

TOWARDS A UNIFICATION OF CERTAIN CHARACTERIZATIONS BY CONDITIONAL EXPECTATIONS

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Abstract. The paper presents a characterization of a general family of distributions by the form of the expectation of an appropriately truncated function of the random variable involved. The obtained result unifies results existing in the literature for specific distributions as well as new results that appear for the first time in this paper. A discrete version is also provided unifying existing characterizations of known discrete distributions.

Key words and phrases: Characterization, conditional expectation, Pareto distribution, exponential distribution, Weibull distribution, Burr distribution, power function distribution, beta distribution, uniform distribution, geometric distribution, Yule distribution, Waring distribution.

1. Introduction

Conditional expectations have been widely used for deducing characterization results concerning probability distributions. The reason for this can be traced in the practical value of such characterizations that stems from the fact that information concerning conditional expectations is easily accessible.

In several instances, the characteristic condition involves the expectation of the truncated form of appropriate strictly monotonic functions of the random variable, that is an expectation of the form $E(h(X) | X > y)$, $y > 0$, where $h(\cdot)$ is a given strictly monotonic function. Knowledge of the form of this expectation can lead to the distribution of the random variable X . Shanbhag's (1970) characterization of the exponential distribution by the assumption that

$$(1.1) \quad E(X | X > y) = y + E(X)$$

is a typical result in this area of characterizations.

Of course, there has been a number of other results in the literature based on the more general condition,

$$(1.2) \quad E(h(X) | X > y) = g(y), y \geq 0$$

for specific forms of the functions $h(\cdot)$ and $g(\cdot)$. So, for various choices of $g(\cdot)$ it is worth mentioning the results by Kotlarski (1973, 1975) Shanbhag and Rao (1975), Gupta (1975) and Dallas (1976).

The theoretical interest, the mathematical elegance and above all the practical value of results based on (1.2) is enhanced when $g(\cdot)$ can be expressed in terms of $h(\cdot)$ and particularly when $h(\cdot)$ is an additive component in the analytical expression of $g(\cdot)$. The reason lies in the interpretation of such an expression as an attempt to approximate $E(h(X) | X > y)$ by $h(y)$, an aim often pursued in connection with applications.

Thus, given such an interpretation, Shanbhag's (1970) simple condition in (1.1) acquires a greater practical value especially since it is connected with the exponential distribution, a distribution with a flood of applications. The same can be said for several other results along these lines:

Using (1.2), Hamdan (1972) characterized a class of distributions which includes the Weibull, the exponential and the uniform distribution by assuming that $g(y) = h(y) + h(c)$, where c is a constant. Under certain restrictions on the function $g(\cdot)$, Laurent (1974) showed that the distribution of a random variable X can be uniquely determined by the condition $E(X | X > y) = y + f(y)$, $y > 0$, where $f(y)$ is an arbitrary function satisfying certain restrictions. It would be worth noting at this point that several of the above mentioned characterizations can be regarded as consequences of Kotlarski's (1973) result which, under certain restrictions, shows that a distribution function $F(\cdot)$ is uniquely determined as the distribution function of a random variable X if and only if

$$(1.3) \quad E[h(X) | X > y] = h(y) + \int_y^b (1 - F(x))dh(x)/(1 - F(y)), \quad a \leq y < b.$$

A natural question that arises is whether a unification of these results is possible.

In the next section of this paper the answer is given in the affirmative. A result is provided that can indeed be regarded as unifying the above cited characterizations upon considering $g(y)$ in (1.2) to be of the form $h(y) + H(y)$. Of course, (1.3) is of this form except that it is somewhat restrictive as $H(\cdot)$ is required to be expressible in terms of a proper probability distribution function. In the result that follows on the contrary, no specific assumptions are made concerning the form of the function $H(\cdot)$: It is merely, the remainder in the approximation of $E(h(X) | X > y)$ by $h(y)$. Since, as mentioned earlier, approximating $E(h(X) | X > y)$ by $h(y)$ is often pursued in connection with applications, characterizing the distribution of a random variable X by a generic form of the remainder $H(y)$ in the above approximation becomes of greater importance. So, the result of Theorem 2.1 in the next section, being in this respect an alternative formulation of Kotlarski's (1973) result, can be thought of as connecting a multitude of characterization results for specific continuous distributions: Some of them are new, others are known.

Section 3 treats the case of discrete distributions where a known result based on a characterizing condition of type (1.2) is Shanbhag's characterization of the geometric distribution. So, discrete versions of the characterizing condition (1.3)

as well as of the main result of Section 2 are provided leading to known results for specific discrete distributions.

2. Characterization of a general class of distributions based on conditional expectations: The continuous case

THEOREM 2.1. *Let X be a real valued random variable with continuous distribution function $F_X(x)$, $x \geq \alpha$. Let $h(\cdot)$ be a strictly monotonic and continuously differentiable function with $E(h(X)) < \infty$ and assume that,*

$$(2.1) \quad E[h(X) | X > y] = h(y) + H(y), \quad y \geq \alpha$$

where $H(y)$ is an arbitrary function. Then the distribution of X is uniquely determined by the form of the function

$$H(y) = E[h(X) | X > y] - h(y), \quad y \geq \alpha.$$

PROOF. From the definition of $E[h(X) | X > y]$ we have

$$\int_y^\infty h(x) dF_X(x) = \bar{F}_X(y)[H(y) + h(y)].$$

Using integration by parts the R.H.S. and $E(h(X)) < \infty$ we find

$$\int_y^\infty h'(x) \bar{F}_X(x) dx = H(y) \bar{F}_X(y).$$

Differentiating the above we find

$$-h'(y) \bar{F}_X(y) = H'(y) \bar{F}_X(y) + H(y) \bar{F}'_X(y)$$

and solving it we find

$$(2.2) \quad \bar{F}_X(y) = c_0 \exp \left\{ - \int \frac{h'(y) + H'(y)}{H(y)} dy \right\}$$

and c_0 is determined from $\bar{F}_X(\alpha) = 1$. This completes the proof of the theorem.

It is therefore obvious that since in practical situations it becomes often important to approximate an expectation of the form $E[h(X) | X > y]$ by $h(y)$, (2.2) provides the population distribution for a variety of problems in terms of the remainder $H(y)$ of this approximation.

Thus, Hamdan's (1972) characterization of the Weibull distribution follows for $h(x) = x^\alpha$, $\alpha > 0$ and $H(x) = c$, $c > 0$, $x \in [0, \infty)$ leading to Shanbhag's (1970) characterization of the exponential distribution when $\alpha = 1$. Laurent's result is also obtained for $h(x) = x$, $x \geq 0$.

Several new characterizations are further obtained. Of these it is worth mentioning (i) a characterization of the power function distribution for $h(x) = -\ln(1 - x^\alpha)$, $\alpha > 0$, $x \in [0, 1]$ and $H(x) = 1$, $x \in [0, 1]$ reducing to Hamdan's (1972) characterization of the uniform distribution in $(0, 1)$ for $\alpha = 1$, (ii) a characterization of the Burr distribution for $h(x) = \ln(1 + x^\alpha)$, $\alpha > 0$, $x \in [0, \infty)$ and $H(x) = c$, $c > 0$, $x \in [0, \infty)$ yielding a characterization of the beta type II distribution for $\alpha = 1$, (iii) a characterization of the Pareto distribution in $[1, \infty)$ for $h(x) = \ln X$ and $H(x) = c$, $c > 0$, $x \in [1, \infty)$ and (iv) a characterization of the Pareto distribution in $[\theta, \infty)$ for $h(x) = x$, $x \in [\theta, \infty)$ and $H(x) = \frac{x}{\alpha-1}$, $\alpha > 0$, $x \in [\theta, \infty)$.

In closing this section, it would be worth noting the following: As already mentioned in the introduction, the particular case where the remainder $H(y)$ in the approximation of $E[h(X) | X > y]$ is a constant and of the particular form $h(c)$, i.e. the case where $g(y)$ in (1.2) equals $h(y) + h(c)$ provides the possibility of extending characterizations of the exponential distribution based on the condition

$$(2.3) \quad E[h(X) | X > y] = h(y) + h(c).$$

This is an alternative form of the condition of constant residual life expectancy whenever $h(\cdot)$ is a monotonic function and leads to a characterization of the distribution of the random variable $h(X)$ as exponential. Indeed, Galambos and Kotz (1978) point out that is a disguised but essentially equivalent form of the characterization of the exponential distribution as the distribution of a random variable X based on the condition

$$(2.4) \quad E[h(X) | X > y] = y + c.$$

This is due to the fact that for strictly increasing $h(\cdot)$ the events $\{X > y\}$ and $\{h(X) > h(y)\}$ are equiprobable. Hence, (2.3) provides a family of characterizations for the exponential distribution as the distribution of $h(X)$ and can only be considered as an extension of (2.4) in the sense that it may lead to meaningful characterizations of distributions whose mean is infinite.

3. Characterization of a general class of distributions based on conditional expectations: The discrete case

Equation (1.1) expresses an elementary characterization of the exponential distribution derived by Shanbhag (1970) while using (1.1) with respect to various monotonic transformations of X leads to characterizations of other continuous distributions which, as seen in the previous section, can be unified through Theorem 2.1. Shanbhag's (1970) characterization of the geometric distribution by the condition $E(X | X \geq y) = [y] + E(X)$, $y \geq 0$ provides a case of a characterization of a discrete distribution by a property of the expectation of the truncated variable X . The relationship that exists between the exponential distribution and the geometric distribution is of course well known and many aspects of it have been revealed through the duality of their properties, the above mentioned characterization being one such instance. As a result of this duality, the geometric

distribution is regarded to be the discrete counterpart of the exponential distribution. The Pareto and the Yule distributions are another pair of distributions linked by the same type of association (Xekalaki and Panaretos (1988)). Since the results that preceded and motivated theorem (2.1) lead, among other results, to characterizations of the Pareto distribution, it would be interesting to examine whether its discrete counterpart, the Yule distribution, can also be characterized by the form of the expectation of an appropriate truncated random variable. It becomes evident from what follows that indeed this duality exists with respect to this characteristic property too. In fact a more general distribution, the Waring distribution, which contains the Yule distribution as a special case, is characterized.

The fact that at least two discrete probability distributions can be characterized by the form of appropriate conditional expectations leads to the question of whether a more general class of discrete distributions can be characterized in the same manner.

As shown in the sequel, in the form of two theorems, this holds true. Specifically it is shown (Theorem 3.2) that a discrete analogue of Kotlarski's (1973) characteristic condition as given by (1.3) is valid and the obtained results are further generalized by the subsequent theorem (Theorem 3.3) which in turn can be regarded as the discrete version of Theorem 2.1.

Before proving the main results we provide some definitions.

DEFINITION 3.1. A non-negative integer valued random variable X is said to have the Waring distribution with parameters α and ρ if its probability function is given by,

$$(3.1) \quad p_x = P(X = x) = \frac{\rho \alpha^{(x)}}{(\alpha + \rho)_{(x+1)}}, \quad x = 0, 1, 2, \dots, \quad \alpha > 0, \quad \rho > 0$$

where

$$a_{(r)} = \Gamma(a + r) / \Gamma(a), \quad r = 0, 1, 2, \dots$$

DEFINITION 3.2. A non-negative integer valued random variable X is said to have the Yule distribution on $\{1, 2, \dots\}$ with parameter ρ if its probability function is given by

$$(3.2) \quad p_x = P(X = x) = \frac{\rho(x-1)!}{(\rho+1)_{(x)}}, \quad x = 1, 2, \dots, \quad \rho > 0.$$

Obviously (3.2) is a shifted one unit to the right Waring distribution with $\alpha = 1$.

LEMMA 3.1. *Let X be a non-negative integer-valued random variable. Then,*

$$(3.3) \quad P(X > r) = \frac{1}{\rho}(\alpha + r)P(X = r), \quad r = 0, 1, 2, \dots, \quad \alpha > 0, \quad \rho > 0$$

if and only if X follows the Waring distribution with probability function given by (3.1).

PROOF.

Necessity: Let X be a random variable having a Waring $(\alpha; \rho)$ distribution. Then

$$\begin{aligned} P(X > r) &= \sum_{x=r+1}^{\infty} \frac{\rho \alpha_{(x)}}{(\alpha + \rho)_{(x+1)}} \\ &= \frac{\rho \alpha_{(r)}}{(\alpha + \rho)_{(r+1)}} \sum_{x=1}^{\infty} \frac{(\alpha + r)_{(x)}}{(\alpha + \rho + r + 1)_{(x)}} \\ &= P(X = r) \frac{1}{\rho} \sum_{x=0}^{\infty} \frac{\rho (\alpha + r)_{(x+1)}}{(\alpha + \rho + r + 1)_{(x+1)}} \\ &= \frac{1}{\rho} (\alpha + r) P(X = r). \end{aligned}$$

Sufficiency: Let

$$(3.4) \quad P(X > r) = \frac{1}{\rho} (\alpha + r) p_r, \quad r = 0, 1, 2, \dots, \quad \alpha > 0, \quad \rho > 0.$$

Specializing (3.4) for $y = r$ and $y = r + 1$ and subtracting the resulting equations we obtain

$$p_{r+1} - \frac{\alpha + r}{\alpha + \rho + r + 1} p_r = 0.$$

The unique solution of the above difference equation is given by

$$\begin{aligned} p_r &= p_0 \prod_{i=0}^{r-1} \frac{\alpha + i}{\alpha + \rho + 1 + i} \\ &= p_0 \frac{\alpha_{(r)}}{(\alpha + \rho + 1)_{(r)}}. \end{aligned}$$

But $\sum_{x=0}^{\infty} p_x = 1$ which leads to $p_0 = \frac{\rho}{\alpha + \rho}$ which completes the proof.

THEOREM 3.1. *Let X be a non-negative integer-valued random variable with a finite expected value. Then X is distributed according to the Waring distribution with probability function given by (3.1) if and only if*

$$(3.5) \quad E[X | X > y] = \mu + (y + 1)\mu_1, \quad y = -1, 0, 1, \dots$$

where $\mu = E(X) = \frac{\alpha}{\rho - 1}$ and $\mu_1 = \frac{\rho}{\rho - 1}$, (i.e. μ_1 is the expected value of the corresponding Yule (ρ) distribution).

PROOF.

Necessity: Let X be a random variable having a Waring $(\alpha; \rho)$ distribution. For $y = -1$ (3.5) is obvious. For $y = 0, 1, 2, \dots$ we have from the definition of

$E[X | X > y]$ and the result of Lemma 3.1 that

$$\begin{aligned}
 E[X | X > y] &= \frac{\rho}{\alpha + y} \frac{1}{P(X = y)} \sum_{x=y+1}^{\infty} xP(X = x) \\
 &= \frac{\rho}{\alpha + y} \frac{1}{P(X = x)} \sum_{x=0}^{\infty} (x + y + 1) \frac{\rho}{\alpha + \rho} \frac{\alpha_{(x+y+1)}}{(\alpha + \rho + 1)_{(x+y+1)}} \\
 &= \frac{\rho}{\alpha + y} \frac{1}{P(X = y)} \\
 &\quad \cdot \sum_{x=0}^{\infty} (x + y + 1) \frac{\rho}{\alpha + \rho} \frac{\alpha_{(y+1)}(\alpha + y + 1)_{(x)}}{(\alpha + \rho + 1)_{(y+1)}(\alpha + \rho + y + 2)_{(x)}} \\
 &= \sum_{x=0}^{\infty} x \frac{\rho}{\alpha + \rho + y + 1} \frac{(\alpha + y + 1)_{(x)}}{(\alpha + \rho + y + 2)_{(x)}} \\
 &\quad + (y + 1) \sum_{x=0}^{\infty} \frac{\rho}{\alpha + \rho + y + 1} \frac{(\alpha + y + 1)_{(x)}}{(\alpha + \rho + y + 2)_{(x)}} \\
 &= \frac{\alpha + y + 1}{\rho - 1} + y + 1 \\
 &= \mu + (y + 1)\mu_1.
 \end{aligned}$$

Sufficiency: Let $E[X | X > y] = \mu + (y + 1)\mu_1$. This relation can be equivalently written as follows:

$$(3.6) \quad \sum_{x=y+1}^{\infty} xP(X = x) = \mu P(X > y) + (y + 1)\mu_1 P(X > y), \quad y = -1, 0, 1, \dots$$

Subtracting (3.6) from the corresponding equation for $y - 1$, we obtain

$$(y - \mu - y\mu_1)P(X = y) = -\mu_1 P(X > y), \quad y = 0, 1, \dots$$

Since $E(X) < \infty$ from (3.5) it follows that $\mu_1 > 1$. Therefore the last relation reduces to

$$P(X > y) = \left(\frac{\mu_1 - 1}{\mu_1} y + \frac{\mu}{\mu_1} \right) P(X = y), \quad y = 0, 1, 2, \dots$$

But this is characteristic for the Waring ($\alpha = \frac{\mu}{\mu_1 - 1}; \rho = \frac{\mu_1}{\mu_1 - 1}$) distribution according to the result of Lemma 3.1.

We observe from the theorem above that $E(X | X > r) = (r + 1)E(X)$ for all $r = 0, 1, 2, \dots$ or equivalently $E(X | X > r) = rE(X)$ for all $r = 1, 2, \dots$ characterizes the Yule distribution with $\rho > 1$.

THEOREM 3.2. *Let X_0 and X be two non-negative integer-valued random variables with corresponding distribution functions F_0 and F_x . Let also $F_x(\alpha) =$*

$F_0(\alpha) = 0$ for some $\alpha \geq 0$. Let $h(\cdot)$ be a real strictly monotonic function such that $E[h(X_0)] < \infty$ and $E[h(X)] < \infty$. Then X_0 and X are identically distributed if and only if the following condition is satisfied

$$(3.7) \quad E[h(X) | X > y] = E[h(X_0) | X_0 > y] \quad \text{for all } y = -1, 0, 1, \dots$$

PROOF.

Necessity: Obvious.

Sufficiency: Observe that by the definition of $E[h(X) | X > y]$ we have

$$E[h(X) | X > y] = \sum_{x=y+1}^{\infty} h(x)P(X = x | X > y).$$

Then, writing

$$P(X = x | X > y) = (\bar{F}_X(x-1) - \bar{F}_X(x)) / \bar{F}_X(y), \quad x = y+1, y+2, \dots$$

we obtain, through an algebraic identity of Abel's lemma type

$$(3.8) \quad E[h(X) | X > y] = h(y+1) + [\bar{F}_x(y)]^{-1} \sum_{x=y+1}^{\infty} \bar{F}_x(x)[h(x+1) - h(x)].$$

A similar expression is obtained for $E[h(X_0) | X_0 > y]$, $y = -1, 0, 1, \dots$. Then, comparing the relations (3.7) and (3.8) it follows that

$$(3.9) \quad [\bar{F}_0(y)]^{-1} \sum_{x=y+1}^{\infty} \bar{F}_0(x)[h(x+1) - h(x)] \\ = [\bar{F}_x(y)]^{-1} \sum_{x=y+1}^{\infty} \bar{F}_x(x)[h(x+1) - h(x)] \quad y = -1, 0, 1, \dots$$

Let

$$(3.10) \quad K_0(y) = \sum_{x=y+1}^{\infty} \bar{F}_0(x)[h(x+1) - h(x)], \quad y = -1, 0, 1, \dots$$

and

$$(3.11) \quad K(y) = \sum_{x=y+1}^{\infty} \bar{F}_x(x)[h(x+1) - h(x)].$$

From (3.10) forming the difference for y and $y-1$ we have, after some calculations

$$(3.12) \quad \bar{F}_0(y) = \frac{K_0(y) - K_0(y-1)}{h(y) - h(y+1)}, \quad y = 0, 1, \dots$$

In a similar way starting from (3.11) we find that

$$(3.13) \quad \bar{F}_x(y) = \frac{K(y) - K(y-1)}{h(y) - h(y+1)}, \quad y = 0, 1, \dots$$

Substituting back in (3.9) it follows that

$$(3.14) \quad \frac{K_0(y)}{K_0(y) - K_0(y-1)} = \frac{K(y)}{K(y) - K(y-1)}, \quad y = 0, 1, 2, \dots$$

From (3.14) it is obvious that $K(y) = CK_0(y)$, where C constant. Using (3.10) and (3.11) this last relation becomes,

$$(3.15) \quad \sum_{x=y+1}^{\infty} \bar{F}_x(x)[h(x+1) - h(x)] = C \sum_{x=y+1}^{\infty} \bar{F}_0(x)[h(x+1) - h(x)],$$

$y = 0, 1, 2, \dots$

Working as with relation (3.10) we have because of the strict monotonicity of $h(\cdot)$

$$\bar{F}_x(y+1) = C\bar{F}_0(y+1), \quad y = -1, 0, 1, \dots$$

i.e.

$$\bar{F}_x(y) = C\bar{F}_0(y), \quad y = 0, 1, 2, \dots$$

Since $F_X(\alpha) = F_0(\alpha) = 0$ it follows $C = 1$, therefore, $F_x(y) = F_0(y)$, $y \geq \alpha$, $\alpha \geq 0$.

Because of the previous theorem we have that the geometric distribution with distribution function $F(x) = 1 - (1-p)^x$ is characterized by $E[X | X > y] = y + \frac{1}{p}$, $y = 1, 2, \dots$ if we take $h(x) = x$.

The same holds true for the Yule distribution with distribution function $F(x) = 1 - \frac{x!}{(\rho+1)_{(x)}}$, $x = 1, 2, \dots$, $\rho > 0$ if $h(x) = x$ and $E[X | X > y] = (y+1)\frac{\rho}{\rho-1}$. This also follows from the corresponding characterization of the Waring distribution by translation and $\alpha = 1$ which can be characterized by relation (3.16)

$$(3.16) \quad E[X | X > y] = \frac{\alpha}{\rho-1} + (y+1)\frac{\rho}{\rho-1}, \quad y = 0, 1, 2, \dots$$

and $h(x) = x$.

A proof of the above is as follows: As $X - y - 1$ is non negative given $X > y$ we have

$$(3.17) \quad E[X | X > y] = (y+1) + \frac{1}{\bar{F}_x(y)} \sum_{x=y+1}^{\infty} \bar{F}_x(x).$$

For the Waring distribution $\bar{F}(x) = \frac{\alpha+x}{\rho} P(X = x)$, $x = 0, 1, \dots$, $\alpha, \rho > 0$, and substitution in (3.17) gives

$$(3.18) \quad \begin{aligned} E[X | X > y] &= (y + 1) + \frac{1}{\bar{F}_x(y)} \frac{\alpha}{\rho} \sum_{x=y+1}^{\infty} P(X = x) \\ &\quad + \frac{1}{\rho} \frac{1}{\bar{F}_x(y)} \sum_{x=y+1}^{\infty} P(X = x) \\ &= y + 1 + \frac{\alpha}{\rho} + \frac{1}{\rho} E[X | X > y]. \end{aligned}$$

Solving (3.18) with respect to $E[X | X > y]$ the result follows.

THEOREM 3.3. *Let X be a discrete random variable defined on $\{m, m + 1, \dots\}$, $m = 0, 1, 2, \dots$ with distribution function $F_X(x)$. Let $h(\cdot)$ be a strictly monotonic function. Then the function*

$$(3.19) \quad H(y) = E[h(X) | X > y] - h(y), \quad y = m, m + 1, \dots, \quad m = 0, 1, 2, \dots$$

uniquely determines the distribution of X .

PROOF. By an argument similar to that used for the proof of Theorem 3.1 it follows that

$$E[h(X) | X > y] = h(y + 1) + \frac{1}{\bar{F}_x(y)} \sum_{x=y+1}^{\infty} [h(x + 1) - h(x)] \bar{F}_x(x)$$

or

$$(3.20) \quad \sum_{x=y+1}^{\infty} h(x + 1) \bar{F}_X(x) - \sum_{x=y+1}^{\infty} h(x) \bar{F}_X(x) = \bar{F}_X(y) [H(y) + h(y) - h(y + 1)].$$

Using the same technique of subtraction we find

$$H(y + 1) \bar{F}_X(y + 1) - [H(y) + h(y) - h(y + 1)] \bar{F}_X(y) = 0.$$

Since $h(\cdot)$ is monotonically increasing the relation (3.19) leads to $H(y) > 0$ for all y . Therefore this last relation can be written,

$$\bar{F}_X(y) - \frac{H(y - 1) + h(y - 1) - h(y)}{H(y - 1)} \bar{F}_X(y - 1) = 0.$$

Since $\bar{F}_X(m - 1) = 1$ the unique solution of this difference equation is given by,

$$(3.21) \quad \bar{F}_X(y - 1) \equiv P(X \geq y) = \prod_{r=m}^{y-1} \frac{H(r - 1) + h(r - 1) - h(r)}{H(r - 1)}.$$

Therefore $P(X \geq y)$ is uniquely determined by $h(y)$ and $H(y)$.

Because of the previous theorem we have that the geometric distribution is characterized by $E[X | X > y] = y + c$, for $c > 1$ and $y = 1, 2, \dots$. Note that this result is a variant of Shanbhag's (1970) characterization of the geometric distribution.

COROLLARY 3.1. (Characterization of the $(m - 1)$ -truncated Yule distribution) *Let X be a discrete random variable taking values on $\{m, m + 1, \dots\}$. Then the distribution of X is the $(m - 1)$ -truncated Yule with parameter $\rho = -\frac{1}{\beta} > 0$ and probability function given by*

$$P(X = x) = \rho \frac{(m + 1)_{(x-m)}}{(\rho + m + 1)_{(x-m+1)}}, \quad x = m, m + 1, \dots; \quad m = 1, 2, \dots, \quad \rho > 0$$

if and only if

$$E[X | X > y] = \alpha + (\beta + 1)y, \quad \text{for } \alpha < -1, \beta < 0, \text{ and } y = m, m + 1, \dots$$

PROOF. Letting $h(x) = x$ and $H(y) = \alpha + \beta y$ the characterizing condition (3.21) becomes

$$P(X \geq y) = \prod_{r=m}^{y-1} \frac{(\alpha - \beta + 1) + \beta r}{(\alpha - \beta) + \beta r}$$

or equivalently,

$$(3.23) \quad P(X \geq y) = \left(\frac{\alpha - \beta + 1}{\beta} + m \right)_{(y-m)} / \left(\frac{\alpha - \beta}{\beta} + m \right)_{(y-m)}, \quad y = m, m + 1, \dots$$

In the special case $\alpha - \beta + 1 = \beta$ the relation (3.22) becomes,

$$\begin{aligned} P(X \geq y) &= \frac{(m + 1)_{(y-m)}}{(1 + \rho + m)_{(y-m)}} \\ &= (\rho + 1 + y) \frac{(m + 1)_{(y-m)}}{(1 + \rho + m)_{(y-m+1)}} \\ &= \left[1 + \frac{1}{\rho}(y + 1) \right] \rho \frac{(m + 1)_{(y-m)}}{(1 + \rho + m)_{(y-m+1)}}. \end{aligned}$$

But this is the necessary and sufficient condition for X to be distributed according to the $(m - 1)$ -truncated Yule (ρ), $\rho = -\frac{1}{\beta} > 0$ (Xekalaki (1984)).

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