Ann. Inst. Statist. Math.
Vol. 48, No. 1, 169–184 (1996)

MOMENT SOLUTION TO AN URN MODEL*

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(Received August 1, 1994; revised February 17, 1995)

Abstract. A subset of Bernard's RD-model (replenishment-depletion) is considered from the viewpoint of the calculus of finite differences. The most general case is considered and includes an urn with balls of many colors, each color being replenished either deterministically or stochastically. Factorial moment generating functions (fmgfs) are employed to define probability generating functions. A new result is given for the two color case defining the fmgf and probability generating function (with probabilities) when the replenishments are positive valued random variables with given factorial moments. This result involves beta integral transforms defining a manifold of discrete distributions. Particular cases relate to hypergeometric discrete distributions.

Key words and phrases: Bernard's urn, beta integral transforms, finite difference calculus, generating function, hypergeometric distributions, hypergeometric functions, moments, replenishment-depletion urn.

1. Introduction

An urn contains c_j balls of color C_j, $j = 1, 2, \ldots, s$. There are m cycles of replenishment (R) and depletion (D). Thus at the i-th cycle, the j-th color ($j = 1, 2, \ldots, s$) is replenished by r_{ji} balls ($\sum_{j=1}^{s} r_{ji} = R_i$); the cycle is completed by randomly removing D_i balls, which may be of different colors; the depletion at any cycle can not in general exceed the urn content at that cycle. This urn model was basically introduced by Bernard (1977). The scheme is displayed in Table 1.

The basic notion of relating a problem in health physics to classical probability urn models is due to Bernard (1977). Bernard's urn was assumed to contain n red balls (representing dangerous material) of radioactive atoms and b white balls of stable structure. This state of the urn is modified by adding r white balls and then randomly removing r balls. The routine of replenishment-depletion is called a cycle and is repeated several times. The obvious question is the decay

* This research was partly supported by Martin Marietta Energy Systems, Inc., under contract DE-AC05-84OR21400 with the U.S. Department of Energy.

Table 1. Urn Schemata.

Color	C_1	C_2	$\cdots$	C_s	R	D	R^*	Accumulation
cycle 0	c_1	c_2	$\cdots$	c_s				
cycle 1	r_{11}	r_{21}	$\cdots$	r_{s1}	R_1	D_1	$R_1 - r_{s1} = R_1^*$	$d_1 = 0$
cycle 2	r_{12}	r_{22}	$\cdots$	r_{s2}	R_2	D_2	$R_2 - r_{s2} = R_2^*$	$d_2 = d_1 + R_1 - D_1$
cycle 3	r_{13}	r_{23}	$\cdots$	r_{s3}	R_3	D_3	$R_3 - r_{s3} = R_3^*$	$d_3 = d_2 + R_2 - D_2$
cycle $\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
cycle i	r_{1i}	r_{2i}	$\cdots$	r_{si}	R_i	D_i	$R_i - r_{si} = R_i^*$	$d_i = \sum_{\lambda=1}^{i-1}(R_\lambda - D_\lambda)$
cycle $\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
cycle m	r_{1m}	r_{2m}	$\cdots$	r_{sm}	R_m	D_m	$R_m - r_{sm} = R_m^*$	d_m
	α_1	α_2	$\cdots$					

(i) R_λ^* refers to the total replenishment at the λ-th cycle excepting the replenishment $(r_{s\lambda})$ to the s-th color.

(ii) The accumulation symbol d_λ at the λ-th cycle represents the excess of input over depletion. Traditionally it is taken to be zero.

(iii) The colors may be arranged in any order in the schemata, but the last one (s) will not be associated with a corresponding parameter in the factorial moment generating function (fmgf). Moreover, in general, the total number of balls in the urn is a constant and this constraint may be used as a check on low-order moments, in particular means.

(iv) The replacement of a particular color at a cycle may be an integer valued random variable relating to event occurrences of $0, 1, 2, \ldots$ events.

(v) In the sequel fmgf is used and parameters involved are taken to be $\alpha_1, \alpha_2, \ldots, \alpha_{s-1}$.

of radioactivity (red balls). In this situation, since atoms are being considered, very large numbers of balls are involved, and cycles when referred to the human condition of one or three a day may be as large 20×365 (30 years) or 90×365. Reference may be made to Shenton (1983).

The first advance was to relate the probability of the state of the system to factorial moment generating functions coupled with finite difference calculus. The use of fmgfs has proved its power historically for many discrete distributions (Poisson, Binomial, hypergeometric for example). First and second factorial moment thus became available using computer facilities. However the evaluation of probabilities, seemed out of reach in general. We now develop a recursive scheme for the fmgf at the m-th cycle in terms of that for the $(m-1)$-th cycle, for the case of an urn containing balls of s colors for which depletion may not necessarily equal replenishment.

The question of recursions for moments is resolved by the relation implied in the recursive scheme for fmgfs.

Generalized urn models may have applications in branches of health physics (including the pathology of dietetics); they may also be of interest as games with urns involving random replenishment or deterministic replenishment.

2. The factorial moment generating function

This relates to $(s-1)$ colors and uses multivariate finite difference calculus. In fact at the m-th cycle

$$(2.1)\qquad f_m(\alpha_1,\alpha_2,\ldots,\alpha_{s-1}) = K_m^{-1} g_m(\underline{E}) A_{s,m}(\underline{\alpha}.\underline{E}) \prod_{i=1}^{m} x_i^{(D_i)} \Bigg|_{\substack{x_i=c_s+r_{si}+d_i \\ i=1 \text{ to } m}}$$

where

$$\text{(a)}\qquad K_m = \prod_{i=1}^{m}(c+R_i+d_i)^{(D_i)},$$

$$c = c_1+c_2+\cdots+c_s, \quad x^{(k)} = x(x-1)\cdots(x-k+1),$$

$$\text{(b)}\qquad g_m(\underline{E}) = \begin{cases} \prod_{i=2}^{m} E_i^{-(R_1^*+R_2^*+\cdots+R_{i-1}^*)}, & m \geq 2, \\ 1, & \text{otherwise.} \end{cases}$$

Here E_{x_i} is the incremental operator: $E_u f(u,v,w,\ldots) = f(1+u,v,w,\ldots)$. ($\mathcal{E}$ to denote mathematical expectation.)

$$\text{(c)}\qquad A_{s,m}(\underline{\alpha},\underline{E}) = \prod_{j=1}^{s-1}\prod_{i=0}^{m-1}\left(\alpha_j + \prod_{\lambda=m-i}^{m} E_{x_\lambda}\right)^{r^*_{j,m-i}}$$

where

$$r^*_{ji} = \begin{cases} r_{ji}, & i \neq 1 \\ r_{ji}+c_j, & i=1. \end{cases}$$

In previous studies of Shenton (1981, 1983), formula (2.1) is relegated to fine print and its symbolic form obscures its mathematical base in the finite-difference calculus; there is also an error in one of the symbols (Shenton (1983), p. 8, R_j). The new version provides the multivariate probability function at any cycle, along with moments. However, it is difficult to handle in some limiting asymptotic cases. Other references are Berg (1974, 1977, 1985), Bowman and Shenton (1985), Johnson and Kotz (1969, 1970, 1977).

We note that (2.1) has an alternative form, derived by extracting all operator terms $E_m, E_{m-1},\ldots,E_1$. Thus there is the user friendly formula

$$(2.2)\qquad f_m(\alpha_1,\alpha_2,\ldots,\alpha_{s-1}) = \mathcal{H}_{s,m}(\underline{\alpha},\underline{E}) \prod_{i=1}^{m}\left(\frac{x_i^{(D_i)}}{S_i^{(D_i)}}\right)\Bigg|_{x_i=S_i}$$

where

(i) $\mathcal{H}_{s,m}(\underline{\alpha},\underline{E}) = \prod_{j=1}^{s-1}\prod_{i=0}^{m-1}\left(1+\frac{\alpha_j}{\prod_{\lambda=m-i}^{m} E_{x_\lambda}}\right)^{r^*_{j,m-i}}$

(ii) $r^*_{j,i} = r_{j,i}$, $i \neq 1$; $r^*_{j,i} = r_{j,i}+c_j.i = 1$;

(iii) D_i is the depletion at the i-th stage; $D_i \leq \sum_{j=1}^{s} r_{j,i}$

(iv) $S_i = c+R_i+d_i$.

We now consider low-order moments.

3. Low order moments

Factorial moments are set up by partial differentiation with respect to the appropriate set of α's. To avoid further complicating notation we consider moments for the colors C_1 and C_2 (see (iii) of Section 1).

3.1 *Mean number of balls of color C_1 at the m-th cycle*

For C_1, the first factorial moment at the m-th cycle is, using N_1 to denote the number of balls of color C_1 at the end of the m-th cycle,

$$\mathcal{E}_m(N_1) = \sum_{\lambda=0}^{m-1} r^*_{1,m-\lambda} \prod_{\mu=1}^{\lambda+1} \frac{(c+R_\mu+d_\mu-D_\mu)}{(c+R_\mu+d_\mu)},$$

with similar expressions for the remaining $s-1$ colors. In this connection note that the number of balls in the urn at the m-th cycle is

$$c+\sum_{\lambda=1}^{m}(R_\lambda - D_\lambda).$$

3.2 *Second order factorial moments*

For the second factorial moment, we have for C_1

$$\begin{aligned}\mathcal{E}_m(N_1)(N_1-1) &= \partial^2 f_m(\underline{\alpha})/\partial\alpha^2 \,|_{\alpha=0} \\ &= \sum_{\lambda=1}^{m} r^*_{1\lambda}(r^*_{1\lambda}-1) \prod_{\mu=\lambda}^{m} \frac{(c+R_\mu+d_\mu-D_\mu)^{(2)}}{(c+R_\mu+d_\mu)^{(2)}} \\ &\quad + \sum_{\substack{\lambda=1 \\ (\lambda\neq\mu)}}^{m} \sum_{\mu=1}^{m} r^*_{1\lambda} r^*_{1\mu} \\ &\quad \times \prod_{\theta=\lambda}^{m} \prod_{\phi=\mu}^{m} \frac{(c+R_\theta+d_\theta-D_\theta)}{(c+R_\theta+d_\theta)} \frac{(c+R_\phi+d_\phi-D_\phi)}{(c+R_\phi+d_\phi)}.\end{aligned}$$

For a cross-product moment, for C_1 and C_2

$$\mathcal{E}_m(N_1N_2) = \sum_{\lambda=1}^{m} \prod_{\mu=1}^{m} r^*_{1\lambda} r^*_{2\mu} \prod_{\theta=\lambda}^{m} \prod_{\phi=\mu}^{m} \frac{(c+R_\theta+d_\theta-D_\theta)}{(c+R_\theta+d_\theta)} \frac{(c+R_\phi+d_\phi-D_\phi)}{(c+R_\phi+d_\phi)}.$$

4. Numerical examples

4.1 *Example* 1: *three colors* (*Table* 2)
First cycle fmgf,

$$f_1(\alpha_1,\alpha_2) = K_1^{-1}(E_{x_1}+\alpha_1)^4(E_{x_1}+\alpha_2)^4 x_1^{(5)} \mid_{x_1=4}$$
$$K_1 = E_{x_1}^8 x_1^{(5)} \mid_{x_1=4} = 12^{(5)}.$$

Third cycle fmgf,

$$\begin{aligned} f_3(\alpha_1,\alpha_2) = K_3^{-1}E_{x_2}^{-5}E_{x_3}^{-14}&(\alpha_1+E_{x_3})^7(\alpha_1+E_{x_3}E_{x_2})^4(\alpha_1+E_{x_3}E_{x_2}E_{x_1})^4 \\ &\times(\alpha_2+E_{x_3})^8(\alpha_2+E_{x_3}E_{x_2})^5 \\ &\times(\alpha_2+E_{x_3}E_{x_2}E_{x_1})^4 x_1^{(5)}x_2^{(13)}x_3^{(23)} \mid_{\substack{x_1=4\\x_2=10\\x_3=15}} \end{aligned}$$

$$K_3 = 12^{(15)}\cdot 22^{(13)}\cdot 30^{(23)}.$$

Table 2. Three colors example.

Cycles	C_1	C_2	C_3	R	D	R^*	
cycle 0	$c_1=1$	$c_2=2$	$c_3=3$				
cycle 1	3	2	1	6	5	5	$d_1=0$
cycle 2	4	5	6	15	13	9	$d_2=1$
cycle 3	7	8	9	24	23	15	$d_3=3$
	α_1	α_2	·				

4.2 *Example* 2
4.2.1 *Two colors-random replenishment* (*Table* 3)
The factorial moment generating function is

$$f_m(\alpha) = K_m^{-1}(\alpha+E_{x_m}E_{x_{m-1}}\cdots E_{x_1})^{c_1}x_1^{(r_1)}x_2^{(r_2)}\cdots x_m^{(r_m)} \mid_{\substack{x_i=c_2+r_i,\\ i=1,2,\ldots,m}} \tag{4.1}$$
$$K_m = (c+r_1)^{(r_1)}(c+r_2)^{(r_2)}\cdots(c+r_m)^{(r_m)} \qquad (c=c_1+c_2).$$

Mean number of C_1 balls at the m-th cycle

$$\mathcal{E}_m(N_1) = c_1\prod_{i=1}^m \frac{c}{c+r_i}.$$

General factorial moment is

$$\mathcal{E}_m(N_1^{(l)}) = c_1^{(l)}\prod_{i=1}^m \frac{c^{(l)}}{(c+r_i)^{(l)}} \qquad (l=1,2,\ldots,c_1).$$

Table 3. Two colors example.

Cycles	C_1	C_2	R	D	R^*	Accumulation
cycle 0	c_1	c_2				
cycle 1	0	r_1	r_1	r_1	0	0
cycle 2	0	r_2	r_2	r_2	0	0
cycle ·	·	·	·	·	·	·
cycle ·	·	·	·	·	·	·
cycle ·	·	·	·	·	·	·
cycle m	0	r_m	r_m	r_m	0	0

(Accumulation $d_i = d_{i-1} + (R_{i-1} - D_{i-1})$, $i = 1, 2, \ldots, m$, $d_1 = 0$.)

For independent random replenishments at the i-th cycle, let r_i have fmgf $G_i(\alpha)$, so that (with k-th factorial moment ν_{ik})

$$G_i(\alpha) = \sum_{k=0}^{\infty} \nu_{ik}\alpha^k/k!,$$

with probability generating function (pgf), $P_i(t) = \sum_{k=0}^{\infty} p_{ik}t^k$, and r_i takes on non-negative integer values. Then

$$\mathcal{E}_m(N_1^{(l)}) = c_1^{(l)} \prod_{i=1}^{m} \left(p_{i0} + \frac{c^{(l)}}{(c+1)^{(l)}} p_{i1} + \frac{c^{(l)}}{(c+2)^{(l)}} p_{i2} + \cdots \right). \tag{4.2}$$

But the component in the product is

$$\begin{aligned} c^{(l)} \sum_{k=0}^{\infty} p_{ik}(1+\Delta_c)^k \frac{1}{c^{(l)}} &= \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \binom{\lambda+l-1}{l-1} \sum_{k=0}^{\infty} \frac{k^{(\lambda)} p_{ik}}{(c+\lambda)^{(\lambda)}} \\ &= \frac{\Gamma(c+1)}{\Gamma(l)\Gamma(c-l+1)} \int_0^1 G_i(-\alpha)\alpha^{l-1}(1-\alpha)^{c-l}d\alpha \\ & \qquad (l = 1, 2, \ldots, c_1). \end{aligned}$$

(Note that $\Delta_x^r(1/x^{(s)}) = \frac{(-1)^r(s+r-1)^{(r)}}{(x+r)^{(s+r)}}$, an analogue to $\Delta_x^r(x^{(s)}) = s^{(r)}x^{(s-r)}$.) Finally then,

$$\mathcal{E}_m(N_1^{(l)}) = c_1^{(l)} \prod_{i=1}^{m} \left[\frac{\Gamma(c+1)}{\Gamma(l)\Gamma(c-l+1)} \int_0^1 G_i(-\alpha)\alpha^{l-1}(1-\alpha)^{c-l}d\alpha \right] \qquad (1 \le l \le c_1). \tag{4.3}$$

Since the random variables involved cannot be negative, $\mathcal{E}_m(N_1^{(l)}) \le c_1^{(l)}$.

4.2.2 *Probability generating function*

We go to equation (4.1) written in the pgf form

$$\sum_{k=0}^{c_1} t^k P_k = \sum_{k=0}^{c_1} t^k \Pr(N_1 = k \mid m)$$
$$= (t + E_{x_1}E_{x_2}\cdots E_{x_m} - 1)^{c_1} \prod_{i=1}^{m} \frac{x_i^{(r_i)}}{(c+r_i)^{(r_i)}}$$

so

$$P_k(c_1, c_2; m) = \binom{c_1}{k}(E_{x_1}\cdots E_{x_m} - 1)^{c_1-k} \prod_{i=1}^{m} \frac{x_i^{(r_i)}}{(c+r_i)^{r_i}}.$$

This consists of terms such as

$$(E_{x_1}E_{x_2}\cdots E_{x_m})^{\omega} \prod_{i=1}^{m} \frac{x_i^{(r_i)}}{(c+r_i)^{(r_i)}}$$
$$= \prod_{i=1}^{m} E_{x_i} \frac{x_i^{(r_i)}}{(c+r_i)^{(r_i)}} = \prod_{i=1}^{m} E_{c_2} \frac{(c_2+r_i)^{(r_i)}}{(c+r_i)^{(r_i)}}$$
$$= \prod_{i=1}^{m} \left[P_{i0} + \frac{(c_2+\omega+1)^{(1)}}{(c+1)^{(1)}} P_{i1} + \frac{(c_2+\omega+2)^{(2)}}{(c+2)^{(2)}} P_{i2} + \cdots\right]$$

and from (4.2)

$$= \mathcal{E}_m N_1^{(c_1-\omega)}$$

which equals the integral form in (4.3). Hence finally,

$$\Pr(N_1 = k \mid m) = \binom{c_1}{k} \sum_{r=0}^{c_1-k} (-1)^r \binom{c_1-k}{r}$$
$$\times \prod_{i=1}^{m} \frac{\Gamma(c+1)}{\Gamma(k+r)\Gamma(c-k-r+1)}$$
$$\times \int_0^1 G_i(-\alpha)\alpha^{k+r-1}(1-\alpha)^{c-k-r} d\alpha \tag{4.4}$$

and the probability of $N_1 = 0$ derived from $\sum_{k=0}^{c_1} \Pr(N_1 = k) = 1$. The components in the product term may be regarded as beta integral transform types.

Specifically

(i) $G_i(\alpha) = 1$ leads to $\sum_{k=1}^{c_1} P_k = 0$, so $P_0 = 1$.

(ii) $G_i(\alpha) = (1+\alpha)^{\omega_i}$, $i = 1, 2, \ldots$, leads to

$$\Pr(N_1 = k \mid m) = \binom{c_1}{k} \sum_{r=0}^{c_1-k} (-1)^r \binom{c_1-k}{r} \prod_{i=1}^{m} \frac{(c-k-r+\omega_i)^{(\omega_i)}}{(c+\omega_i)^{(\omega_i)}}$$
$$(k = 1, 2, \ldots, c_1).$$

(iii) Defining the product term in (4.4) as $\prod_{i=1}^{m} H(G_i; k; r)$, the i-th component turns out to be a hypergeometric function in cases where the replacement fmgf refers to a classical form. Thus

		$G_i(\alpha)$	$H(G_i; k; r)$
(a)	Binomial:	$(1+p_i\alpha)^{n_i}$	$F(-n_i, k+r; c+1; p_i)$
(b)	Poisson:	$\exp(\alpha\theta_i)$	$M(k+r; c+1; -\theta_i)$
(c)	Negative Binomial:	$(1-p_i\alpha)^{-\kappa_i}$	$F(\kappa_i, k+r; c+1; p_i)$.

Here we use the notation $F(a, b; c; z)$ for the hypergeometric function, and $M(a; b; z)$ for the confluent form; see National Bureau of Standard, Applied Maths Series 55, Chapters 13 and 15 (1964), i.e.,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1.$$

No simplification is possible for the Neyman Type A distribution when $G(\alpha) = \exp[-\theta_1(1-\exp(\theta_2\alpha))]$, or for the logarithmic series distribution when $G(\alpha) = \ln(1-p\alpha)/\ln(1-p)$.

For the case, $c_1 = 2$, $c_2 = 4$, $G_i(\alpha) = \exp(\alpha\theta)$, $\theta = 1$ and $m = 5$, we have

$$\begin{aligned}
\Pr(N_1 = 2 \mid 5) &= \sum_{r=0}^{0} (-1)^r \binom{1}{r} \left[30 \int_0^1 e^{-\alpha\theta} \alpha(1-\alpha)^4 d\alpha\right]^5 \\
&= \left[30 \int_0^1 e^{-\alpha\theta} \alpha(1-\alpha)^4 d\alpha\right]^5 \\
\Pr(N_1 = 1 \mid 5) &= 2\left[\sum_{r=0}^{1} (-1)^r \binom{1}{r} \frac{6!}{\Gamma(r+1)\Gamma(6-r)} \int_0^1 e^{-\alpha\theta} \alpha^r (1-\alpha)^{5-r} d\alpha\right]^5 \\
&= 2\left\{\left[6 \int_0^1 e^{-\alpha\theta}(1-\alpha)^5 d\alpha\right]^5 - \left[30 \int_0^1 e^{-\alpha\theta} \alpha(1-\alpha)^4 d\alpha\right]^5\right\}
\end{aligned}$$

giving

$$P_2 = 0.25491352, \qquad P_1 = 0.50546918, \qquad P_0 = 0.23961730.$$

4.2.3 *Further illustration*

In Table 4 we give illustrations of the four moments of N_1 (mean, standard deviation, skewness and kurtosis) when the random replenishments have Poisson, binomial, and negative binomial forms. (It is fairly obvious that when replenishments are deterministic with a constant amount the mean (when an integer) of the stochastic case, there will be a tendency to less variability. Here

$$\mathcal{E}(N_1^{(l)}) = c_1^{(l)} \left\{\frac{c^{(l)}}{(c+r)^{(l)}}\right\}^m.$$

Table 4(a). Two color urn model with Poisson random replenishments.

θ	c_1	c_2	m	$\mathcal{E}_m(N_1)$	$\sigma_m(N_1)$	$\sqrt{\beta_1}(N_1)$	$\beta_2(N_1)$
0.1	100	1	100	90.67	2.91	−0.28	1.81
1.0	100	1	100	37.69	4.83	0.05	2.98
5.0	50	5	30	3.83	1.87	0.44	3.15

(The r-th factorial moment of the Poisson is θ^r).

Table 4(b). Two color urn model with binomial random replenishments.

n	p	c_1	c_2	m	$\mathcal{E}_m(N_1)$	$\sigma_m(N_1)$	$\sqrt{\beta_1}(N_1)$	$\beta_2(N_1)$
5	0.20	100	1	100	37.62	4.55	0.04	2.98
10	0.10	100	1	100	37.66	4.69	0.05	2.98
20	0.05	100	1	100	37.67	4.76	0.05	2.98
25	0.20	50	5	30	3.80	1.83	0.42	3.12
50	0.10	50	5	30	3.81	1.85	0.43	3.14
100	0.05	50	5	30	3.82	1.86	0.44	3.14

Table 4(c). Two color urn model with negative binomial random replenishments.

p	κ	c_1	c_2	m	$\mathcal{E}_m(N_1)$	$\sigma_m(N_1)$	$\sqrt{\beta_1}(N_1)$	$\beta_2(N_1)$
0.1	10.0	100	1	100	37.73	4.97	0.05	2.98
1.0	1.0	100	1	100	38.04	6.03	0.09	2.96
10.0	0.1	100	1	100	40.81	11.85	0.21	2.79

In particular for the third example in Table 4(a) ($\theta = 5$, $c_1 = 50$, $c_2 = 5$, $m = 30$) we have 3.8298 for the mean and 1.8669 for the s.d. Also see Fig. 1 which illustrate decay of the mean as number of cycles increases.)

In view of the ramifications of the possible forms involved we can only outline general tendencies.

(a) The mean number of C_1 balls tends to zero as $m \to \infty$. For from (4.3) with $l = 1$, note that $0 < G_i(\alpha) \leq 1$, equality being rejected since here it would require $G(\alpha) = 1$ entailing zero input. That the inequality holds stems from $\mathcal{E}(1-\alpha)^x < 1$ for non-negative random variables, and $0 < \alpha < 1$. Hence components in the product in (4.3) ensure a convergence to zero.

(b) A similar property for the variance does not in general hold. An illustration is given in Fig. 2.

(c) All examples in Table 4 from the skewness and kurtosis values are of beta-type (Pearson type 1) and may be J-shaped. Note that since at any cycle

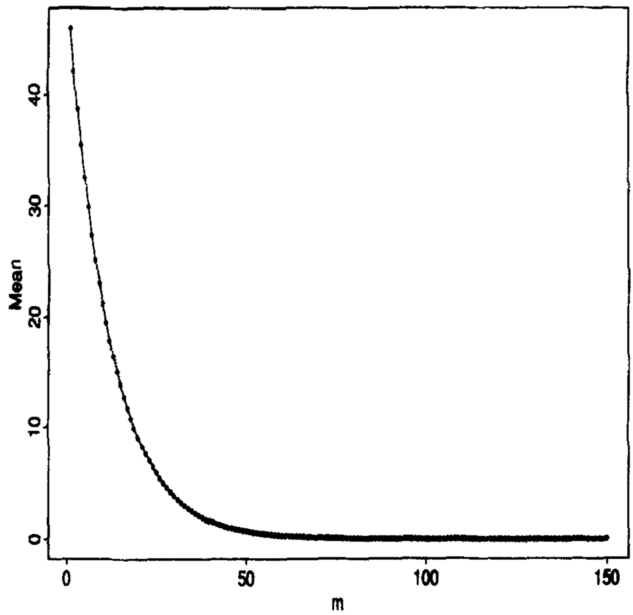

Fig. 1. Mean when $c_1 = 50$, $c_2 = 5$, $\theta = 5$ and varying m.

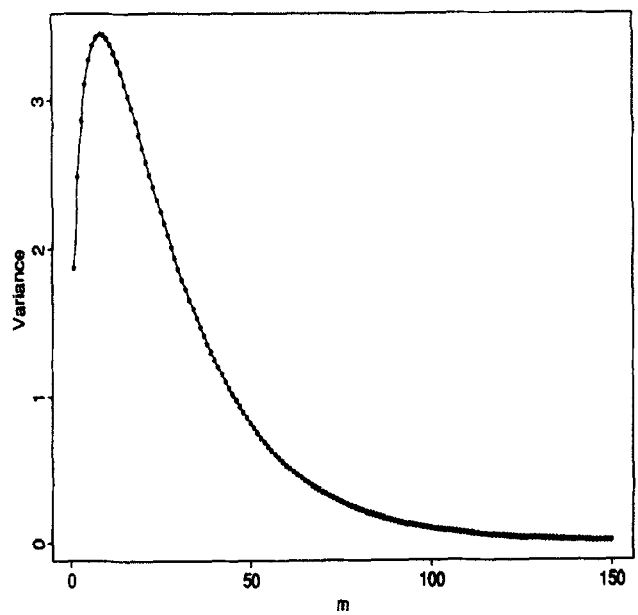

Fig. 2. Variance when $c_1 = 50$, $c_2 = 5$, $\theta = 5$ and varying m

$N_1 + N_2 = c$, $\mathrm{Var}(N_2) = \mathrm{Var}(N_1)$, $\sqrt{\beta}_1(N_2) = -\sqrt{\beta}_1(N_1)$, and $\beta_2(N_1) = \beta_2(N_2)$.

(d) When the number of balls in the urn is large, we may approximate the distribution of c_1 balls by fitting a binomial $(1+p^*\alpha)^{c_1} = [p^*t + (1-p^*)]^{c_1}$, where at the m-th cycle

$$p^* = \left(\frac{c+1}{c+1+np}\right)^m,$$

with moments

$$\mu_1'(N_1) \sim c_1 p^*, \qquad \sigma(N_1) \sim \sqrt{(}c_1 p^*(1-p^*)),$$
$$\sqrt{\beta}_1(N_1) \sim (1-2p^*)/\sigma(N_1), \qquad \beta_2(N_1) \sim 3 + \frac{(1-6p^*+6p^{*2})}{\sigma^2(N_1)}.$$

For the Poisson replenishment, np is taken to be θ, and for the negative binomial replenishments np is taken to be κp. Comparisons for the cases in Table 4(a) are given in Table 5.

Table 5. Comparisons.

Distribution		$\mu_1'(N_1)$	$\sigma(N_1)$	$\sqrt{\beta}_1(N_1)$	$\beta_2(N_1)$
	(1)	90.67	2.91	0.28	3.06
Poisson	(2)	37.70	4.85	0.05	2.98
	(3)	3.84	1.88	0.45	3.16
Binomial	(1)–(3)	37.70	4.85	0.05	2.98
	(4)–(6)	3.84	1.88	0.15	3.16
Neg. Binomial	(1)–(3)	37.70	4.85	0.05	2.98

There is good agreement for Poisson and binomial input, but for the negative binomial case the standard deviation for the 2nd and 3rd entries deviate markedly; for the latter the input variances are 2 and 11 respectively.

In general one should use formula (4.3) for the factorial moments, and reach some idea of the shape and type of distribution of c_1 balls using the first four moments.

5. Recursive schemes for fmgf and factorial moments

5.1 *General*

From (2.2) and Table 1, we have for the fmgfs

$$(5.1) \qquad f_m(\alpha_1, \alpha_2, \ldots, \alpha_{s-1}) = \left(1+\frac{\alpha_1}{E_{x_m}}\right)^{r^*_{1,m}} \left(1+\frac{\alpha_2}{E_{x_m}}\right)^{r^*_{2,m}} \cdots \left(1+\frac{\alpha_{s-1}}{E_{x_m}}\right)^{r^*_{s-1,m}} \times f_{m-1}\left(\frac{\alpha_1}{E_{x_m}}, \frac{\alpha_2}{E_{x_m}}, \ldots, \frac{\alpha_{s-1}}{E_{x_m}}\right) \frac{x_m^{(D_m)}}{S_M^{(D_m)}}\Bigg|_{x_m=S_m}.$$

Defining $f_m(\alpha_1, \alpha_2, \dots, \alpha_{s-1})$ as a multivariate polynomial in $\alpha_1, \alpha_2, \dots, \alpha_{s-1}$ and equating coefficients in (5.1) leads to complete factorial moment information. Note that if, $\mu_{(s)}$ denotes an s-th factorial moments, then for example

$$\mu_{(2)} = \mu_2' - \mu_1',$$
$$\mu_{(3)} = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'$$

etc., where μ_s' is a non-central moment. These and their generalizations involve Stirling numbers of the first kind.

5.2 *fmgf featuring one color* ($s = 2$)

Here

$$f_m(\alpha) = \left(1 + \frac{\alpha}{E_{x_m}}\right)^{r_{1,m}^*} f_{m-1}\left(\frac{\alpha}{E_{x_m}}\right) \frac{x_m^{(D_m)}}{S_m^{(D_m)}}\Bigg|_{x_m = S_m} \qquad (m = 1, 2, \dots; f_0(\alpha) = 1) \tag{5.2}$$

where we define

$$f_m(\alpha) = f_{m,0} + \alpha f_{m,1} + \alpha^2 f_{m,2} + \cdots$$

with factorial moments $f_{m,1}$, $2!f_{m,2}$, $3!f_{m,3}$, etc.

Equating coefficients of powers of α, we have the relations,

$$f_{m,1} = \left[\binom{r_{1,m}^*}{1} f_{m-1,0} + f_{m-1,1}\right] \frac{(S_m - 1)^{(D_m)}}{S_m^{(D_m)}},$$
$$f_{m,2} = \left[\binom{r_{1,m}^*}{2} f_{m-1,0} + \binom{r_{1,m}^*}{1} f_{m-1,1} + f_{m-1,2}\right] \frac{(S_m - 2)^{(D_m)}}{S_m^{(D_m)}},$$

and in general

$$f_{m,s} = \left[\sum_{j=0}^{s} \binom{r_{j,m}^*}{s-j} f_{m-1,j}\right] \frac{(S_m - s)^{(D_m)}}{S_m^{(D_m)}}, \qquad (m = 1, 2, \dots; f_0(\alpha) = 1).$$

Recursive schemes, for central moments would appear to be complicated especially in view of the non-linear forms involved. For example, for the variance,

$$\mu_2 = \mu_{(2)} + \mu_{(1)} - (\mu_{(1)})^2$$

where $\mu_{(1)}$ is the mean and $\mu_{(2)}$ the 2nd factorial moment.

Note also from (5.2) that $f_m(\alpha)$ is of degree $\lambda_m = \sum_{j=1}^{m} r_{1,j}^*$ in α, so that factorial moments higher than λ_m will be zero. Hence the complete fmgf $f_m(\alpha)$ is now defined recursively, so that expressed in terms of $t = 1 + \alpha$ yields the pgf. Notwithstanding repetitions, recursive relations for probabilities would almost certainly be more complicated than those for factorial moments.

5.3 *Bivariate relations*

Now take $s = 3$ in (5.1). Any two desired colors of balls in Table 1 can be translated to a 3 color urn situation. Here

$$f_m(\alpha_1,\alpha_2) = \left(1+\frac{\alpha_1}{E_{x_m}}\right)^{r^*_{1,m}} \left(1+\frac{\alpha_2}{E_{x_m}}\right)^{r^*_{2,m}} \times f_{m-1}\left(\frac{\alpha_1}{E_{x_m}}, \frac{\alpha_2}{E_{x_m}}\right) \frac{x_m^{(D_m)}}{S_m^{(D_m)}}\bigg|_{x_m=S_m}$$

and for example, for the correlation we need $f_{m,1,1}$ where

$$f_m(\alpha_1,\alpha_2) = f_{m,0,0} + \alpha_1 f_{m,1,0} + \alpha_2 f_{m,0,1} + \alpha^2 f_{m,2,0} + \alpha_1\alpha_2 f_{m,1,1} + \alpha_2^2 f_{m,0,2} + \cdots + .$$

The relation is

$$f_{m,1,1} = (r^*_{1,m} r^*_{2,m} f_{m-1,0,0} + r^*_{1,m} f_{m-1,0,1} + r^*_{2,m} f_{m-1,1,0} + f_{m-1,1,1}) \times \frac{(S_m-2)^{(D_m)}}{S_m^{(D_m)}} \qquad (m = 1,2,\ldots; f_0(\alpha_1,\alpha_2) = 1)$$

5.4 *Numerical example on the moments*

Example 1. Depletion replenishment schemata (Table 6).

For C_1 balls using the formulation in (2.1),

Table 6. Depletion replenishment schemata.

Color	C_1	C_2	R	D	R^*	Accumulation
cycle 0	2	6				
cycle 1	0	4	4	4	$r_1^* = 0$	$d_1 = 0$
cycle 2	0	5	5	5	$r_2^* = 0$	$d_2 = 0$
fmgf	α					

Cycle 1.

$$\begin{aligned} f_1(\alpha) &= K_1^{-1}(E_{x_1}+\alpha)^2 x_1^{(4)}\,|_{x_1=10} \qquad (K_1 = 12^{(4)}) \\ &= A_{10} + \alpha A_{11} + \alpha^2 A_{12} \\ &= \frac{1}{12^{(4)}}(12^{(4)} + 2\cdot 11^{(4)}\alpha + 10^{(4)}\alpha^2). \end{aligned}$$

Cycle 2.

$$\begin{aligned}
f_2(\alpha) &= K_2^{-1}(E_{x_2}+\alpha)^0(E_{x_2}E_{x_1}+\alpha)^2 x_1^{(4)} x_2^{(5)} \,|_{x_1=10, x_2=11} \qquad (K_2 = 13^{(5)}) \\
&= \frac{K_2^{-1}}{K_1^{-1}}(E_{x_2}+\alpha)^0 E_{x_2}^2 f_1\left(\frac{\alpha}{E_{x_2}}\right) x_2^{(5)} \\
&= \frac{1}{13^{(5)}}(A_{10}E_{x_2}^2 + A_{11}\alpha E_{x_2} + A_{12}\alpha^2)x_2^{(5)} \\
&= 1 + \alpha A_{11}\frac{12^{(5)}}{13^{(5)}} + \alpha^2 A_{12}\frac{11^{(5)}}{13^{(5)}}.
\end{aligned}$$

At the second cycle, the mean is

$$\mu_1' = \frac{12^{(5)}}{13^{(5)}} \cdot 2 \cdot \frac{11^{(4)}}{12^{(4)}} = \frac{32}{39} = 0.8205.$$

The second factorial moment is

$$\mu_{[2]}' = 2 \cdot \frac{11^{(5)}}{13^{(5)}} \frac{10^{(4)}}{12^{(4)}} = \frac{2 \cdot 8 \cdot 7}{13 \cdot 12} \frac{8 \cdot 7}{12 \cdot 11} = 0.3046$$

and the standard deviation is 0.6722.

The pgf is $(196t^2 + 664t + 427)/1287 = 0.1523t^2 + 0.5159t + 0.3318$.

Example 2. Shenton (1981) (Table 7).
For C_1 balls:

Table 7. Replenishments exceed depletions.

Color	C_1	C_2	R	D	R^*	Accumulation
cycle 0	1	5				
cycle 1	1	2	3	1	$r_1^* = 0$	$d_1 = 0$
cycle 2	2	1	3	1	$r_2^* = 0$	$d_2 = 2$
fmgf	α					

Cycle 1.

$$\begin{aligned}
f_1(\alpha) &= K_1^{-1}(E_{x_1}+\alpha)^2 7^{(1)} \qquad (K_1 = 9) \\
&= \frac{1}{9}\left(1 + \frac{\alpha}{E_{x_1}}\right)^2 9^{(1)} \\
&= \left(1 + \frac{2 \cdot 8}{9}\alpha + \frac{7}{9}\alpha^2\right).
\end{aligned}$$

Cycle 2.

$$f_2(\alpha) = K_2^{-1}E_{x_2}^{-1}(E_{x_2}+\alpha)^2(E_{x_2}E_{x_1}+\alpha)^2 x_1^{(1)}x_2^{(1)}\,|_{x_1=7,x_2=8} \qquad (K_2 = 9\cdot 8)$$
$$= \frac{1}{11}\left(1+\frac{\alpha}{E_{x_2}}\right)^2\left(1+\frac{16\alpha}{9E_{x_2}}+\frac{7\alpha^2}{9E_{x_2}^2}\right)11^{(1)}$$
$$= \frac{1}{11}\left(1+\frac{2\alpha}{E_{x_2}}+\frac{\alpha^2}{E_{x_2}^2}\right)\left(1+\frac{16\alpha}{9E_{x_2}}+\frac{7\alpha^2}{9E_{x_2}^2}\right)11^{(1)}$$
$$= \frac{1}{11}\left(1+\frac{34\alpha}{9E_{x_2}}+\frac{48\alpha^2}{9E_{x_2}^2}+\frac{30\alpha^3}{9E_{x_2}^3}+\frac{7\alpha^4}{9E_{x_2}^4}\right)11^{(1)}$$
$$= (6t^2+44t^3+49t^4)/99$$

as indicated in the paper.

Example 3. Shenton (1981), pp. 333–334 (Table 8).
Consider setting up the mean and variance for color 4 in this paper.

Table 8. Replenishments equal deplations.

Color	C_1	C_2	R	D	R^*	Accumulation
cycle 0	3	16				
cycle 1	6	5	11	11	$r_1^*=0$	$d_1=0$
cycle 2	4	10	14	14	$r_2^*=0$	$d_2=0$
cycle 3	1	11	12	12	$r_3^*=0$	$d_3=0$
fmgf	α					

Cycle 1.

$$f_1(\alpha) = K_1^{-1}(E_{x_1}+\alpha)^9 21^{(11)}$$
$$= 1+\frac{9\cdot 19\alpha}{30}+\frac{36\cdot 19\cdot 18\alpha^2}{30\cdot 29}+\cdots.$$

Cycle 2.

$$f_2(\alpha) = \frac{1}{33^{(10)}}\left(1+\frac{\alpha}{E_{x_2}}\right)^4\left(1+\frac{9\cdot 19\alpha}{30E_{x_2}}+\frac{36\cdot 19\cdot 18\alpha^2}{30\cdot 29E_{x_2}^2}+\cdots\right)33_2^{(14)}$$
$$= 1+\left(4+\frac{9\cdot 19}{30}\right)\frac{19}{33}\alpha+\left(6+\frac{49\cdot 19}{30}+\frac{36\cdot 19\cdot 18}{30\cdot 29}\right)\frac{19\cdot 18}{33\cdot 32}\alpha^2+\cdots.$$

Cycle 3.

$$f_3(\alpha) = \frac{1}{31^{(12)}}\left(1+\frac{\alpha}{E_{x_2}}\right)^1\left(1+\frac{f_{21}\alpha}{E_{x_2}}+\frac{f_{22}\alpha^2}{E_{x_2}^2}+\cdots\right)31_2^{(12)}$$
$$= 1+4.035875\alpha+7.169258\alpha^2+\cdots.$$

Then the mean $=$ 4.0359 and the variance $= 2\times 7.169258+4.035875$ $4.035875^2 = 2.0861$ in an agreement with previoous work.

6. Conclusion

A computer program to set up iterations for the first four moments for a particular color is being studied. This approach is clearly needed if cycles of 1000 or so are being considered, and such cycles are certainly possible for health physics applications.

Acknowledgements

We appreciate the helpful remarks of both referees and the Editor of the Journal.

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