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THE LIMITING NORMALITY OF THE TEST STATISTIC FOR THE TWO-SAMPLE PROBLEM INDUCED BY A CONVEX SUM DISTANCE

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Abstract. Critchlow (1992, *J. Statist. Plann. Inference*, **32**, 325–346) proposed a method of a unified construction of a class of rank tests. In this paper, we introduce a convex sum distance and prove the limiting normality of the test statistics for the two-sample problem derived by his method.

Key words and phrases: Convex sum distance, transposition property, limiting normality.

1. Introduction

Let F and G be continuous distribution functions of the first population and the second population. We consider rank tests for testing $\mathrm{H} : F(x) \equiv G(x)$ against $\mathrm{K} : F(x) \geq G(x)$, with strict inequality for some x. Critchlow (1992) proposed a method of a unified construction of a class of rank tests. For the preceding two-sample problem, his method is summarized as follows; let $Z_1, \ldots, Z_m$ be the observations from the first population and $Z_{m+1}, \ldots, Z_{m+n}$ be the observations from the second population, and let $\pi(i)$ be the rank of Z_i ($i = 1, \ldots, N$, $N = m+n$) among $Z_1, \ldots, Z_N$. Then π is a member of the permutation group denoted by S_N. The two permutations π and σ are defined to be equivalent if and only if $\{\pi(1), \ldots, \pi(m)\} = \{\sigma(1), \ldots, \sigma(m)\}$, then the equivalence class including π is $[\pi] = \pi(S_m \times S_n)$, where $S_m \times S_n = \{\sigma \in S_N; \sigma(i) \leq m, \forall i \leq m\}$, and where $\pi(S_m \times S_n)$ is the left coset $\{\pi \circ \sigma; \sigma \in S_m \times S_n\}$. Now the class of permutations, which are most in agreement with K, is $S_m \times S_n$. Let d be a metric on S_N and define a metric d^* on $S_N/(S_m \times S_n)$ by

$$d^*([\pi], [\sigma]) = \max\left\{\max_{\beta\in[\sigma]} \min_{\alpha\in[\pi]} d(\alpha, \beta), \max_{\alpha\in[\pi]} \min_{\beta\in[\sigma]} d(\alpha, \beta)\right\}.$$

Critchlow suggested using $d^*([\pi], S_m \times S_n)$ as a test statistic, where π is the observed ranking.

Now we introduce the convex sum distance on S_N. Take a strictly increasing, convex, twice differentiable function f satisfying $f(0) = 0$. For any π, σ in S_N, we define

$$d(\pi, \sigma) = \sum_{i=1}^{N} f\left(\frac{|\pi(i) - \sigma(i)|}{N}\right).$$

The purpose of this paper is to study the asymptotic distribution of $d^*([\pi], S_m \times S_n)$, induced by the convex sum distance.

2. Representation of the test statistic

Now we rewrite the m observations from the first population as $X_1, \dots, X_m$ and the n observations from the second population as $Y_1, \dots, Y_n$.

DEFINITION 2.1. The metric d on S_N is called right-invariant, if and only if $d(\alpha, \beta) = d(\alpha \circ \gamma, \beta \circ \gamma)$ for all $\alpha, \beta, \gamma \in S_N$.

Note that if d is a right-invariant metric, then the formula given for $d^*([\pi], [\sigma])$ simplifies to

$$\min_{\alpha \in [\pi], \beta \in [\sigma]} d(\alpha, \beta).$$

DEFINITION 2.2. Let d be a metric on S_N. Let $\alpha, \beta, \gamma \in S_N$ be permutations such that α and β differ by a single transposition; that is, there exist integers $p, q \in \{1, \dots, N\}$ such that $\alpha(p) = \beta(q)$, $\alpha(q) = \beta(p)$, $\alpha(i) = \beta(i)$, $\forall i \neq p, q$. Suppose further that $\alpha(p) \leq \alpha(q)$ and $\gamma(p) \leq \gamma(q)$. If the preceding conditions imply that $d(\alpha, \gamma) \leq d(\beta, \gamma)$, then the metric d is said to possess the transposition property.

LEMMA 2.1. *The convex sum distance is right-invariant, and possesses transposition property.*

This lemma is the generalization of Lemma 2 of Critchlow ((1985), pp. 52–53), for the "case" $d = R$.

THEOREM 2.1. *The test statistic for two-sample location problem d^* induced by the convex sum distance is represented by*

$$T_N = \sum_{i=1}^{m} f\left(\frac{i - r_{(i)}}{N}\right) + \sum_{j=1}^{n} f\left(\frac{m + j - s_{(j)}}{N}\right),$$

where $r_{(1)} \leq \cdots \leq r_{(m)}$ are the ordered ranks of the X's in the pooled sample $X_1, \dots, X_m$, $Y_1, \dots, Y_n$, and similarly $s_{(1)} \leq \cdots \leq s_{(n)}$ are the ordered ranks of the Y's.

PROOF. Critchlow (1992) showed that if the metric on the permutation group is right-invariant and possesses the transposition property, then the test statistic induced by it can be written in the form of Theorem 2.1 using $r_{(i)}$ and $s_{(j)}$. Thus by the lemma, we have the theorem.

Example. Spearman's footrule: $F(\pi,\sigma) = N\sum_{i=1}^{N}|\frac{\pi(i)-\sigma(i)}{N}|$ and Spearman's rho: $R(\pi,\sigma) = N(\sum_{i=1}^{N}(\frac{\pi(i)-\sigma(i)}{N})^2)^{1/2}$, where $\pi,\sigma \in S_N$, are convex sum distances. Note that the test statistic induced by F is equivalent to Wilcoxon test statistic, and that the test statistic induced by R is equivalent to $\sum_{i=1}^{m} ir_{(i)} + \sum_{j=1}^{n}(m+j)s_{(j)}$. Fueda (1993) showed that this test statistic is asymptotically normal and superior to Wilcoxon test statistic for a wide class of asymmetric underlying distributions.

3. Limiting normality of the test statistic

The following theorem states the limiting normality of the test statistic induced by a convex sum distance.

We use the following assumption.

(a) There exists λ_0 such that for all N, $0 < \lambda_0 \le \lambda_N \le 1-\lambda_0 < 1$, where $\lambda_N = \frac{m}{N}$.

(b) There exist K and δ such that for all $\lambda_0 \le \lambda \le 1-\lambda_0$, $i = 0, 1$ and $0 < x < 1$, $|\lambda^i f^{(i)}(\lambda x)| \le K(x(1-x))^{-i-1/2+\delta}$.

THEOREM 3.1. *Under assumptions* (a) *and* (b), $(T_N - \mu_{T_N})/\sigma_{T_N}$ *converges to the standard normal distribution.*

We put $h_1(x) = f((1-\lambda_N)x)$ *and* $h_2(x) = f(\lambda_N(1-x))$, *then* μ_{T_N} *and* σ_{T_N} *are given as follows*:

$$
\begin{aligned}
\mu_{T_N} &= m\int h_1(G(x))dF(x) + n\int h_2(F(x))dG(x),\\
\sigma_{T_N}^2 &= \frac{2}{m}\left(m^2 \iint_{-\infty<x<y<\infty} F(x)(1-F(y))h_1'(G(x))h_1'(G(y))dG(x)dG(y)\right.\\
&\qquad - mn\iint_{-\infty<x<y<\infty} F(x)(1-F(y))h_1'(G(x))h_2'(F(y))dG(x)dG(y)\\
&\qquad - mn\iint_{-\infty<y<x<\infty} (1-F(x))F(y)h_1'(G(x))h_2'(F(y))dG(x)dG(y)\\
&\qquad \left. + n^2\iint_{-\infty<x<y<\infty} F(x)(1-F(y))h_2'(F(x))h_2'(F(y))dG(x)dG(y)\right)\\
&\quad + \frac{2}{n}\left(m^2 \iint_{-\infty<x<y<\infty} G(x)(1-G(y))h_1'(G(x))h_1'(G(y))dF(x)dF(y)\right.\\
&\qquad - mn\iint_{-\infty<x<y<\infty} G(x)(1-G(y))h_1'(G(x))h_2'(F(y))dF(x)dF(y)\\
&\qquad - mn\iint_{-\infty<y<x<\infty} (1-G(x))G(y)h_1'(G(x))h_2'(F(y))dF(x)dF(y)\\
&\qquad \left. + n^2\iint_{-\infty<x<y<\infty} G(x)(1-G(y))h_2'(F(x))h_2'(F(y))dF(x)dF(y)\right),
\end{aligned}
$$

and these are the asymptotic expectation and variance of T_N.

PROOF. The statistics $r_{(i)}$ and $s_{(j)}$ may be represented as follows;

$$\begin{aligned} r_{(i)} &= \#\{k; X_k \le X_{(i)}\} + \#\{l; Y_l < X_{(i)}\} \\ &= i + \#\{l; Y_l < X_{(i)}\}, \\ s_{(j)} &= j + \#\{k; X_k < Y_{(j)}\}. \end{aligned}$$

Then we have

$$f\left(\frac{|i - r_{(i)}|}{N}\right) = f\left(\frac{\#\{l; Y_l < X_{(i)}\}}{N}\right) = f\left(\frac{n}{N} G_n(X_{(i)})\right),$$
$$f\left(\frac{|m + j - s_{(j)}|}{N}\right) = f\left(\frac{m - \#\{k; X_k < Y_{(j)}\}}{N}\right) = f\left(\frac{m}{N}(1 - F_m(Y_{(j)}))\right),$$

where F_m and G_n are the empirical distribution functions of $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$, respectively.

Let $h_1(x) = f((1 - \lambda_N)x)$, $h_2(x) = f(\lambda_N(1 - x))$, then

$$\begin{aligned} \frac{1}{\sqrt{N}} T_N &= \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{m} h_1(G_n(X_i)) + \sum_{j=1}^{n} h_2(F_m(Y_j)) \right) \\ &= \lambda_N \sqrt{N} \int h_1(G_n(x)) dF_m(x) + (1 - \lambda_N)\sqrt{N} \int h_2(F_m(y)) dG_n(y). \end{aligned}$$

Similar to Chernoff-Savage, we may expand T_N as follows:

$$T_N = \sum_{i=1}^{2} \mu_{iN} + \sum_{i=1}^{4} B_{iN} + \sum_{i=1}^{4} C_{iN},$$

where

$$\begin{aligned} \mu_{1N} &= \lambda_N \sqrt{N} \int h_1(G(x)) dF(x), \\ B_{1N} &= \lambda_N \sqrt{N} \int h_1(G(x)) d(F_m(x) - F(x)), \\ B_{2N} &= \lambda_N \sqrt{N} \int (G_n(x) - G(x)) h_1'(G(x)) dF(x), \\ C_{1N} &= \lambda_N \sqrt{N} \int (G_n(x) - G(x)) h_1'(G(x)) d(F_m(x) - F(x)), \\ C_{2N} &= \lambda_N \sqrt{N} \int (h_1(G_n(x)) - h_1(G(x)) - (G_n(x) - G(x)) h_1'(G(x))) dF_m(x), \\ \mu_{2N} &= (1 - \lambda_N)\sqrt{N} \int h_2(F(x)) dG(x), \\ B_{3N} &= (1 - \lambda_N)\sqrt{N} \int h_2(F(x)) d(G_n(x) - G(x)), \end{aligned}$$

$$B_{4N} = (1-\lambda_N)\sqrt{N}\int (F_m(x) - F(x))h_2'(F(x))dG(x),$$

$$C_{3N} = (1-\lambda_N)\sqrt{N}\int (F_m(x) - F(x))h_2'(F(x))d(G_n(x) - G(x)),$$

$$\begin{aligned} C_{4N} = (1-\lambda_N)\sqrt{N}\int &(h_2(F_m(x)) - h_2(F(x)) \\ &- (F_m(x) - F(x))h_2'(F(x)))dG_n(x). \end{aligned}$$

Obviously μ_{1N} and μ_{2N} are non-random. We show the theorem in the following two steps.

Step 1. We show $C_{1N}, \ldots, C_{4N}$ are all $o_p(1)$.

Step 1.1. To begin with, we show $C_{1N} = o_p(1)$. We use the following lemma.

LEMMA 3.1. (Puri and Sen (1971)) *For all* $\varepsilon > 0$ *and for all* $0 < \delta' < \frac{1}{2}$, *there exists* $c(\varepsilon, \delta')$ *such that*

$$\Pr\left\{\sup_x \frac{\sqrt{n}|G_n(x) - G(x)|}{(G(x)(1-G(x)))^{1/2-\delta'}} \le c(\varepsilon, \delta')\right\} \ge 1 - \varepsilon.$$

By Lemma 3.1 and assumption (b), we have for all x,

$$\sqrt{n}|G_n(x) - G(x)|h'(G(x)) \le c(\varepsilon, \delta')K(G(x)(1-G(x)))^{\delta^*-1}$$

with probability larger than $1-\varepsilon$, where $\delta^* = \delta - \delta'$.

Because of arbitrariness of δ', we can get δ' such that $\delta' < \frac{\delta}{4}$ and $\delta^* > 0$,

$$|C_{1N}| \le \frac{\lambda_N}{\sqrt{1-\lambda_N}} Kc(\varepsilon, \delta') \int (G(x)(1-G(x)))^{\delta^*-1} dV_{(F_m - F)}(x),$$

where $V_{(F_m - F)}$ is the total variation of $F_m - F$. By the law of large numbers, $V_{(F_m - F)} \to 0$ as $m \to \infty$. Therefore $C_{1N} = o_p(1)$.

Similarly, we prove $C_{3N} = o_p(1)$.

Step 1.2. Next we show $C_{2N} = o_p(1)$.

$$\begin{aligned} |C_{2N}| &\le \frac{1}{\sqrt{N}} \sum_{i=1}^{m} |h_1(G_n(X_i)) - h_1(G(X_i)) - (G_n(X_i) - G(X_i))h_1'(G(X_i))| \\ &\le \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |h_1(G_n(Z_i)) - h_1(G(Z_i)) - (G_n(Z_i) - G(Z_i))h_1'(G(Z_i))|, \end{aligned}$$

where $Z_1, \ldots, Z_N$ are the order statistics for the pooled sample $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$.

Now let $K_N = \min\{[\frac{N}{2}], [N^{\delta'}]\}$, $0 < 2\delta' < \delta < \frac{1}{2}$, and

$$C_{2N}^{(1)} = \frac{1}{\sqrt{N}} \sum_{i=1}^{K_N} |h_1(G_n(Z_i)) - h_1(G(Z_i)) - (G_n(Z_i) - G(Z_i))h_1'(G(Z_i))|,$$

$$C_{2N}^{(2)} = \frac{1}{\sqrt{N}} \sum_{i=K_N+1}^{N-K_N} |h_1(G_n(Z_i)) - h_1(G(Z_i)) - (G_n(Z_i) - G(Z_i))h_1'(G(Z_i))|,$$

$$C_{2N}^{(3)} = \frac{1}{\sqrt{N}} \sum_{i=N-K_N+1}^{N} |h_1(G_n(Z_i)) - h_1(G(Z_i)) - (G_n(Z_i) - G(Z_i))h_1'(G(Z_i))|.$$

We note that $|C_{2N}| \leq C_{2N}^{(1)} + C_{2N}^{(2)} + C_{2N}^{(3)}$. We consider $C_{2N}^{(1)}$. Because h_1 is strictly increasing and convex, we have

$$h_1(G(Z_i)) \leq h_1(1), \qquad \text{and} \qquad h_1'(G(Z_i)) \leq h_1'(1-).$$

Then

$$\begin{aligned}
C_{2N}^{(1)} &\leq \frac{1}{\sqrt{N}} \sum_{i=1}^{K_N} (|h_1(G_n(Z_i))| + |h_1(G(Z_i))| + (|G_n(Z_i)| + |G(Z_i)|)|h_1'(G(Z_i))|) \\
&\leq N^{-1/2} K_N (2h_1(1) + 2h_1'(1-)) \\
&< N^{-1/4}(2h_1(1) + 2h_1'(1-)).
\end{aligned}$$

The last inequality follows from $K_N \leq N^{\delta'}$ and $0 < \delta' < \frac{\delta}{4} < \frac{1}{4}$. Therefore $C_{2N}^{(1)} = o_p(1)$, similarly, $C_{2N}^{(3)} = o_p(1)$. Next we consider $C_{2N}^{(2)}$.

$$C_{2N}^{(2)} = \sqrt{N} \frac{1}{N} \sum_{i=K_N+1}^{N-K_N} |h_1(G_n(Z_i)) - h_1(G(Z_i)) - (G_n(Z_i) - G(Z_i))h_1'(G(Z_i))|.$$

Let

$$\begin{aligned}
S_N^{(1)}(\tau) &= \{x; G^{-1}(\tau) \leq x \leq G^{-1}(1-\tau)\}, \\
S_N^{(2)}(\tau) &= \{x; Z_{K_N} < x < G^{-1}(\tau)\}, \\
S_N^{(3)}(\tau) &= \{x; G^{-1}(1-\tau) < x < Z_{N-K_N+1}\},
\end{aligned}$$

then

$$C_{2N}^{(2)} = \sum_{j=1}^{3} \int_{S_N^{(j)}(\tau)} |h_1(G_n(z)) - h_1(G(z)) - (G_n(z) - G(z))h_1'(G(z))| dG(z).$$

By definition of the derivative of h_1, for $^\forall|v| \leq \tau$, $\tau \leq {}^\forall u \leq 1 - \tau$, we have

$$\begin{aligned}
&\sqrt{N} \left| h_1\left(u + \frac{v}{\sqrt{N}}\right) - h_1(u) - \frac{v}{\sqrt{N}} h_1'(u) \right| \\
&= \sqrt{N} o\left(\frac{v}{\sqrt{n}}\right) \\
&= o(1).
\end{aligned}$$

Thus,

$$\sup_{|v|\le\tau}\sup_{\tau\le u\le 1-\tau}\sqrt{N}\left|h_1\left(u+\frac{v}{\sqrt{N}}\right)-h_1(u)-\frac{v}{\sqrt{N}}h_1'(u)\right|\to 0 \qquad (N\to\infty).$$

Hence, we have with probability larger than $1-\varepsilon$,

$$\begin{aligned}&\int_{S_N^{(1)}(\tau)}\sqrt{N}|h_1(G_n(z))-h_1(G(z))-(G_n(z)-G(z))h_1'(G(z))|dG(z)\\&\quad=\int_{S_N^{(1)}(\tau)}\sqrt{N}\left|h_1\left(G(z)+\frac{\sqrt{N}(G_n(z)-G(z))}{\sqrt{N}}\right)\right.\\&\qquad\qquad\left.-h_1(G(z))-\frac{\sqrt{N}(G_n(z)-G(z))}{\sqrt{N}}h_1'(G(z))\right|dG(z)<\frac{\varepsilon}{2}.\end{aligned}$$

There exists $0<\phi<1$ for $j=2$, 3, such that

$$\begin{aligned}&\int_{S_N^{(j)}(\tau)}\sqrt{N}|h_1(G_n(z))-h_1(G(z))-(G_n(z)-G(z))h_1'(G(z))|dG(z)\\&\quad=\int_{S_N^{(j)}(\tau)}\sqrt{N}|G_n(z)-G(z)|\\&\qquad\qquad\cdot|h_1'(\phi G(z)+(1-\phi)G_n(z))-h_1'(G(z))|dG(z).\end{aligned}$$

We have

$$\begin{aligned}&\int_{S_N^{(j)}(\tau)}\sqrt{N}|G_n(z)-G(z)||h_1'(\phi G(z)+(1-\phi)G_n(z))-h_1'(G(z))|dG(z)\\&\quad\le\int_{S_N^{(j)}(\tau)}\sqrt{N}|G_n(z)-G(z)|\\&\qquad\qquad\cdot(|h_1'(\phi G(z)+(1-\phi)G_n(z))|+|h_1'(G(z))|)dG(z).\end{aligned}$$

Now by Lemma 3.1, we have with probability larger than $1-\varepsilon$,

$$\sqrt{N}|G_n(z)-G(z)|\le c(\varepsilon,\delta')(G(z)(1-G(z)))^{1/2-\delta'},$$

and by assumption (b), we have

$$\begin{aligned}&|h_1'(\phi G(z)+(1-\phi)G_n(z))|\\&\quad\le K((\phi G(z)+(1-\phi)G_n(z))(1-(\phi G(z)+(1-\phi)G_n(z))))^{-1-1/2+\delta},\end{aligned}$$

and

$$|h_1'(G(z))|\le K(G(z)(1-G(z)))^{-1-1/2+\delta}.$$

So

$$\begin{aligned}
&\int_{S_N^{(j)}(\tau)} \sqrt{N}|G_n(z)-G(z)|(|h_1'(\phi G(z)+(1-\phi)G_n(z))|+|h_1'(G(z))|)dG(z) \\
&\quad \le \int_{S_N^{(j)}(\tau)} c(\varepsilon,\delta')(G(z)(1-G(z)))^{1/2-\delta'} \\
&\qquad \times (K((\phi G(z)+(1-\phi)G_n(z))(1-(\phi G(z)+(1-\phi)G_n(z))))^{-1-1/2+\delta} \\
&\qquad + K(G(z)(1-G(z)))^{-1-1/2+\delta})dG(z) \\
&\quad = \int_{S_N^{(j)}(\tau)} c(\varepsilon,\delta')K(G(z)(1-G(z)))^{-1+\delta-\delta'} \\
&\qquad \times \left(\left(\frac{(\phi G(z)+(1-\phi)G_n(z))(1-(\phi G(z)+(1-\phi)G_n(z)))}{G(z)(1-G(z))}\right)^{-1-1/2+\delta}\right. \\
&\qquad \left. +1\right)dG(z).
\end{aligned}$$

Since

$$\frac{\phi G(z)+(1-\phi)G_n(z)}{G(z)} = \phi+(1-\phi)\frac{G_n(z)}{G(z)}$$

and

$$\frac{1-(\phi G(z)+(1-\phi)G_n(z))}{1-G(z)} = \phi+(1-\phi)\frac{1-G_n(z)}{1-G(z)},$$

there exists β such that

$$\begin{aligned}
&\frac{\inf_{z\in S_N^{(j)}(\tau)}(\phi G(z)+(1-\phi)G_n(z))(1-(\phi G(z)+(1-\phi)G_n(z)))}{G(z)(1-G(z))} \\
&\quad > \beta^2\left(\frac{N}{N+1}\right)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(\frac{(\phi G(z)+(1-\phi)G_n(z))(1-(\phi G(z)+(1-\phi)G_n(z)))}{G(z)(1-G(z))}\right)^{-1-1/2+\delta}+1 \\
&\quad < \left(\beta_1\frac{N}{N+1}\right)^{-3+2\delta}+1.
\end{aligned}$$

Now, it follows that

$$\begin{aligned}
&\sum_{j=2}^{3}\int_{S_N^{(j)}(\tau)} (G(z)(1-G(z)))^{\delta-\delta'-1}dG_n(z) \\
&\quad \le \left(\int_0^\tau+\int_{1-\tau}^1\right)(G(z)(1-G(z)))^{\delta-\delta'-1}dG_n(z) \\
&\quad \le \left(\frac{2^{1-\delta-\delta'}}{\delta-\delta'}\right)\tau^{\delta-\delta'}.
\end{aligned}$$

Since for all $\varepsilon > 0, \delta'$ and τ may be so chosen that

$$c(\varepsilon, \delta')K\left(\frac{2^{1-\delta-\delta'}}{\delta-\delta'}\right)\tau^{\delta-\delta'}\left(\left(\beta_1\frac{N}{N+1}\right)^{-3+2\delta}+1\right)<\frac{\varepsilon}{2},$$

we have $C_{2N}^{(2)} = o_p(1)$, and $C_{2N} = o_p(1)$.

$C_{4N} = o_p(1)$ can be shown similarly.

Step 2. Finally, we show that $B_{1N} + B_{2N} + B_{3N} + B_{4N}$ has the normal distribution in the limit. We obtain

$$\begin{aligned}B_{1N} &= \lambda_N\sqrt{N}\int h_1(G(x))d(F_m(x)-F(x))\\&= \lambda_N\sqrt{N}\int B_1(x)d(F_m(x)-F(x)),\end{aligned}$$

and

$$\begin{aligned}B_{2N} &= \lambda_N\sqrt{N}\int (G_n(x)-G(x))h_1'(G(x))dF(x)\\&= \lambda_N\sqrt{N}\left(\left[(G_n(x)-G(x))\int_{x_0}^{x} h_1'(G(y))dF(y)\right]_{-\infty}^{\infty}\right.\\&\qquad\left.-\int_{-\infty}^{\infty}\int_{x_0}^{x} h_1'(G(x))dF(y)d(G_n(x)-G(x))\right)\\&= -\lambda_N\sqrt{N}\int_{-\infty}^{\infty} B_2(x)d(G_n(x)-G(x)),\end{aligned}$$

where

$$B_1(x) = h_1(G(x)) \quad\text{and}\quad B_2(x) = \int_{x_0}^{x} h_1'(G(y))dF(y).$$

Thus $B_{1N} + B_{2N}$ is represented by

$$\lambda_N\sqrt{N}\left(\frac{1}{m}\sum_{i=1}^{m}(B_1(X_i)-E[B_1(X_i)]) - \frac{1}{n}\sum_{j=1}^{n}(B_2(Y_j)-E[B_2(Y_j)])\right).$$

Similarly we obtain

$$\begin{aligned}&B_{3N}+B_{4N}\\&\quad= (1-\lambda_N)\sqrt{N}\left(-\frac{1}{m}\sum_{i=1}^{m}(B_3(X_i)-E[B_1(X_i)])\right.\\&\qquad\qquad\left.+\frac{1}{n}\sum_{j=1}^{n}(B_4(Y_j)-E[B_2(Y_j)])\right),\end{aligned}$$

where

$$B_3(x) = \int_{x_0}^{x} h_2'(F(y))dG(y),$$
$$B_4(x) = h_2(F(x)) = \int_{x_0}^{x} h_2'(F(y))dF(y).$$

Therefore,

$$\begin{aligned}
&B_{1N} + B_{2N} + B_{3N} + B_{4N} \\
&\quad = \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{m} \left((B_1(X_i) - E[B_1(X_i)]) - \frac{n}{m}(B_3(X_i) - E[B_3(X_i)]) \right) \right. \\
&\qquad \left. + \sum_{j=1}^{n} \left(-\frac{m}{n}(B_2(Y_j) - E[B_2(Y_j)]) + (B_4(Y_j) - E[B_4(Y_j)]) \right) \right)
\end{aligned}$$

We shall show that the variances of

$$(B_1(X) - E[B_1(X)]) - \frac{n}{m}(B_3(X) - E[B_3(X)])$$

and

$$-\frac{m}{n}(B_2(Y) - E[B_2(Y)]) + (B_4(Y) - E[B_4(Y)])$$

are finite. Since

$$\begin{aligned}
B_1(X) - E[B_1(X)] &= \int_{-\infty}^{\infty} h_1(G(x))d(F_1(x) - F(x)) \\
&= -\int_{-\infty}^{\infty} (F_1(x) - F(x))h_1'(G(x))dG(x), \\
B_3(X) - E[B_3(X)] &= \int_{-\infty}^{\infty} \int_{x_0}^{x} h_2'(F(y))dG(y)d(F_1(x) - F(x)) \\
&= -\int_{-\infty}^{\infty} (F_1(x) - F(x))h_2'(F(x))dG(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{Var}(B_1(X)) \\
&\quad = E\left[\iint (F_1(x) - F(x))(F_1(y) - F(y))h_1'(G(x))h_1'(G(y))dG(x)dG(y) \right] \\
&\quad = 2 \iint_{-\infty<x<y<\infty} F(x)(1 - F(y))h_1'(G(x))h_1'(G(y))dG(x)dG(y), \\
&\mathrm{Cov}(B_1(X), B_3(X)) \\
&\quad = \iint_{-\infty<x<y<\infty} F(x)(1 - F(y))h_1'(G(x))h_2'(F(y))dG(x)dG(y) \\
&\qquad + \iint_{-\infty<y<x<\infty} (1 - F(x))F(y)h_1'(G(x))h_2'(F(y))dG(x)dG(y),
\end{aligned}$$

and

$$\mathrm{Var}(B_3(X)) = 2\iint_{-\infty<x<y<\infty} F(x)(1-F(y))h_2'(F(x))h_2'(F(y))dG(x)dG(y).$$

Therefore, we have

$$\begin{aligned}
&\mathrm{Var}\left((B_1(X)-E[B_1(X)]) - \frac{n}{m}(B_3(X)-E[B_3(X)])\right) \\
&= \mathrm{Var}(B_1(X)) - 2\frac{n}{m}\,\mathrm{Cov}(B_1(X),B_3(X)) + \frac{n^2}{m^2}\,\mathrm{Var}(B_3(X)) \\
&< \infty.
\end{aligned}$$

Similarly

$$\mathrm{Var}\left(-\frac{m}{n}(B_2(Y)-E[B_2(Y)]) + (B_4(Y)-E[B_4(Y)])\right) < \infty.$$

Thus we complete the proof.

Example. (continued) Let F_N^* and R_N^* be the test statistics induced by Spearman's footrule and Spearman's rho. Under null hypothesis H : $F(x) \equiv G(x)$, we get

$$\begin{aligned}
\mu_{F_N^*} &= \frac{mn}{2N}, \\
\sigma^2_{F_N^*} &= \frac{mn}{3N}, \\
\mu_{R_N^{*}{}^2} &= \frac{m^2n+mn^2}{3N^2}, \\
\sigma^2_{R_N^{*}{}^2} &= \frac{mn(4m^2+7mn+4n^2)}{45N^3}.
\end{aligned}$$

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References

Critchlow, D. E. (1985). Metric methods for analyzing partially ranked data, *Lecture Notes in Statist.*, **34**, Springer, New York.

Critchlow, D. E. (1992). On rank statistics: an approach via metrics on the permutation group, *J. Statist. Plann. Inference*, **32**, 325–346.

Fueda, K. (1993). Spearman's rank correlation type two-sample test, *Mem. Fac. Sci. Kyushu Univ. Ser. A*, **47**(1), 27–39.

Puri, M. L. and Sen, P. K. (1971). *Nonparametric Methods in Multivariate Analysis*, Wiley, New York.