-1+0} f_0'(x) = -ce^{-1/2}$ since $f_0'(x) = -cxe^{-x^2/2}$ for $|x| < 1$. Since the conditions (A.1) to (A.3) are satisfied, it follows from Theorem 3.3 that the second order asymptotic loss of information of the statistic $(Z_1^*/(\sqrt{n}I_0), \overline{\theta}, \underline{\theta})$ vanishes, where $Z_1^* = \sqrt{n}(\overline{X} - \hat{\theta}^*)$, $\overline{X} = \sum_{i=1}^n X_i/n$, $\hat{\theta}^* = (\overline{\theta} + \underline{\theta})/2$ with $\underline{\theta} = \max_{1\le i\le n} X_i - 1$ and $\overline{\theta} = \min_{1\le i\le n} X_i + 1$, and $I_0 = 1 - 2ce^{-1/2}$.

Example 4.2. (Truncated Cauchy distribution) Let $X_1, \dots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} \dfrac{c}{1+(x-\theta)^2} & \text{for} \quad |x-\theta| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where θ is a real-valued unknown parameter and c is some positive constant. Then it is easily seen that $\lim_{x\to -1+0} f_0(x) = \lim_{x\to 1-0} f_0(x) = c/2$ and that $h = \lim_{x\to 1-0} f_0'(x) = -\lim_{x\to -1+0} f_0'(x) = -c/2$ since $f_0'(x) = -2cx/(1+x^2)^2$ for $|x| < 1$. Since the conditions (A.1) to (A.3) are satisfied, it follows from Theorem 3.3 that the second order asymptotic loss of information of the statistic $(Z_1^*/(\sqrt{n}I_0), \overline{\theta}, \underline{\theta})$ vanishes, where

$$Z_1^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{X_i - \hat{\theta}^*}{1 + (X_i - \hat{\theta}^*)^2}, \qquad \hat{\theta}^* = (\overline{\theta} + \underline{\theta})/2$$

with $\underline{\theta} = \max_{1\le i\le n} X_i - 1$ and $\overline{\theta} = \min_{1\le i\le n} X_i + 1$, and

$$I_0 = 8c \int_0^1 \frac{x^2}{(1+x^2)^3} dx.$$

Example 4.3. Let $X_1, \dots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} c\exp[\{1-(x-\theta)^2\}^p] & \text{for} \quad |x-\theta| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where θ is a real-valued unknown parameter and $p > 2$ and c is some constant. Then it is seen that $\lim_{x\to -1+0} f_0(x) = \lim_{x\to 1-0} f_0(x) = c$, and that $h = \lim_{x\to -1+0} f_0'(x) = -\lim_{x\to 1-0} f_0'(x) = 0$, since $f_0'(x) = -2cpx(1-x^2)^{p-1} \cdot e^{-(1-x^2)^p}$ for $|x| < 1$. Since the conditions (A.1) to (A.3) are satisfied, it follows from Theorem 3.3 that the second order asymptotic loss of the statistic $(Z_1^*/(\sqrt{n}I_0), \overline{\theta}, \underline{\theta})$ vanishes, where

$$Z_1^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n 2p(X_i - \hat{\theta}^*)\{1 - (X_i - \hat{\theta}^*)^2\}^{p-1}, \qquad \hat{\theta}^* = (\overline{\theta} + \underline{\theta})/2$$

with $\underline{\theta} = \max_{1\le i\le n} X_i - 1$ and $\overline{\theta} = \min_{1\le i\le n} X_i + 1$, and

$$I_0 = 8p^2 \int_0^1 x^2(1-x^2)^{2p-2} ce^{(1-x^2)^p} dx.$$

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