Ann. Inst. Statist. Math.
Vol. 48, No. 1, 185–199 (1996)

# LIFETIME DISTRIBUTION AND ESTIMATION PROBLEMS OF CONSECUTIVE-$k$-OUT-OF-$n$:F SYSTEMS*

SIGEO AKI[1] AND KATUOMI HIRANO[2]

[1] *Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Machikaneyama-cho, Toyonaka, Osaka 560, Japan*
[2] *The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan*

(Received February 13, 1995; revised July 13, 1995)

**Abstract.** Explicit formula is given for the lifetime distribution of a consecutive-$k$-out-of-$n$:F system. It is given as a linear combination of distributions of order statistics of the lifetimes of $n$ components. We assume that the lifetimes are independent and identically distributed. The results should make it possible to treat the parametric estimation problems based on the observations of the lifetimes of the system. In fact, we take up, as some examples, the cases where the lifetimes of the components follow the exponential, the Weibull, and the Pareto distributions, and obtain feasible estimators by moment method. In particular, it is shown that the moment estimator is quite good for the exponential case in the sense that the asymptotic efficiency is close to one.

*Key words and phrases*: Consecutive-$k$-out-of-$n$:F system, system reliability, failure time, discrete distributions of order $k$, order statistics, exponential distribution, Weibull distribution, Pareto distribution, moment estimator.

## 1. Introduction and general result

Many papers on consecutive-$k$-out-of-$n$:F system have been published. Recently, these works were surveyed by Hirano (1994) and Chao *et al.* (1995).

Let $k$ and $n$ be positive integers satisfying $n \geq k \geq 1$. A consecutive-$k$-out-of-$n$:F system (Con/$k$/$n$/F) is a system of $n$ components in sequence where the system fails if and only if $k$ consecutive components fail. For $i = 1, \dots, n$, let $X_i$ be a random variable such that $X_i = 1 (= 0)$ if the $i$-th component is functioning (resp. fails). Let $\phi(\boldsymbol{X}) = \phi(X_1, \dots, X_n)$ be the structure function of the Con/$k$/$n$/F system. Then, we can obtain that

$$\phi(\boldsymbol{X}) = \prod_{j=1}^{n-k+1} \left\{ 1 - \prod_{i=j}^{j+k-1} (1 - X_i) \right\}.$$

* This research was partially supported by the ISM Cooperative Research Program (94-ISM-CRP-5).

From this formula, we see that the Con/$k$/$n$/F system is a coherent system. For the definition of a coherent system, see Barlow and Proschan (1981).

Reliability of the Con/$k$/$n$/F system has been investigated by many researchers (cf. Derman *et al.* (1982), Lambiris and Papastavridis (1985), Hwang (1986, 1991), Hirano and Aki (1993)). When $X_1, X_2, \dots, X_n$ are independent identically distributed (i.i.d.) random variables with $P(X_i = 1) = p$ and $P(X_i = 0) = q(= 1 - p)$, the reliability of the system is given by

$$P(\phi(\boldsymbol{X}) = 1) = \sum_{r=0}^{[n/k]} (-1)^r (pq^k)^r \left\{ \binom{n-rk}{r} + \binom{n-k(r+1)}{r} (-1)q^k \right\}.$$

Recently, discrete distributions of order $k$ have been studied very actively. The distributions are closely related to distributions of runs in various random sequences. Relationships between the reliability of the Con/$k$/$n$/F system and discrete distributions of order $k$ were shown (cf. Aki (1985), Hirano (1986), Philippou (1986), Aki and Hirano (1989) and Johnson *et al.* (1992)). The binomial distribution of order $k$, which is denoted by $B_k(n,p)$, is the distribution of the number of occurrences of consecutive $k$ successes in $n$ independent Bernoulli trials $X_1, X_2, \dots, X_n$ with success probability $P(X_i = 1) = p$. Let $B_k(n,p;x)$, $x = 0, 1, \dots, [n/k]$ be the probability function of the binomial distribution of order $k$. Then, we see that $P(\phi(\boldsymbol{X}) = 1) = B_k(n, q; 0)$ where $q = 1 - p$. The corresponding relation still holds if the distribution of $(X_1, X_2, \dots, X_n)$ is supposed to be a Markov chain or a binary sequence of order $k$ (Aki (1985), Aki and Hirano (1988, 1993)).

Suppose that the random variables $X_1, \dots, X_n$ are independently and identically distributed with $P(X_i = 1) = p$ and $P(X_i = 0) = q(= 1 - p)$. Then, the reliability of the system is a function of $p$, so we set $h(p) = P(\phi(\boldsymbol{X}) = 1)$. Now, we consider the case that $p$ depends on time $t$. Let $\xi_i$ be the failure time of the $i$-th component and let $G_i(t)$ be its cumulative distribution function, i.e. $P(\xi_i \le t) = G_i(t)$. Assume further that $\xi_1, \xi_2, \dots, \xi_n$ are i.i.d. random variables. Then, since $G_1(t), \dots, G_n(t)$ are the same, we can write $G(t) = G_i(t)$, $i = 1, \dots, n$. Let $T$ be the failure time of the system. Then, the reliability of the Con/$k$/$n$/F system is given by $P(\phi(\boldsymbol{X}) = 1) = h(1 - G(t))$. Thus, the probability that the system fails until $t$ is written as $P(T \le t) = 1 - h(1 - G(t))$.

The lifetime distribution of the Con/$k$/$n$/F system was discussed by Derman *et al.* (1982), Shantikumar (1985) , Chen and Hwang (1985) and Iyer (1990, 1992). Since the Con/$k$/$n$/F system is a system of $n$ components in sequence, we identify the system with the set $\{1, 2, \dots, n\}$. Then a subsystem corresponds to the subset of $\{1, 2, \dots, n\}$. A subset of $\{1, 2, \dots, n\}$ is called $m$-cutset if and only if the subset contains just $m$ elements and the Con/$k$/$n$/F system fails if the corresponding components of the subset fail. A subsequence of $\{1, 2, \dots, n\}$ of length $m$ is called $m$-cutsequence if it is a permutation of an $m$-cutset. An $m$-cutsequence $\{c_1, c_2, \dots, c_m\}$ is said to be minimal if the subsequence $\{c_1, c_2, \dots, c_{m-1}\}$ is not an $(m-1)$-cutsequence. Under the assumption that the failure time of each component follows an exponential distribution, Bollinger and Salvia (1985) gave the lifetime distribution of the system by means of the number of minimal $m$-cutsequences, $r_{mk}$. Hwang (1986) studied the number $r_{mk}$. Hwang (1991) gave the following explicit expression of $r_{mk}$

$$r_{mk} = (n-m+1)\cdot(m-1)!\sum_{i=0}^{\gamma}(-1)^i\binom{n-m+2}{i}\binom{n-ki}{n-m+1} - m!\sum_{i=0}^{\gamma}(-1)^i\binom{n-m+1}{i}\binom{n-ki}{n-m};$$
$$n \ge m \ge k \ge 1, \qquad \gamma = [n/k].$$

The expression can be interpreted as follows:

Let $N_k(n,m)$ be the number of subsets (of $\{1,2,\dots,n\}$) which contain just $m$ elements and are not $m$-cutset. It is easy to see that $N_k(n,m)$ is equal to the number of ways of putting $m$ identical objects into $(n-m+1)$ different cells when each of $(n-m+1)$ different cells may contain no more than $k-1$ objects. Then, we can see that

$$N_k(n,m) = \sum_{i=0}^{[n/k]}(-1)^i\binom{n-m+1}{i}\binom{n-ki}{n-m},$$

(cf. e.g. Riordan (1958), p. 104). Since $r_{mk}$ can be written as

$$r_{mk} = (n-m+1)((m-1)!)N_k(n,m-1) - m!N_k(n,m),$$

we see that the above expression holds.

Now, we let $\xi_1,\xi_2,\dots,\xi_n$ be the lifetimes of each component of the Con/$k$/$n$/F system. We assume that $\xi_1,\xi_2,\dots,\xi_n$ are i.i.d. random variables with cumulative distribution function (cdf) $G(t)$. Let $\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)}$ be the order statistics of $\xi_1,\xi_2,\dots,\xi_n$ and let $G_{(i)}(t)$ be the cdf of $\xi_{(i)}$. Then, we have

THEOREM 1.1. *Under the above assumptions, the cdf of the lifetime of the* Con/$k$/$n$/F *system is given by*

$$F(t) = \sum_{i=1}^{n}\omega_i G_{(i)}(t),$$

*where*

$$\omega_i = \frac{r_{ik}}{n(n-1)\cdots(n-i+1)},$$

$r_{jk}$ *is the number of minimal $j$-cutsequences given above and*

$$G_{(i)}(t) = \sum_{j=i}^{n}\binom{n}{j}G^j(t)(1-G(t))^{n-j}.$$

PROOF. Let $D_1,D_2,\dots,D_n$ denote the antiranks of $\xi_1,\xi_2,\dots,\xi_n$ defined by $\xi_{D_i} = \xi_{(i)}$, for $i=1,\dots,n$. The system fails when one of components fails. Thus, the lifetime $T$ of the system coincides with $\xi_{(j)}$ for some $j$. When $T = \xi_{(j)}$ holds, we see that the sequence $\{D_1, D_2, \ldots, D_j\}$ is a minimal $j$-cutsequence. Note that the antiranks and the order statistics are independent under the assumption that $\xi_1, \xi_2, \ldots, \xi_n$ are i.i.d. random variables. Then we have

$$\begin{aligned}
P(T \le t) &= \sum_{j=1}^{n} P(T \le t, T = \xi_{(j)}) \\
&= \sum_{j=1}^{n} P(\xi_{(j)} \le t, \{D_1, D_2, \ldots, D_j\} \text{ is a minimal } j\text{-cutsequence}) \\
&= \sum_{j=1}^{n} P(\xi_{(j)} \le t) \cdot P(\{D_1, D_2, \ldots, D_j\} \text{ is a minimal } j\text{-cutsequence}) \\
&= \sum_{j=1}^{n} \frac{r_{jk}}{n(n-1)\cdots(n-j+1)} P(\xi_{(j)} \le t) \\
&= \sum_{j=1}^{n} \omega_j P(\xi_{(j)} \le t).
\end{aligned}$$

For the expression of $G_{(i)}(t)$, see e.g. David (1981). This completes the proof. □

COROLLARY 1.1. *If $G(t)$ has density $g(t)$, the distribution of the lifetime of the* Con/$k$/$n$/F *system has density*

$$f(t) = \sum_{i=1}^{n} \omega_i g_{(i)}(t),$$

*where*

$$g_{(i)}(t) = \frac{n!}{(i-1)!(n-i)!} G^{i-1}(t)(1 - G(t))^{n-i} g(t).$$

*Remark.* The lifetime distribution function $F(t)$ has another simple expression,

$$\begin{aligned}
F(t) = 1 - &\sum_{r=0}^{[n/k]} (-1)^r G^{kr}(t)(1 - G(t))^r \\
&\cdot \left\{ \binom{n - rk}{r} + \binom{n - k(r+1)}{r} (-1) G^k(t) \right\}.
\end{aligned}$$

This expression is a direct consequence of the result of relationship between the lifetime of the system and the binomial distribution of order $k$. For any fixed $t$, let $Y(t)$ be a random variable distributed as the binomial distribution of order $k$, $B_k(n, G(t))$. Then, the relation $F(t) = P(Y(t) \ge 1) = 1 - P(Y(t) = 0)$ holds. Let $Z(t)$ be the number of success runs of length $k$ or more in independent Bernoulli trials of length $n$ with success probability $G(t)$. Then, clearly, $P(Y(t) = 0) =$

$P(Z(t) = 0)$ holds and hence we have $F(t) = 1 - P(Z(t) = 0)$. By setting $t = 0$ in the probability generating function of the distribution of $Z(t)$ given by Hirano and Aki ((1993), formula (5)), the above expression is easily derived.

The expression of $F(t)$ given in Remark is very convenient for computation both algebraically and numerically. The expression of $F(t)$ given in Theorem 1.1 is useful for theoretical purposes as well as for computation. From Theorem 1.1, $F(t)$ is expressed by means of distribution functions of order statistics and hence we can use many results for characteristics of order statistics in order to derive characteristics of the lifetime of the system.

In Section 2, we shall apply our result to estimation problems of Con/$k$/$n$/F systems.

## 2. Parametric estimation of Con/$k$/$n$/F system

### 2.1 *Estimation problem*

Main purpose of this section is to illustrate that parameters of the Con/$k$/$n$/F system, which are supposed to be parameters of distribution of i.i.d. lifetimes of $n$ components of the system, can be estimated based on observations of the lifetime of the system when $n$ and $k$ are fixed adequately.

We shall study the following three cases that lifetimes of $n$ components are i.i.d. with the exponential, the Weibull and the Pareto distributions, respectively. In every case, we estimate the parameters by using the method of moments for adequately selected $k$ and $n$, since estimation by maximum likelihood is too difficult to be practical.

We examine the performance of the moment estimator only in case of exponential distribution by using numerical integration. It seems, however, that moment estimation can be used in the other cases based on our simulation study.

Throughout the section, we assume the following:

Let $T_1, T_2, \ldots, T_m$ be lifetimes of a Con/$k$/$n$/F system. We assume that $T_1, T_2, \ldots, T_m$ are i.i.d. random variables. We also assume that lifetimes of $n$ components in the system are i.i.d. random variables with cdf $G(t)$. Then, for $i = 1, \ldots, m$, $T_i$ is distributed as $F(t)$ given in Theorem 1.1 in Section 1. The statistical problem we treat is to estimate the parameter in $G(t)$ based on independent observations $T_1, T_2, \ldots, T_m$.

### 2.2 *Exponential components*

In this subsection, we assume that the lifetime of every component of the system follows the exponential distribution with parameter $\mu$, i.e. $G(t) = 1 - e^{-\mu t}$. Then, by using Corollary 1.1 of Theorem 1.1, the probability density function (pdf) of $T_1$ can be written as

$$f(t)(= f(t; \mu)) = \sum_{i=1}^{n} \omega_i \frac{n!}{(i-1)!(n-i)!} \{1 - e^{-\mu t}\}^{i-1} \mu e^{-\mu(n-i+1)t}.$$

Figure 1 shows the graphs of density functions of lifetime of the Con/2/5/F system with exponential components ($\mu = 1$, $1/2$ and $1/3$).

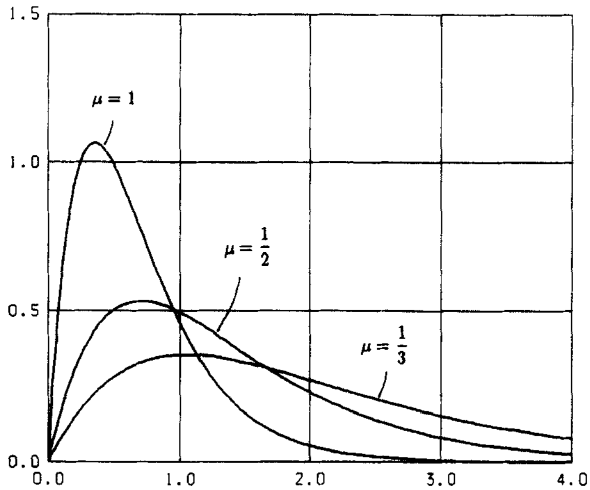


Fig. 1. Density functions of lifetime of Con/2/5/F system with exponential components ($\mu = 1$, $\frac{1}{2}$ and $\frac{1}{3}$).

In order to construct a moment estimator of $\mu$, we derive mean and variance of $T_1$. Let $\xi_1, \dots, \xi_n$ be lifetimes of $n$ components of the system. $\xi_1, \dots, \xi_n$ are assumed to be distributed as i.i.d. exponential distribution with mean $1/\mu$. Let $\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)}$ be the order statistics. Then, the following result on moments of the order statistics is well known (cf. e.g. David (1981));

$$E(\xi_{(r)}) = \frac{1}{\mu} \sum_{i=n-r+1}^{n} \frac{1}{i},$$

and

$$\mathrm{Var}(\xi_{(r)}) = \mathrm{Cov}(\xi_{(r)}, \xi_{(s)}) = \frac{1}{\mu^2} \sum_{i=n-r+1}^{n} \frac{1}{i^2}, \qquad \text{for} \quad r < s.$$

From the result, we see that

$$E(T_1) = \frac{A}{\mu} \qquad \text{and} \qquad \mathrm{Var}(T_1) = \frac{B}{\mu^2},$$

where

$$A = \omega_1 \frac{1}{n} + \omega_2 \left( \frac{1}{n-1} + \frac{1}{n} \right) + \cdots + \omega_n \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right)$$

and

$$B = \omega_1 \left[ \left( \frac{1}{n} \right)^2 + \left( \frac{1}{n} \right)^2 \right] + \omega_2 \left[ \left( \frac{1}{n-1} \right)^2 + \left( \frac{1}{n} \right)^2 + \left( \frac{1}{n-1} + \frac{1}{n} \right)^2 \right]$$
$$+ \cdots + \omega_n \left[ \left( \frac{1}{1} \right)^2 + \cdots + \left( \frac{1}{n} \right)^2 + \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right)^2 \right] - A^2.$$

Now, letting $\bar{T}$ be the sample mean of $T_1, T_2, \ldots, T_m$, i.e., $\bar{T} = \frac{1}{m}\sum_{i=1}^{m} T_i$, we can define a moment estimator $\hat{\mu}_{MM} = \frac{A}{\bar{T}}$.

PROPOSITION 2.1. *$\sqrt{m}(\hat{\mu}_{MM} - \mu)$ converges in distribution to normal distribution with mean 0 and variance $\frac{B}{A^2}\mu^2$ as $m$ tends to infinity.*

PROOF. From the central limit theorem, we have $\sqrt{m}(\bar{T} - f(\mu)) \to N(0, \sigma^2(\mu))$, where $f(\mu) = \frac{A}{\mu}$ and $\sigma^2(\mu) = \mathrm{Var}(T_1) = \frac{B}{\mu^2}$. If $g(x)$ is a real valued function having a nonzero differential at $x = f(\mu)$, $\sqrt{m}(g(\bar{T}) - g(f(\mu)))$ converges in distribution to $N(0, \sigma^2(\mu) \cdot \{g'(f(\mu))\}^2)$ (cf. e.g. Serfling (1980)). By setting $g = f^{-1}$, we complete the proof. □

We tabulate the values of asymptotic variances of the moment estimators of the parameters of the systems for some selected values of $k$ and $n$, where the parameter $\mu$ is fixed to be one. For obtaining asymptotic variance for another value of $\mu$, it suffices to multiply $\mu^2$ to the value in Table 1. This is easily seen from the form of the moment estimator and from the fact that $\mu$ is a scale parameter in $f(t;\mu)$, i.e. we can write

$$f(t;\mu) = \mu h(\mu t),$$

where $h(t) = f(t;1)$. The values of Fisher information at $\mu = 1$ in Table 1 are obtained by means of numerical integration. Since the parameter $\mu$ is a scale parameter of $f(t;\mu)$, the values of Fisher information at $\mu$ can be obtained by multiplying $1/\mu^2$ to the values in Table 1. We can get the values of asymptotic efficiency of the moment estimators by dividing the reciprocal of the Fisher information by the asymptotic variance of the estimator. Thus, the values of asymptotic efficiency do not depend on the value of $\mu$. Asymptotic efficiency of the moment estimator is very close to one.

For reader's convenience, we give several corresponding values of $\omega_i$ in Table 2.

We have studied the distribution of the moment estimator by Monte Carlo experiments when the sample size is small. We exhibit in Table 3 the estimated mean and variance of the moment estimator when the sample size is 10, 20 or 30. In each cell, the uppermost figure is the estimated mean ($E.M.$), the middle figure is the estimated variance ($E.V.$), and the lowermost figure is the normalized estimated variance by the sample size ($N.E.V. \equiv m \times E.V.$) of the moment estimator based on 1000000 Monte Carlo samples.

### 2.3 *Weibull components*

In this subsection, the lifetime distribution of each component of the Con/$k$/$n$/F system is assumed to be the Weibull distribution with parameters $\alpha$ and $c$, $W(\alpha, c)$, where the cdf and pdf of $W(\alpha, c)$ are given respectively as

$$G(t) = 1 - \exp\left\{-\left(\frac{t}{\alpha}\right)^c\right\}$$

and

$$g(t) = \frac{c}{\alpha} \cdot \left(\frac{t}{\alpha}\right)^{c-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^c\right\}.$$

Table 1. Asymptotic property of moment estimator of $\mu$.

| System | Asymptotic Variance ($\mu = 1$) | Fisher Information ($\mu = 1$) | Asymptotic Efficiency |
|---|---|---|---|
| Con/2/4/F | 13/25 | 1.924683 | 0.999165 |
| Con/2/5/F | 11/21 | 1.921233 | 0.993680 |
| Con/3/5/F | 31/64 | 2.206517 | 0.935645 |
| Con/2/10/F | 116784/267289 | 2.292570 | 0.998332 |
| Con/3/10/F | 6401/20480 | 3.203936 | 0.998616 |
| Con/4/10/F | 3641/13467 | 3.739892 | 0.988988 |
| Con/5/10/F | 61/261 | 4.308889 | 0.992991 |

Table 2. Values of weights $\omega_i$.

| | $k=2$ | $k=3$ | $k=4$ | $k=5$ |
|---|---|---|---|---|
| $n=4$ | $\omega_1=0$ | $\omega_1=0$ | $\omega_1=0$ | |
| | $\omega_2=1/2$ | $\omega_2=0$ | $\omega_2=0$ | |
| | $\omega_3=1/2$ | $\omega_3=1/2$ | $\omega_3=0$ | |
| | $\omega_4=0$ | $\omega_4=1/2$ | $\omega_4=1$ | |
| $n=5$ | $\omega_1=0$ | $\omega_1=0$ | $\omega_1=0$ | $\omega_1=0$ |
| | $\omega_2=4/10$ | $\omega_2=0$ | $\omega_2=0$ | $\omega_2=0$ |
| | $\omega_3=5/10$ | $\omega_3=3/10$ | $\omega_3=0$ | $\omega_3=0$ |
| | $\omega_4=1/10$ | $\omega_4=5/10$ | $\omega_4=2/5$ | $\omega_4=0$ |
| | $\omega_5=0$ | $\omega_5=2/10$ | $\omega_5=3/5$ | $\omega_5=1$ |
| $n=10$ | $\omega_1=0$ | $\omega_1=0$ | $\omega_1=0$ | $\omega_1=0$ |
| | $\omega_2=42/210$ | $\omega_2=0$ | $\omega_2=0$ | $\omega_2=0$ |
| | $\omega_3=70/210$ | $\omega_3=14/210$ | $\omega_3=0$ | $\omega_3=0$ |
| | $\omega_4=63/210$ | $\omega_4=35/210$ | $\omega_4=7/210$ | $\omega_4=0$ |
| | $\omega_5=30/210$ | $\omega_5=56/210$ | $\omega_5=23/210$ | $\omega_5=1/42$ |
| | $\omega_6=5/210$ | $\omega_6=60/210$ | $\omega_6=45/210$ | $\omega_6=4/42$ |
| | $\omega_7=0$ | $\omega_7=38/210$ | $\omega_7=65/210$ | $\omega_7=9/42$ |
| | $\omega_8=0$ | $\omega_8=7/210$ | $\omega_8=56/210$ | $\omega_8=14/42$ |
| | $\omega_9=0$ | $\omega_9=0$ | $\omega_9=14/210$ | $\omega_9=14/42$ |
| | $\omega_{10}=0$ | $\omega_{10}=0$ | $\omega_{10}=0$ | $\omega_{10}=0$ |

Let $\xi_1, \dots, \xi_n$ be lifetimes of $n$ components of the system. $\xi_1, \dots, \xi_n$ are assumed to be distributed as i.i.d. Weibull distribution with parameters $\alpha$ and $c$. Let $\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)}$ be the order statistics. Then the pdf of $\xi_{(i)}$ can be written as

$$g_{(i)}(t) = \frac{n! c t^{c-1}}{(i-1)!(n-i)!\alpha^c} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \exp\left[-(n-i+1+j)\left(\frac{t}{\alpha}\right)^c\right].$$

Table 3. Finite sample property of moment estimator of $\mu$.

| System | | $m = 10$ | $m = 20$ | $m = 30$ |
|---|---|---|---|---|
| Con/2/4/F | $E.M.$ | 1.054876 | 1.026671 | 1.017747 |
| | $E.V.$ | 0.064066 | 0.028729 | 0.018592 |
| | $N.E.V.$ | 0.640658 | 0.574578 | 0.557748 |
| Con/2/5/F | $E.M.$ | 1.055115 | 1.026638 | 1.017793 |
| | $E.V.$ | 0.063852 | 0.028865 | 0.018623 |
| | $N.E.V.$ | 0.638523 | 0.577293 | 0.558693 |
| Con/3/5/F | $E.M.$ | 1.048967 | 1.024190 | 1.016042 |
| | $E.V.$ | 0.054820 | 0.025702 | 0.016801 |
| | $N.E.V.$ | 0.548201 | 0.514044 | 0.504036 |
| Con/2/10/F | $E.M.$ | 1.045644 | 1.022090 | 1.014673 |
| | $E.V.$ | 0.052803 | 0.023969 | 0.015453 |
| | $N.E.V.$ | 0.528032 | 0.479379 | 0.463593 |
| Con/3/10/F | $E.M.$ | 1.032516 | 1.015980 | 1.010493 |
| | $E.V.$ | 0.035400 | 0.016641 | 0.010869 |
| | $N.E.V.$ | 0.354003 | 0.332820 | 0.326055 |
| Con/5/10/F | $E.M.$ | 1.023943 | 1.011847 | 1.007628 |
| | $E.V.$ | 0.025398 | 0.012180 | 0.008001 |
| | $N.E.V.$ | 0.253984 | 0.243607 | 0.240037 |

The $l$-th moment of the $i$-th order statistic is expressed as

$$E[\xi_{(i)}^l] = \frac{n!\alpha^l}{(i-1)!(n-i)!}\Gamma\left(1+\frac{l}{c}\right)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j(n-i+1+j)^{-(1+l/c)}.$$

Let $T_1, T_2, \ldots, T_m$ be i.i.d. observations of the lifetime of a Con/$k$/$n$/F system By using Theorem 1.1 of the previous section, we have

$$E[T_1] = \sum_{i=1}^{n}\omega_i\int_0^\infty t g_{(i)}(t)dt = \alpha\cdot A(c)$$

where

$$A(c) = \sum_{i=1}^{n}\frac{\omega_i n!}{(i-1)!(n-i)!}\Gamma\left(1+\frac{1}{c}\right)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j(n-i+1+j)^{-(1+1/c)},$$

and

$$E[T_1^2] = \sum_{i=1}^{n}\omega_i\int_0^\infty t^2 g_{(i)}(t)dt = \alpha^2\cdot B(c)$$

where

$$B(c) = \sum_{i=1}^{n} \frac{\omega_i n!}{(i-1)!(n-i)!} \Gamma\left(1+\frac{2}{c}\right) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j (n-i+1+j)^{-(1+2/c)}.$$

We set

$$M_1 = \frac{1}{m}\sum_{i=1}^{m} T_i \quad \text{and} \quad M_2 = \frac{1}{m}\sum_{i=1}^{m} T_i^2.$$

Then, the solutions $\hat{\alpha}$ and $\hat{c}$ of the following equations are moment estimators of $\alpha$ and $c$, respectively,

$$\begin{cases} \alpha A(c) = M_1 \\ \alpha^2 B(c) = M_2. \end{cases}$$

By dividing $M_2$ by $M_1^2$, we can eliminate $\alpha$ from the above equations. Define

$$\phi(c) = \frac{B(c)}{\{A(c)\}^2}.$$

If the equation $\phi(c) = M_2/M_1^2$ can be solved, and if the solution is unique, then the above equations can be solved by substituting this solution.

Fortunately, the function $\phi(c)$ is smooth and tractable for some selected values of $k$ and $n$. The equation $\phi(c) = M_2/M_1^2$ can be solved by using standard techniques of numerical analysis, such as the Newton-Raphson method or the bisection method. To illustrate this fact we give an example.

*Example* 1. We consider a Con/2/5/F system with the Weibull components. Figure 2 shows the graphs of density functions of lifetime of Con/2/5/F system with Weibull components ($W(1,c)$, $c = 1,2,3$). The function $\phi(c)$ is written as

$$\phi(c) = \frac{\Gamma\left(1+\frac{2}{c}\right)}{\left(\Gamma\left(1+\frac{1}{c}\right)\right)^2} \cdot \frac{f_2(c)}{(f_1(c))^2},$$

where

$$f_1(c) = 2^{-1/c} + 3 \cdot 3^{-1/c} - 4 \cdot 4^{-1/c} + 5^{-1/c}$$

and

$$f_2(c) = 2^{-2/c} + 3 \cdot 3^{-2/c} - 4 \cdot 4^{-2/c} + 5^{-2/c}.$$

This function $\phi(c)$ is a very smooth monotonically decreasing function (see Fig. 3). Further, derivative of this function is also good in the sense that it is easily evaluated numerically. It is easy to see that

$$f_1'(c) = -\frac{1}{c^2}(2^{-1/c} \cdot \log 2 + 3 \cdot 3^{-1/c} \cdot \log 3 - 4 \cdot 4^{-1/c} \cdot \log 4 + 5^{-1/c} \cdot \log 5)$$

and

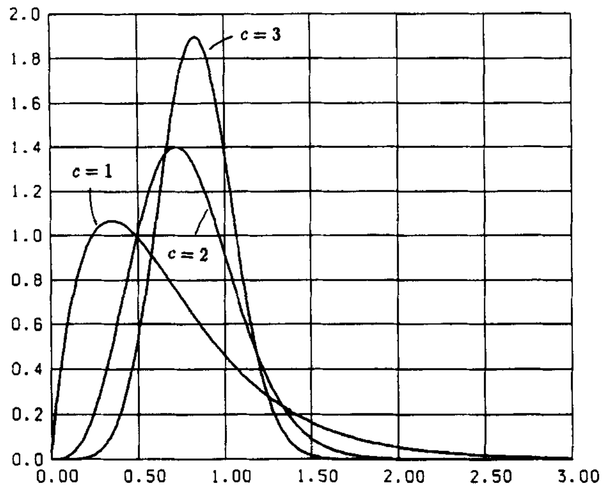


Fig. 2. Density functions of lifetime of Con/2/5/F system with Weibull components ($W(1,c)$, $c=1,2,3$).

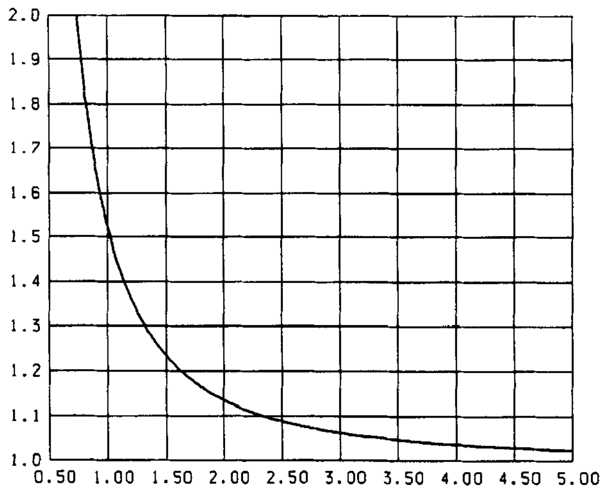


Fig. 3. Graph of $\phi(c)$ of Con/2/5/F system with Weibull components ($W(\alpha,c)$).

$$f_2'(c) = -\frac{2}{c^2}(2^{-2/c}\cdot\log 2 + 3\cdot 3^{-2/c}\cdot\log 3 - 4\cdot 4^{-2/c}\cdot\log 4 + 5^{-2/c}\cdot\log 5).$$

Noting that

$$\left(\frac{\Gamma\left(1+\frac{2}{c}\right)}{\left(\Gamma\left(1+\frac{1}{c}\right)\right)^2}\right)' = \frac{2\Gamma\left(1+\frac{2}{c}\right)}{c^2\Gamma\left(1+\frac{1}{c}\right)^2}\left[\psi\left(1+\frac{1}{c}\right)-\psi\left(1+\frac{2}{c}\right)\right],$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$, we can write

$$\phi'(c) = \frac{\Gamma\left(1+\frac{2}{c}\right)}{\Gamma\left(1+\frac{1}{c}\right)^2}\left[\frac{2f_2(c)}{c^2 f_2(c)^2}\left\{\psi\left(1+\frac{1}{c}\right)-\psi\left(1+\frac{2}{c}\right)\right\} + \frac{f_2'(c)}{f_1(c)^2} - \frac{2f_1'(c)f_2(c)}{f_1(c)^3}\right].$$

## 2.4 *Pareto components*

We consider in this subsection the case that lifetime of each component of a Con/$k$/$n$/F system follows the Pareto distribution with parameters $\alpha$ and $\beta$, *Pareto*$(\alpha,\beta)$. The cdf and pdf of *Pareto*$(\alpha,\beta)$ are given as

$$G(t) = 1 - \left(\frac{\alpha}{t}\right)^{\beta-1}$$

and

$$g(t) = \frac{1}{\alpha}\cdot\frac{\beta-1}{\left(\frac{t}{\alpha}\right)^{\beta}},$$

where $t > \alpha$, $\alpha > 0$ and $\beta > 1$. For $\beta > 2$, the distribution has mean $\frac{\alpha(\beta-1)}{\beta-2}$ and for $\beta > 3$, the distribution has variance $\frac{\alpha^2(\beta-1)}{(\beta-3)(\beta-2)^2}$.

Let $\xi_1,\dots,\xi_n$ be lifetimes of $n$ components of the system. $\xi_1,\dots,\xi_n$ are assumed to be distributed as i.i.d. Pareto distribution with parameters $\alpha$ and $\beta$. Let $\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)}$ be the order statistics. Then the pdf of $\xi_{(i)}$ can be written as

$$g_i(t) = \frac{n!}{(i-1)!(n-i)!}\left\{1-\left(\frac{\alpha}{t}\right)^{\beta-1}\right\}^{i-1}\cdot\frac{\beta-1}{\alpha}\cdot\left(\frac{\alpha}{t}\right)^{(\beta-1)(n-i)+\beta},$$

where $t > \alpha$. The $l$-th moment of the $i$-th order statistic is expressed as

$$E[\xi_{(i)}^l] = \frac{n!\alpha^l}{(n-i)!}\cdot\frac{\Gamma\left(n-i-\frac{l}{\beta-1}+1\right)}{\Gamma\left(n-\frac{l}{\beta-1}+1\right)}.$$

Let $T_1, T_2, \dots, T_m$ be i.i.d. observations of the lifetime of a Con/$k$/$n$/F system. Then, from Theorem 1.1 in Section 1, we can write the pdf and the $l$-th moment of $T_1$ respectively as

$$\sum_{i=1}^{n}\frac{n!\omega_i}{(i-1)!(n-i)!}\left\{1-\left(\frac{\alpha}{t}\right)^{\beta-1}\right\}^{i-1}\cdot\frac{\beta-1}{\alpha}\cdot\left(\frac{\alpha}{t}\right)^{(\beta-1)(n-i)+\beta},$$

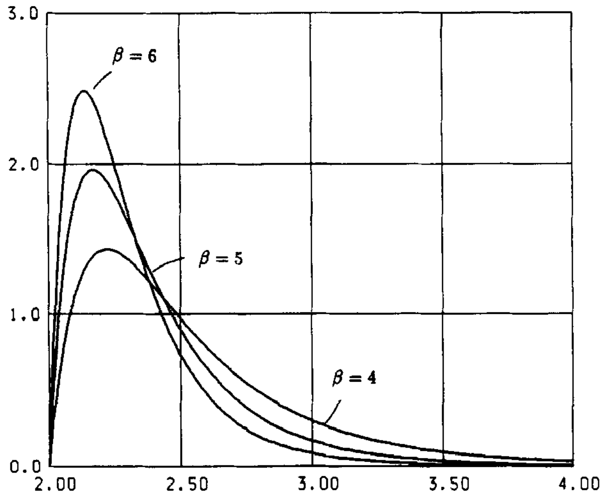


Fig. 4. Density functions of lifetime of Con/2/5/F system with Pareto components ($Pareto(2, \beta)$, $\beta = 4, 5, 6$).

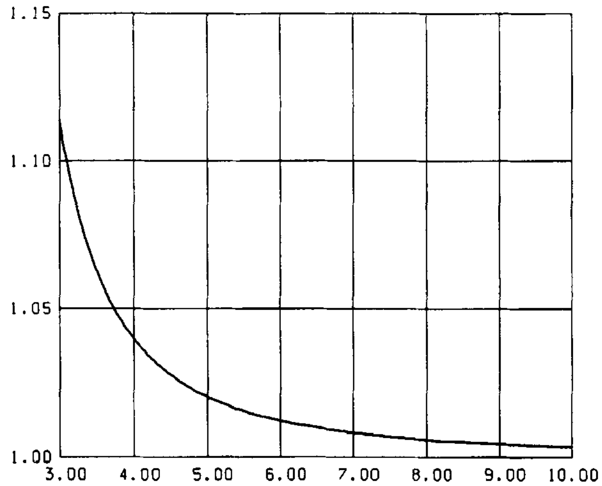


Fig. 5. Graph of $\phi_1(\beta)$ of Con/2/5/F system with Pareto components ($Pareto(\alpha, \beta)$).

and

$$E[T_1^l] = (\beta - 1)\alpha^l \sum_{j=1}^{n} \sum_{i=0}^{j-1} \frac{r_{jk}}{(j-i-1)!i!}(-1)^{j-i-1} \cdot \frac{1}{(\beta-1)(n-i)-l}.$$

*Example* 2. We consider a Con/2/5/F system with Pareto components. Figure 4 shows graphs of density functions of lifetime of Con/2/5/F system with Pareto components ($Pareto(2, \beta)$, $\beta = 4, 5, 6$). Since $n = 5$ and $k = 2$, we have, that $r_{12} = 0$, $r_{22} = 8$, $r_{32} = 30$, $r_{42} = 12$ and $r_{52} = 0$. Then we can write

$$m_1 \equiv E[T_1] = (\beta - 1)\alpha \cdot a(\beta)$$

and

$$m_2 \equiv E[{T_1}^2] = (\beta - 1)\alpha^2 \cdot b(\beta)$$

where

$$a(\beta) = \left\{ \frac{5}{5(\beta-1)-1} - \frac{16}{4(\beta-1)-1} + \frac{9}{3(\beta-1)-1} + \frac{2}{2(\beta-1)-1} \right\}$$

and

$$b(\beta) = \left\{ \frac{5}{5(\beta-1)-2} - \frac{16}{4(\beta-1)-2} + \frac{9}{3(\beta-1)-2} + \frac{2}{2(\beta-1)-2} \right\}.$$

Similarly, as in Example 1, we can eliminate $\alpha$ by dividing $m_2$ by $m_1^2$. By letting

$$\phi_1(\beta) = \frac{b(\beta)}{(\beta-1)\{a(\beta)\}^2},$$

we can estimate $\beta$ by solving $\phi_1(\beta) = M_2/M_1^2$, where $M_1 = \frac{1}{m}\sum_{i=1}^m T_i$ and $M_2 = \frac{1}{m}\sum_{i=1}^m T_i^2$. Figure 5 exhibits that the graph of $\phi_1(\beta)$ is a monotonically decreasing smooth function.

## References


Aki, S. (1985). Discrete distributions of order $k$ on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205–224.

Aki, S. and Hirano, K. (1988). Some characteristics of the binomial distribution of order $k$ and related distributions, *Statistical Theory and Data Analysis II*; *Proceedings of the Second Pacific Area Statistical Conference* (ed. K. Matusita), 211–222, North-Holland, Amsterdam.

Aki, S. and Hirano, K. (1989). Estimation of parameters in the discrete distributions of order $k$, *Ann. Inst. Statist. Math.*, **41**, 47–61.

Aki, S. and Hirano, K. (1993). Discrete distributions related to succession events in a two-state Markov chain, *Statistical Sciences and Data Analysis*; *Proceedings of the Third Pacific Area Statistical Conference* (eds. K. Matusita, M. L. Puri and T. Hayakawa), 467–474, VSP, Utrecht.

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing*, To Begin With, Silver Spring, MD.

Bollinger, R. C. and Salvia, A. A. (1985). Consecutive-$k$-out-of-$n$:F system with sequential failures, *IEEE Transactions on Reliability*, **R-34**, 43–45.

Chao, M. T., Fu, J. C. and Koutras, M. V. (1995). Survey of reliability studies of consecutive-$k$-out-of-$n$:F & related systems, *IEEE Transactions on Reliability*, **40**, 120–127.

Chen, R. W. and Hwang, F. K. (1985). Failure distributions of consecutive-$k$-out-of-$n$:F systems, *IEEE Transactions on Reliability*, **R-34**, 338–341.

David, H. A. (1981). *Order Statistics*, 2nd ed., Wiley, New York.

Derman, C., Lieberman, G. J. and Ross, S. M. (1982). On the consecutive-$k$-of-$n$:F system, *IEEE Transactions on Reliability*, **R-31**, 57–63.

Hirano, K. (1986). Some properties of the distributions of order $k$, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 43–53, Reidel, Dordrecht.

Hirano, K. (1994). Consecutive-$k$-out-of-$n$:F Systems, *Proc. Inst. Statist. Math.*, **42**, 45–61 (in Japanese).

Hirano, K. and Aki, S. (1993). On number of occurrences of success runs of specified length in a two-state Markov chain, *Statistica Sinica*, **3**, 313–320.

Hwang, F. K. (1986). Simplified reliabilities for consecutive-$k$-out-of-$n$:F systems, *SIAM J. Algebraic Discrete Methods*, **7**, 258–264.

Hwang, F. K. (1991). An explicit solution for the number of minimal $p$-cutsequences in a consecutive-$k$-out-of-$n$:F system, *IEEE Transactions on Reliability*, **R-40**, 553–554.

Iyer, S. (1990). Distribution of time to failure of consecutive $k$-out-of-$n$:F systems, *IEEE Transactions on Reliability*, **R-39**, 97–101.

Iyer, S. (1992). Distribution of the lifetime of consecutive $k$-within-$m$-out-of-$n$:F systems, *IEEE Transactions on Reliability*, **R-41**, 448–450.

Johnson, N. L., Kotz, S. and Kemp, A. W. (1992). *Univariate Discrete Distributions*, 2nd ed., Wiley, New York.

Lambiris, M. and Papastavridis, S. (1985). Exact reliability formulas for linear and circular consecutive-$k$-out-of-$n$:F systems, *IEEE Transactions on Reliability*, **R-34**, 124–126.

Philippou, A. N. (1986). Distributions and Fibonacci polynomials of order $k$, longest runs and reliability of consecutive-$k$-out-of-$n$:F system, *Fibonacci Numbers and Their Applications* (eds. A. N. Philippou, G. E. Bergum and A. F. Horadam), 203–227, Reidel, Dordrecht.

Riordan, J. (1958). *An Introduction to Combinatorial Analysis*, Wiley, New York.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, Wiley, New York.

Shantikumar, J. G. (1985). Lifetime distribution of consecutive-$k$-out-of-$n$:F systems with exchangeable life times, *IEEE Transactions on Reliability*, **R-34**, 480–483.