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RELATIONSHIPS BETWEEN PURE TAIL ORDERINGS OF LIFETIME DISTRIBUTIONS AND SOME CONCEPTS OF RESIDUAL LIFE

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Abstract. The concepts of pure-tail orderings as defined by *sq*- and D-orderings are shown to order the family of reliability life distributions which age smoothly in a natural way. This ordering extends to comparisons regarding the limiting behavior of the residual life, mean residual life, sojourn time between perfect repairs in repairable systems, failure rate and, through the preservation of *sq*- and D-orderings by various reliability operations, to certain coherent systems of components that age smoothly. Possible applications of the results to the industrial practice of cannibalization are also noted.

Key words and phrases: Residual life, mean residual life, increasing failure rate, regular variation, D-ordering, coherent systems, decreasing failure rate (DFR), age-smooth life distributions.

1. Introduction

The concepts of life distributions with aging or anti-aging properties play a central role in the theory and practice of reliability. Thus, for example, the concepts of Increasing Failure Rate (IFR), Increasing Failure Rate on the Average (IFRA), and New Better than Used (NBU), life distributions have received well-deserved attention in the reliability literature (see e.g., Barlow and Proschan (1975)), and have proven to be quite useful in many applications in reliability and maintenance. Intricately related are the concepts of partial orderings of life distributions, examples of which we mention convex ordering, star-shaped ordering, and superadditive ordering. Indeed, it is well-known that a survival function S is IFR if and only if S is smaller, according to the convex ordering, than the exponential survival function. Similarly, S is IFRA (NBU) if and only if S is smaller than the exponential survival function according to the star-shaped (superadditive) ordering. Other partial orderings of survival functions with interesting reliability applications include stochastic-, failure rate-, and likelihood ratio-ordering (see e.g., Singh and Vijayasree (1991), Lehmann and Rojo (1992)). One common feature of these partial orderings is that they order the whole distribution. In some industrial situations, however, the relevant comparisons among survival functions involve only the right tail of the survival functions. An example of such a situation is provided by the industrial practice of "cannibalization" (see Isaacson *et al.* (1988)) whereby the used parts from a given failed system (judged to be beyond repair) are used as replacement parts in the repair and maintenance of another repairable system.

In defining an optimal replacement policy for failed items, various costs must be taken into account. However, there are situations in which the only practical available replacement policy is cannibalization. One such situation is the escalating need to maintain proper emergency medical services during and after natural catastrophic events. The increase in load on emergency power generators, for example, will be abrupt and demanding during the initial stages of the catastrophe. Typically, the variability in the demand for spare parts under such conditions is time dependent and difficult to predict. See, for example, Crawford (1987). It is thus likely that spare parts will not be available in some instances. The final outcome of the catastrophe depends heavily on the ability to maintain a minimum number of emergency medical services and communication systems in a fully capable status; the cannibalization strategy may be the only viable replacement policy under these conditions.

Thus, in replacing a failed component in a repairable system, the choice between two spare parts of the same age would surely be influenced by a comparison of the survival functions in the right tail in terms of either residual lifetime or mean residual life for example.

Recently, Rojo (1988, 1992) introduced q- and D-ordering which order survival functions according to their right tail behavior. A detailed comparison of the q- and D-ordering was carried out in Rojo (1988) and Rojo (1992), and closure properties under reliability operators were discussed in Rojo (1993). Klüppelberg (1990) provides an interesting treatment of (right) tail equivalence. The related and interesting concept of age-smoothness of failure distributions which captures properties which are essentially tail dependent has been proposed by Bhattacharjee (1986).

The purpose of the present paper is to demonstrate that sq- and D-orderings order the family of reliability life distributions which age-smoothly in a natural way. As a consequence, sq- and D-orderings impose an ordering on comparisons regarding the limiting behavior of residual lifetime, mean residual lifetime, and sojourn time between perfect repairs in repairable systems.

2. Ordering the family of age-smooth life distributions

The notions of life distributions being IFR, IFRA, NBU, etc. are central to the theory and practice of reliability, and have been helpful in developing bounds on system reliability and studying functionals of interest in maintenance policies. The characteristic of aging notions, such as IFR, IFRA and NBU, is that these notions impose a requirement on the whole distribution. For some applications, such as cannibalization, this may be too strong a requirement. Bhattacharjee (1986) successfully weakened the notions of IFR and IFRA by introducing the notion of life distributions which age smoothly and showed its connection to the notion of regular variation. In what follows the hazard function of a life distribution F will be denoted by H_F. That is, $H_F(x) = -\ln \bar{F}(x)$. It will be assumed throughout that $F(x) < 1$ for all $x > 0$.

DEFINITION 1. A life distribution F is age-smooth if

$$\lim_{t\to\infty} (H_F(t+x) - H_F(t)) \quad \text{exists for each} \quad x > 0. \tag{2.1}$$

Thus, the life distribution F is age-smooth if the hazard function of the residual lifetime converges in the extended sense as age increases. Also, note that (2.1) is equivalent to the existence of $\lim_{t\to\infty} \bar{F}_t(x)$ for each $x > 0$, where $\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t)$ denotes the survival function of the residual life at age t.

It is clear from Definition 1 that a life distribution F is age-smooth if the function $\bar{F}_t(x)$ does not oscillate in a neighborhood of $t = \infty$, and hence the name age-smooth. Examples of distribution functions which are not age-smooth are provided next.

Example 1. Let $\bar{F}(x) = c(1 + \frac{1}{(1+x)^{1/2}} + \sin((1+x)^{1/2}))e^{-x}$, where $c = (2+\sin(1))^{-1}$. Also, $\bar{G}(x) = (\frac{1}{(\ln x)^{1/3}} + 1 + \sin((\ln x)^{1/3}))x^{-1/3}$, $x > x_1 > 0$ where $\bar{G}(x_1) = 1$. Then it is not difficult to verify that $\lim_{t\to\infty} \bar{F}_t(x)$ and $\lim_{t\to\infty} \bar{G}_t(x)$ do not exist due to the oscillating nature of $\bar{F}_t(x)$ and $\bar{G}_t(x)$.

Let $\mathcal{F}$ denote the family of age-smooth distributions. This family is of great interest in reliability since, under some mild regularity conditions (see Bhattacharjee (1986)), the family $\mathcal{F}$ contains the IFR and DFR classes, while having a nonempty intersection with the IFRA, NBU, DMRL, and NBUE classes and their duals. Recall the following definition (Bingham *et al.* (1987), p. 18),

DEFINITION 2. A measurable function $f > 0$ such that $f(ax)/f(x) \to a^\rho$, as $x \to \infty$, for all $a > 0$ is called regularly varying of index ρ.

Bhattacharjee (1986) showed that

$$F \in \mathcal{F} \Leftrightarrow \bar{F}(\ln x) \quad \text{is regularly varying with index } -\rho, \ 0 \le \rho \le \infty. \tag{2.2}$$

Moreover, F is age-smooth with $0 \le \rho < \infty$ if and only if it can be represented in a unique manner as $\bar{F}(t) = e^{-\rho t}L(e^t)$ for some slowly varying L. When F is age-smooth with $\rho = \infty$, then F has the unique representation $\bar{F}(\ln L(1+x)) = (1+x)^{-1}$, $x > 0$ for some slowly varying L as demonstrated by Bhattacharjee (1986). Since the family $\mathcal{F}$ is motivated by considerations of pure-tail behavior of life distributions, it is of interest to introduce a notion of ordering in $\mathcal{F}$, which should reflect pure-tail properties of these life distributions. It is the characterization given by (2.2) that paves the way for the introduction of a simple pure-tail partial ordering on $\mathcal{F}$. This is done by introducing the following concepts: Let $F^{-1}(u) = \inf\{t : F(t) \ge u\}$, $0 < u < 1$ and define, DEFINITION 3. Let F, G be life distributions. Then

$$\begin{array}{lll} F \leq_D G (F \leq_q G) & \text{if} & \overline{\lim}_{x\to\infty} \dfrac{\bar{F}(x)}{\bar{G}(x)} < \infty \left(\overline{\lim}_{u\to 1} \dfrac{F^{-1}(u)}{G^{-1}(u)} < \infty \right) \\ F <_D G (F <_q G) & \text{if} & F \leq_D G \text{ and } G \not\leq_D F (F \leq_q G \text{ and } G \not\leq_q F) \\ F \sim_D G (F \sim_q G) & \text{if} & F \leq_D G \text{ and } G \leq_D F (F \leq_q G \text{ and } G \leq_q F). \end{array}$$

The relationship between q-ordering and D-ordering is considered in Rojo (1992). Closure properties of q- and D-orderings under various reliability operations are considered by Rojo (1993). See also the nice treatment of equivalent tailed distributions by Klüppelberg (1990).

Next, the structure of $\mathcal{F}$, as characterized by (2.2), is exploited to delineate the consequences of ordering $\mathcal{F}$ by using $\leq_D$. For that purpose, let F and G be distributions in $\mathcal{F}$, and let $-\rho_F$ and $-\rho_G$ represent the index of regular variation of F and G, respectively. The following result concludes that if F and G are ordered according to D-ordering then the limiting values, as age increases, of the failure rates, mean residual lifetimes, and residual lifetimes of F and G are also ordered in a natural way. The proofs of all the results are included in the Appendix. In what follows, if F denotes a life distribution with density f, $r_F(t)$, $m_F(t)$, and F_t denote, respectively, its failure rate function defined by $f(t)/\bar{F}(t)$, its mean residual life at time t defined by $\int_t^\infty \bar{F}(x)dx/\bar{F}(t)$, and residual life at time t. The proofs of the main results hinge on the following lemma.

LEMMA 2.1. *Let F be a life distribution in $\mathcal{F}$ with density f. Then,*
(i) $\lim_{t\to\infty} r_F(t) = \rho_F$
(ii) $\lim_{t\to\infty} m_F(t) = 1/\rho_F$,
where ρ_F is the index of regular variation defined by (2.2).

The following result relating the comparison of the limiting value of r_F and r_G to the D-ordering of F and G is the main result of this section.

THEOREM 2.1. *Let F and G be life distributions in $\mathcal{F}$. Then,*
(i) $F \leq_D (\sim_D) G$ *implies* $\lim_{t\to\infty} r_F(t) \geq (=) \lim_{t\to\infty} r_G(t)$, *and* $\lim_{t\to\infty} r_F(t) > \lim_{t\to\infty} r_G(t)$ *implies* $F <_D G$.
(ii) $F \leq_D (\sim_D) G$ *implies* $\lim_{t\to\infty} m_F(t) \leq (=) \lim_{t\to\infty} m_G(t)$, *and* $\lim_{t\to\infty} m_F(t) < \lim_{t\to\infty} m_G(t)$ *implies* $F <_D G$.
(iii) $F \leq_D (\sim_D, <_D) G \Leftrightarrow F_t \leq_D (\sim_D, <_D) G_t$ *for all* t.

The converse to (i) and (ii) does not hold as shown in the following example.

Example 2. Let $\bar{F} = xe^{-(x-1)}$, $x > 1$ and $\bar{G} = \frac{1}{4}x^2e^{-(x-2)}$, $x > 2$. Then $r_F(t) = 1 - \frac{1}{t} \to 1$ and $r_G(t) = 1 - \frac{2}{t} \to 1$ while, also, $m_F(t) \to 1$ and $m_G(t) \to 1$. However, $\overline{\lim}_{x\to\infty} \bar{F}(x)/\bar{G}(x) = 0$ and hence $F <_D G$ while $\lim_{t\to\infty} r_F(t) = \lim_{t\to\infty} r_G(t)$ and $\lim_{t\to\infty} m_F(t) = \lim_{t\to\infty} m_G(t)$.

Note that q-ordering as defined by Definition 3 is not strong enough to order $\mathcal{F}$. Indeed, let $F, G \in \mathcal{F}$, with $\bar{F}(x) = e^{-\rho_F x} L_1(e^x)$ and $\bar{G}(x) = e^{-\rho_G x} L_2(e^x)$ where $0 < \rho_F, \rho_G < \infty$. Then,

$$\overline{\lim_{u \to 1}} \frac{F^{-1}(u)}{G^{-1}(u)} = \frac{\rho_G}{\rho_F} \overline{\lim_{u \to 1}} \left(\frac{1 - \ln L_1(e^{F^{-1}(u)})/\ln(1-u)}{1 - \ln L_2(e^{G^{-1}(u)})/\ln(1-u)} \right). \tag{2.3}$$

Now,

$$\lim_{u \to 1} \frac{\ln L_1(e^{F^{-1}(u)})}{\ln(1-u)} = \lim_{t \to \infty} \frac{\ln L_1(t)}{\ln \bar{F}(\ln t)} = \lim_{t \to \infty} \frac{\ln L_1(t)}{\ln L_1(t) - \rho_F \ln t} = 0 \tag{2.4}$$

and similarly $(\ln L_2(e^{G^{-1}(u)}))/\ln(1-u) \to 0$, so that $F \sim_q G$ whenever $0 < \rho_F, \rho_G < \infty$.

A partial ordering stronger than q-ordering is now introduced in the family $\mathcal{F}$.

Definition 4. Let $F, G \in \mathcal{F}$. Then

$$F \leq_{sq} G \quad \text{if} \quad \overline{\lim_{u \to 1}} \frac{F^{-1}(u)}{G^{-1}(u)} \leq 1$$

$$F <_{sq} G \quad \text{if} \quad \overline{\lim_{u \to 1}} \frac{F^{-1}(u)}{G^{-1}(u)} < 1$$

$$F \sim_{sq} G \quad \text{if} \quad \overline{\lim_{u \to 1}} \frac{F^{-1}(u)}{G^{-1}(u)} = 1.$$

Note that the argument given by (2.3) and (2.4) shows that $\lim_{u \to 1} \frac{F^{-1}(u)}{G^{-1}(u)} = \frac{\rho_G}{\rho_F}$ for $F, G \in \mathcal{F}$. Therefore, the following theorem follows immediately by using Lemma 2.1.

Theorem 2.2. *Let $F, G \in \mathcal{F}$ with $\rho_F > 0$ and $\rho_G < \infty$ or $\rho_F < \infty$ and $\rho_G > 0$. Then,*

(i) $F \leq_{sq}, \sim_{sq}, <_{sq} G \Leftrightarrow \lim_{t \to \infty} r_F(t) \geq, =, > \lim_{t \to \infty} r_G(t)$

(ii) $F \leq_{sq}, \sim_{sq}, <_{sq} G \Leftrightarrow \lim_{t \to \infty} m_F(t) \leq, =, < \lim_{t \to \infty} m_G(t)$

(iii) $F \leq_{sq}, \sim_{sq}, <_{sq} G \Leftrightarrow F_t \leq_{sq}, \sim_{sq}, <_{sq} G_t$ *for each* t.

Note that Theorem 2.2 does not hold if $\rho_F = \rho_G = 0$ or $\rho_F = \rho_G = \infty$. To see this, take $\bar{F}(x) = 1/(1+x)$ and $\bar{G}(x) = 1/(1+x^2)$, $x > 0$, so that $\rho_F = \rho_G = 0$. It follows that $G <_{sq} F$ while $\lim_{t \to \infty} r_F(t) = \lim_{t \to \infty} r_G(t) = 0$. Also, taking $\bar{F}(x) = \exp(-\exp(x))$ and $\bar{G}(x) = \exp(-\exp(x^{1/2}))$, it follows that $F^{-1}(u) = \ln\ln(1/(1-u))$ and $G^{-1}(u) = 2\ln\ln(1/(1-u))$ so that $F^{-1}(u)/G^{-1}(u) = 1/2$ and hence $F <_{sq} G$. However, $\rho_F = \rho_G = \infty$.

3. Ordering of systems with age-smooth components

Of more interest to applications in reliability is the ordering of systems with age-smooth components. For this purpose, closure properties of the family of age-smooth life distributions $\mathcal{F}$ under various reliability operations must be considered. One such property was discussed by Bhattacharjee (1986), and it is stated in the following lemma.

LEMMA 3.1. *Let $F \in \mathcal{F}$. Then $F_\alpha = 1 - \bar{F}^\alpha \in \mathcal{F}$, $\alpha > 0$.*

Note that for α an integer, F_α is the lifetime distribution of a series system of i.i.d. components. It follows that the age-smooth property is preserved by the formation of series systems of i.i.d. components. On the other hand, for arbitrary $\alpha > 0$, F_α represents the distribution of the sojourn time between perfect repairs in the imperfect maintenance model of Brown and Proschan (1980) in which the repair of a failed equipment is minimal or perfect (i.e. post recovery lifetime equals residual life or original life, respectively) with probabilities α and $(1-\alpha)$, respectively. It follows then, from Lemma 3.1, that age-smoothness is inherited by the sojourn time between perfect repairs in the model of Brown and Proschan (1980).

Bhattacharjee (1986) also pointed out that other closure properties of $\mathcal{F}$, such as convolution and finite mixtures, were not known. However, there have appeared results in the literature which imply in particular the closure of $\mathcal{F}$ under convolution. We state this result and others in the following theorem.

THEOREM 3.1.

(i) *Let $F, G \in \mathcal{F}$ with $\rho_F, \rho_G \geq 0$. Then, $H = F * G \in \mathcal{F}$, with $\rho_H = \min(\rho_F, \rho_G)$.*

(ii) *Let $F_i \in \mathcal{F}$, $i = 1, \ldots, n$ with $\rho_i \geq 0$. Then $\sum_{i=1}^n \alpha_i F_i \in \mathcal{F}$, where $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$, so that $\mathcal{F}$ is closed under finite mixtures.*

(iii) *Consider an coherent system with component lifetimes given by $F_1, \ldots, F_n \in \mathcal{F}$, and suppose $\lim_{x\to\infty} \bar{F}_j(x)/\bar{F}_{i_0}(x) < \infty$, $j = 1, \ldots, n$ for some i_0. Let G denote the system's lifetime distribution. Then, $G \in \mathcal{F}$.*

Note that, in particular, when the components of the coherent system are identical, the assumption that $\lim_{x\to\infty} \bar{F}_j(x)/\bar{F}_{i_0}(x) < \infty$, $j = 1, \ldots, n$ for some i_0 holds. As a consequence of Theorems 2.1, 2.2 and 3.1, and Lemma 3.1, we obtain the following.

COROLLARY 3.1. *Denote the two-fold convolution of the life distribution function F by $F^{(2)}$. Then,*

(i) *Theorems 2.1 and 2.2 hold with r_F, r_G, m_F, m_G, F_t, G_t replaced by $r_F^{(2)}$, $r_G^{(2)}$, $m_F^{(2)}$, $m_G^{(2)}$, $F_t^{(2)}$, $G_t^{(2)}$.*

(ii) *Let H_1 and H_2 be the lifetime distributions of a coherent system with i.i.d. components and lifetimes F and $G \in \mathcal{F}$, respectively. Then Theorems 2.1 and 2.2 hold with r_F, r_G, m_F, m_G, F_t, G_t replaced by r_{H_1}, r_{H_2}, m_{H_1}, m_{H_2}, $H_{1,t}$, $H_{2,t}$, respectively, where $\bar{H}_{i,t} = \bar{H}_i(t+x)/\bar{H}_i(t)$.*

(iii) *Let* F_α *and* G_α *be the lifetime distributions given by* $F_\alpha = 1 - \bar{F}^\alpha$, $G_\alpha = 1 - \bar{G}^\alpha$ *for* $F, G \in \mathcal{F}$, $\alpha > 0$. *Then Theorems* 2.1 *and* 2.2 *hold with* r_F, r_G, m_F, m_G, F_t, G_t *replaced by* r_{F_α}, r_{G_α}, m_{F_α}, m_{G_α}, $F_{\alpha,t}$, $G_{\alpha,t}$, *respectively.*

Similar results may be stated for finite mixtures of elements of $\mathcal{F}$. The statement of the result and its simple proof are left to the reader.

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Appendix

PROOF OF LEMMA 2.1. The case for which $0 \le \rho < \infty$ is considered first. Using the representation (1.5.2) on page 21 of Bingham *et al.* (1987), it follows that

$$r_F(t) = \frac{d}{dt} - \ln \bar{F}(t) = \rho_F - L'(e^t)e^t/L(e^t), \tag{A.1}$$

where L is slowly varying at ∞. Now, by a Theorem of de Bruijin (1959) (see Bingham *et al.* (1987), Theorem 1.3.3), there is a slowly varying function L_1, with $L_1 \in C^\infty[0,\infty)$, and such that $L_1'(e^t)e^t/L_1(e^t) \to 0$ as $t \to \infty$, and $L(t)/L_1(t) \to 1$ as $t \to \infty$. It follows that $\ln L(e^t) - \ln L_1(e^t) \to 0$ as $t \to \infty$, and since $L_1'(e^t)e^t/L_1(e^t) \to 0$, $d \ln L(e^t)/dt \to 0$ as $t \to \infty$. It follows from (A.1) that $r_F(t) \to \rho_F$. To show (ii), note that by L'Hopital's rule $\lim_{t\to\infty} m_F(t) = \lim_{t\to\infty} \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \lim_{t\to\infty} \frac{1}{r_F(t)} = \frac{1}{\rho_F}$.

If $\rho = \infty$, then by Bhattacharjee (1986), $\bar{F}(\ln L(1+x)) = 1/(1+x)$ for some slowly varying function L. It follows that $r_F(\ln L(1+x)) = L(1+x)/((1+x)L'(1+x))$. One more application of Theorem 1.3.3 in Bingham *et al.* (1987), yields the result.

PROOF OF THEOREM 2.1. To prove (i) and (ii), note that as a consequence of (2.2), $F \le_D G$ implies $\rho_F \ge \rho_G$. Then, by using Lemma 2.1 the conclusion follows.

(iii) This follows immediately from the definitions of D-ordering, and F_t. □

PROOF OF THEOREM 3.1. (i) The case $0 < \rho_F < \rho_G$ follows immediately from Breiman (1965) Proposition 3. The case $\rho_F = \rho_G$ follows from Theorem 3 of Embrechts and Goldie (1980) or Feller ((1971), p. 278). The case where $\rho_F \ne \rho_G$ with either $\rho_F = 0$ or $\rho_G = 0$, also follows from Feller.

(ii) To show that $G(t) = \sum_{i=1}^{n} \alpha_i F_i(t) \in \mathcal{F}$, using (2.2), it is enough to show that $\bar{G}(\ln t)$ is regularly varying. Now, $\bar{G}(\ln t) = \sum_{i=1}^{n} \alpha_i \bar{F}_i(\ln t)$ with each $\bar{F}_i(\ln t)$ regularly varying. It then follows from Proposition 1.5.7 of Bingham *et al.* (1987), that $\bar{G}(\ln t)$ is regularly varying.

(iii) Let $h : R^n \to [0,1]$ denote the reliability function of the system. Let $\bar{H}(x)$ be the vector $(\bar{F}_1(x), \dots, \bar{F}_n(x))$, we use the notation $h(0_i, \bar{F}_1(x), \dots, \bar{F}_n(x)) = h(\bar{F}_1(x), \dots, \bar{F}_{i-1}(x), 0, \bar{F}_{i+1}(x), \dots, \bar{F}_n(x))$ and $h(1_i, \bar{F}_1(x), \dots, \bar{F}_n(x)) = h(\bar{F}_1(x), \bar{F}_2(x), \dots, \bar{F}_{i-1}(x), 1, \bar{F}_{i+1}(x), \dots, \bar{F}_n(x))$. Then,

$$\frac{\bar{G}(\ln(tx))}{\bar{G}(\ln(x))} = \frac{\bar{F}_{i_0}(\ln(tx))}{\bar{F}_{i_0}(\ln(x))} \cdot \left\{ \frac{h(1_{i_0}, \bar{H}(\ln(tx))) + (F_{i_0}(\ln(tx))/\bar{F}_{i_0}(\ln(tx)))h(0_{i_0}, \bar{H}(\ln(tx)))}{h(1_{i_0}, \bar{H}(\ln(x))) + (F_{i_0}(\ln(x))/\bar{F}_{i_0}(\ln(x)))h(0_{i_0}, \bar{H}(\ln(x)))} \right\}.$$

Since $\lim_{x\to\infty} \bar{F}_j(x)/\bar{F}_{i_0}(x) < \infty$, $j = 1, \dots, n$, then $\lim_{x\to\infty} h(1_{i_0}, \bar{H}(\ln(x))) + (F_{i_0}(\ln(x))/\bar{F}_{i_0}(\ln(x)))h(0_{i_0}, \bar{H}(\ln(x))) < \infty$, and hence $\lim_{x\to\infty} \bar{G}(\ln(tx))/\bar{G}(\ln(x)) = \lim_{x\to\infty} \bar{F}_{i_0}(\ln(tx))/\bar{F}_{i_0}(\ln(x))$ and the result follows. $\square$

Proof of Corollary 3.1. (i) Follows immediately from (i) of Theorem 3.1 and Theorems 2.1 and 2.2.

(ii) In the case of i.i.d. components $\bar{H}_1(\ln t)$ and $\bar{H}_2(\ln t)$ are polynomials in $\bar{F}(\ln t)$ and $\bar{G}(\ln t)$ of the same degree, say k, and hence $\bar{H}_1(\ln t)$ is $(-\rho_F k)$-regularly varying while $\bar{H}_2(\ln t)$ is $(-\rho_G k)$-regularly varying. The result then follows from Theorems 2.1 and 2.2.

(iii) This follows immediately after observing that since $\bar{F}(\ln t)$ and $\bar{G}(\ln t)$ are $(-\rho_F)$- and $(-\rho_G)$-regularly varying, then $\bar{F}_\alpha(\ln t)$ and $\bar{G}_\alpha(\ln t)$ are $(-\alpha\rho_F)$- and $(-\alpha\rho_G)$-regularly varying. Using Theorems 2.1 and 2.2 the result follows. $\square$

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