Ann. Inst. Statist. Math.
Vol. 48, No. 1, 127–144 (1996)

THE MAXIMUM SIZE OF THE PLANAR SECTIONS OF RANDOM SPHERES AND ITS APPLICATION TO METALLURGY

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(Received June 17, 1994; revised April 12, 1995)

Abstract. A theorem of this paper proves that if the size distribution of random spheres is generalized gamma, its Wicksell transform and other related distributions belong to the domain of attraction of the Gumbel distribution. The theorem also shows the attraction coefficients of the distributions. The fatigue strength of high-strength steel is closely related to the maximum size of nonmetallic inclusions in the region of maximum stress of the steel. Murakami and others developed a method, making use of the Gumbel QQ-plot, for predicting the maximum size from the size distribution of inclusion circles in microscopic view-fields. Based on the Gumbel approximation of the maximum of Wicksell transforms, a modified and extended version of Murakami's method is justified, and its performance is evaluated by simulation.

Key words and phrases: Extreme value theory, generalized gamma distribution, Gumbel distribution, metal fatigue, stereology, Wicksell's corpuscle problem.

1. Introduction

Metal fatigue of high-strength steel starts from defects due to nonmetallic inclusions. It is known by experience that the fatigue strength is strongly related to the maximum size of inclusions in a high stress region of the steel. Hence, it is necessary to predict the maximum size based on the size data measured in microscopic view-fields of planar sections of a specimen.

In a series of papers and a book, Yukitaka Murakami and his coauthors developed a prediction method to control the quality of high-strength steel. See, e.g., Murakami and Usuki (1989), Murakami (1993, 1994) and Uemura and Murakami (1990). In his problem, nonmetallic inclusions are round and scattered within the steel. To estimate the size distribution and the spatial density of random spheres from those of circles on the sectional plane is Wicksell's corpuscle problem

(Wicksell (1925)). See, for example, Ripley (1981), Stoyan *et al.* (1987) or Reiss (1993).

A characteristic of Murakami's method is to use *only* the maximum sectional sizes of nonmetallic inclusions in microscopic view-fields. Relying on the extreme-value theory, he uses the Gumbel QQ-plot (probability paper) of the sectional maximum data for the prediction of, or the extrapolation to, the maximum size of spheres in the specific part of a test piece. For tension and compression, narrow sections of metallic pieces are critical. For bending and torsion, central surface parts are critical. The former case is more complicated and is of main concern, but the latter case will also be studied.

Murakami's method was developed based on metallurgic research, experiments and simulations. In this paper, a modified and extended version of Murakami's method is justified based on a statistical parametric model, and its performance is examined mainly by simulation. The examination and analysis of available data reveal that the current engineering practice is reasonable but not sufficiently accurate. In the concluding discussions, a way to improve the situation is suggested.

In Section 2, Wicksell transform in terms of the areas, rather than diameters, and its power transformations are mentioned. In Section 3, a general result by Drees and Reiss (1992) on the maximum of Wicksell transforms is restated to give a theoretical overview of the problem.

In Section 4, the generalized gamma distribution is assumed as the distribution of the size of inclusions. Under the assumption, the main theorem and its corollaries prove that all the related size distributions belong to the domain of attraction of the Gumbel distribution, and show the attraction coefficients of the distributions. In Section 5, a generalized and modified version of Murakami's method for prediction is proposed.

In Section 6, a main source of the biases of the estimates is discussed, and the asymptotic variances of the estimates are evaluated. In Section 7, datasets by Murakami (1993) are analyzed, and the results of Section 6 are checked by simulations. The results of this section show that the method of the current use is unsatisfactory, and some alternatives are suggested for a sequel work.

The final Section 8 is supplementary. In the Subsection 8.1, reports on the size distribution of inclusions are surveyed. In the Subsection 8.2, the proofs of the Theorem 4.1 and its Corollary 4.2 of Section 4 are given, and in the Subsection 8.3, a mathematical aspect of our Gumbel approximation is discussed. In the Subsection 8.4, the original version of Murakami's method is examined.

2. Wicksell's corpuscle problem

Let $\{[x_n, y_n, z_n; s_{Vn}]\}$ be a marked Poisson process in the xyz-space. (x_n, y_n, z_n) is the center of a sphere with the corresponding intensity λ_V. s_{Vn} is the size of the sphere at (x_n, y_n, z_n) with the p.d.f. (probability density function) f_V, which is independent of (x_n, y_n, z_n). We assume that the mean of f_V is finite and small enough compared with λ_V, and the spheres are actually disjoint. The spheres are cut by a sectional plane, say, the xy-plane. Let the point process $\{[x_n, y_n; s_{An}]\}$ describe the sectional circles. (x_n, y_n) is the center of circles with the corresponding intensity λ_A, and s_{An} is the size of the circle at (x_n, y_n) with the p.d.f. f_A, which is independent of (x_n, y_n).

As the sizes s_{Vn} and s_{An} the diameter is usually considered. Here, we adopt the area of the great circle of spheres as s_{Vn} and the area of sectional circles as s_{An}, and let S_V and S_A denote the areas as random variables. We need, for the theory and practice, to consider the area S_C of great circle of spheres which cross the sectional plane. Thus, there are three random variables S_ω, $\omega = V$, C and A, with p.d.f. $f_\omega(s)$, d.f. (distribution function) F_ω and survival function $\bar{F}_\omega(s) = 1 - F_\omega(s)$. It is known that

$$\lambda_V = \sqrt{\pi}\lambda_A/(2\mu_0), \qquad \mu_0 = E(\sqrt{S_V}), \tag{2.1}$$

$$f_C(s) = \frac{\sqrt{s}}{\mu_0} f_V(s), \qquad 0 < s < \infty, \tag{2.2}$$

$$S_A = S_C(1 - U^2), \tag{2.3}$$

$$\bar{F}_A(s) = \frac{1}{2\mu_0}\int_s^\infty \frac{1}{\sqrt{v-s}}\bar{F}_V(v)dv, \tag{2.4}$$

and

$$f_A(s) = \frac{1}{2\mu_0}\int_s^\infty \frac{1}{\sqrt{v-s}} f_V(v)dv. \tag{2.5}$$

In (2.3), U is the uniform random variable on $(0, 1)$ and independent of S_C. Note that $1 - U^2$ has the beta distribution $\mathsf{Be}(1, 1/2)$. The p.d.f. (2.5) (or the probability distribution of S_A) is called 'Wicksell transform of the p.d.f. f_V' (or of the probability distribution of S_V).

PROPOSITION 2.1. *From* (2.2) *and* (2.3),

$$E(S_C^r) = E(S_V^{r+1/2})/\mu_0, \qquad \textit{and} \qquad E(S_A^r) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r+1)}{\Gamma(r+3/2)}\frac{E(S_V^{r+1/2})}{\mu_0}.$$

PROPOSITION 2.2. *The d.f.'s F_V, F_C and F_A, defined above, are equivariant with respect to their scales. That is, for any positive ξ, the d.f.'s $F_V(s/\xi)$, $F_C(s/\xi)$ and $F_A(s/\xi)$, are the triple of a Wicksell transformation.*

PROPOSITION 2.3. *If F_V is an exponential distribution, or if F_A is exponential, then $F_V = F_A$, and vice versa. This is the unique invariant d.f. of the Wicksell transformation.*

We prefer areas S_ω, $\omega = V$, C and A, to diameters because the original data are areas: they are numbers of pixels of digital microscopic images with gray levels within some limits. Another reason is that (2.5) (or (2.4)) is simpler than the more popular transform for the diameter distributions. This fact was utilized by Hall and Smith (1988) in the nonparametric estimation of F_V by a random sample from F_A. Other than these, there is no reason to choose length or area as the size of spheres. Even a size without geometric meaning will work, and we shall use later $T_\omega = S_\omega^\beta$, $\beta > 0$, $\omega = V$, C and A.

PROPOSITION 2.4. *Let ϕ_ω and Φ_ω denote the p.d.f. and d.f. of $T_\omega = S_\omega^\beta$, respectively. Corresponding to* (2.2)–(2.5), *the Wicksell transformation in terms of them is as follows.*

$$\phi_C(t) = t^{1/2\beta}\phi_V(t)/E(T_V^{1/2\beta}), \qquad (E(T_V^{1/2\beta}) = E(\sqrt{S_V})), \tag{2.6}$$

$$T_A = (1-U^2)^\beta T_C, \tag{2.7}$$

$$\bar{\Phi}_A(t) = \frac{1}{2E(T_V^{1/2\beta})}\int_{t^{1/\beta}}^\infty \frac{1}{\sqrt{v - t^{1/\beta}}}\bar{\Phi}_V(v^\beta)dv, \tag{2.8}$$

and

$$\phi_A(t) = \frac{t^{1/\beta - 1}}{2\beta E(T_V^{1/2\beta})}\int_{t^{1/\beta}}^\infty \frac{1}{\sqrt{v - t^{1/\beta}}} f_V(v)dv. \tag{2.9}$$

These return to popular expressions for diameters if $\beta = 1/2$.

3. Limit distribution of the maximum of Wicksell transforms

Let $(X_j)_{j=1}^\infty$ be a sequence of i.i.d. (independent and identically distributed) random variables with a common d.f. H. If a normalized maximum of $(X_j)_{j=1}^n$ has a limit distribution, that is, if there exists a sequence of pairs of coefficients $((a_n, b_n), a_n > 0)_{n=1}^\infty$ and a nondegenerate d.f. L such that

$$\lim_{n\to\infty} H^n(a_n x + b_n) = L(x),$$

then 'H belongs to the domain of attraction of L' ($H \in \mathcal{D}(L)$, by symbol), and (a_n, b_n)'s are 'the attraction coefficients of H'. In fact, L is limited to the following d.f.'s.

$$L_{ic}(x) = \begin{cases} \exp(-x^{-c}), & x \geq 0, \quad c > 0, \ i = 1, \\ \exp(-(-x)^c), & x \leq 0, \quad c > 0, \ i = 2, \\ \exp(-\exp(-x)), & -\infty < x < \infty, \quad c = 1, \ i = 3. \end{cases}$$

L_{1c}, L_{2c} and L_{31} are called Fréchet, Weibull and Gumbel distributions, respectively. See, for example, Galambos (1987), Castillo (1988) or Reiss (1993) for the statistical extreme-value theory.

For the maximum of Wicksell transforms, a general result was obtained by Drees and Reiss (1992). They gave their result in terms of the d.f.'s of the diameters. Here, it is restated in terms of the d.f. Φ_ω of $T_\omega = S_\omega^\beta$, $\beta > 0$, $\omega = V$ and A, which are introduced at the end of Section 2.

PROPOSITION 3.1.

$$\Phi_A \in \mathcal{D}(L_{1,c-1/2\beta}) \quad \textit{if } \Phi_V \in \mathcal{D}(L_{1c}), \quad c > \frac{1}{2\beta},$$

$$\Phi_A \in \mathcal{D}(L_{2,c+1/2}) \quad \textit{if } \Phi_V \in \mathcal{D}(L_{2c}), \quad c > 0,$$

and

$$\Phi_A \in \mathcal{D}(L_{31}) \qquad\qquad \textit{if } \Phi_V \in \mathcal{D}(L_{31}).$$

PROOF. In the case of L_{31}, use Theorem 1 of Drees and Reiss (1992) and Theorem 1.5.6 of de Haan (1970), to obtain the proposition. The proofs of the other cases are similar. □

Note that these are for F_ω if $\beta = 1$ and for diameters if $\beta = 1/2$. Drees and Reiss (1992) gave, for $\beta = 1/2$, a condition on Φ_A such as $\Phi_V \in \mathcal{D}(L_{ic})$, and discussed the case where Φ_V is the generalized Pareto distribution.

In the following, we shall discuss only the case of L_{31}. Instead of L_{31}, the d.f. of the Gumbel distribution is denoted by

$$\Lambda(x) = \exp(-\exp(-x)), \qquad -\infty < x < \infty.$$

We observe the maximum area of circles in a view-field. The number N of the circles is the Poisson variable with mean $\lambda_A A$, where A is the area of the view-field.

In general, let $(X_j)_{j=1}^{\infty}$ be a sequence of i.i.d. random variables with d.f. H, and let N be the Poisson variable with mean θ and independent of X_j's. Under the condition $N > 0$, $Y = \max(X_1, \dots, X_N)$ has the d.f.

$$H(y;\theta) := \frac{1}{1-e^{-\theta}}(e^{-\theta \bar{H}(y)} - e^{-\theta}) \sim e^{-\theta \bar{H}(y)}, \qquad \theta \to \infty,$$

where $\bar{H}(y) = 1 - H(y)$. Further if $H \in \mathcal{D}(\Lambda)$ with attraction coefficients (a_n, b_n), then

$$H(a_\theta y + b_\theta; \theta) \to \Lambda(y), \qquad \theta \to \infty.$$

See, for example, Corollary 2.4.1 of de Haan (1970). This is a special case of a more general result in Section 6.2 of Galambos (1987).

4. Generalized gamma model

A distribution with the p.d.f.

$$h(x;\alpha,\gamma,\xi) = \frac{1}{\Gamma(\alpha)}\frac{\gamma}{\xi^{\alpha\gamma}} x^{\alpha\gamma-1} e^{-(x/\xi)^{\gamma}} 1[0 < x < \infty], \qquad \alpha, \gamma > 0, \tag{4.1}$$

is denoted by $\mathsf{Ga}(\alpha,\gamma,\xi)$ and called the generalized gamma distribution. See, e.g., Bain (1983). It is one of the most general families of distributions of positive random variables, and covers typical distributions except for those with shorter or longer tails. $\mathsf{Ga}(\alpha,1,\xi)$ is the gamma distribution, $\mathsf{Ga}(1,\gamma,\xi)$ is the Weibull distribution, and $\mathsf{Ga}(1,1,\xi)$ is the exponential distribution. Ga is a limit of the generalized F distribution, and the log-normal distribution is its limit.

We assume, in Wicksell's corpuscle problem, that the area S_V of the great circle of the sphere follows $\mathsf{Ga}(\alpha,\gamma,\xi)$. Some engineering reports justifying the assumption are reviewed in the Subsection 8.1. This model has the following nice properties. One is that if the distribution of any size of spheres, diameter, area of great circle or volume, is generalized gamma then that of another size is also generalized gamma. S_C follows $\mathsf{Ga}(\alpha + (2\gamma)^{-1}, \gamma, \xi)$ and $T_C = S_C^\beta$ follows $\mathsf{Ga}(\alpha + (2\gamma)^{-1}, \gamma/\beta, \xi^\beta)$. Another is that under the assumption, all F_V, F_C and F_A belong to $\mathcal{D}(\Lambda)$ as will be shown soon. Moreover, the random numbers with d.f. F_C are easily generated, and they are converted to those with F_A by (2.3).

PROPOSITION 4.1. *If S_V follows* $\mathsf{Ga}(\alpha, \gamma, \xi)$.

$$\mu_0 = E(\sqrt{S_V}) = \sqrt{\xi}\,\Gamma\left(\alpha + \frac{1}{2\gamma}\right) \Big/ \Gamma(\alpha) \tag{4.2}$$

and

$$E(\sqrt{S_A}) = \frac{\pi}{4}\sqrt{\xi}\,\Gamma\left(\alpha + \frac{1}{\gamma}\right) \Big/ \Gamma\left(\alpha + \frac{1}{2\gamma}\right). \tag{4.3}$$

Moreover, all the moments of S_V, S_C and S_A are expressed in terms of the gamma function, and this is another merit of adopting the generalized gamma distribution.

THEOREM 4.1. *Wicksell transforms F_A of the d.f. F_V, and f_A of the p.d.f. f_V, have the following asymptotic expressions if F_V follows* $\mathsf{Ga}(\alpha, \gamma, 1)$.

$$1 - F_A(s) \sim \frac{\sqrt{\pi/\gamma}}{2\,\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} s^{(\alpha-3/2)\gamma+1/2} \exp(-s^\gamma), \qquad s \to \infty, \tag{4.4}$$

and

$$f_A(s) \sim \frac{\sqrt{\pi\gamma}}{2\,\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} s^{(\alpha-1/2)\gamma-1/2} \exp(-s^\gamma), \qquad s \to \infty. \tag{4.5}$$

The proof is given in Subsection 8.2. If $(\alpha, \gamma) = (1, 1)$ the asymptotic expressions are the exponential survival function and p.d.f., and these are exact ones. See, Proposition 2.3.

COROLLARY 4.1. *If S_V follows $\mathsf{Ga}(\alpha, \gamma, \xi)$, all F_V, F_C and F_A belong to $\mathcal{D}(\Lambda)$, and their attraction coefficients are as follows. The scale coefficient is common for all F_V, F_C and F_A, and*

$$a_n = \frac{\xi}{\gamma}(\log n)^{1/\gamma - 1}. \tag{4.6}$$

Their location coefficients are

$$b_{Vn} = \xi(\log n)^{1/\gamma} + a_n\{(\alpha - 1)\log\log n - \log\Gamma(\alpha)\}, \tag{4.7}$$

$$b_{Cn} = \xi(\log n)^{1/\gamma} + a_n\left\{\left(\alpha + \frac{1}{2\gamma} - 1\right)\log\log n - \log\Gamma\left(\alpha + \frac{1}{2\gamma}\right)\right\}, \tag{4.8}$$

and

$$(4.9)\quad b_{An} = \xi(\log n)^{1/\gamma} + a_n \left\{ \left(\alpha + \frac{1}{2\gamma} - \frac{3}{2}\right) \log\log n - \log \frac{2\,\Gamma\left(\alpha + \frac{1}{2\gamma}\right)}{\sqrt{\pi/\gamma}} \right\}.$$

PROOF. This is based on the Theorem 4.1 and Proposition 1 in Takahashi (1987). (There is a typographical error in Takahashi's proposition; the term $\log \alpha$ in b_n should read $\log a$.) □

COROLLARY 4.2. *Under the condition of Corollary 4.1, the d.f.'s Φ_ω of $T_\omega = S_\omega^\gamma$, $\omega = V$, C and A, also belong to $\mathcal{D}(\Lambda)$, and their attraction coefficients are as follows. The scale coefficient a_n is common and independent of n.*

$$(4.10)\quad a_n = \xi^\gamma,$$

$$(4.11)\quad b_{Vn} = \xi^\gamma\{\log n + (\alpha - 1)\log\log n - \log\Gamma(\alpha)\},$$

$$(4.12)\quad b_{Cn} = \xi^\gamma \left\{ \log n + \left(\alpha + \frac{1}{2\gamma} - 1\right) \log\log n - \log\Gamma\left(\alpha + \frac{1}{2\gamma}\right) \right\},$$

and

$$(4.13)\quad b_{An} = \xi^\gamma \left\{ \log n + \left(\alpha + \frac{1}{2\gamma} - \frac{3}{2}\right) \log\log n - \log \frac{2\,\Gamma\left(\alpha + \frac{1}{2\gamma}\right)}{\sqrt{\pi/\gamma}} \right\}.$$

The proof is given in Subsection 8.2. The coincidence of a_n in Corollary 4.2, that is, the parallelism of the QQ-plots of the maximums of S_ω^γ, $\omega = V$, C and A, was conjectured by Uemura and Murakami (1990), in the case (α, γ)=(1, 1/2), based on simulation. They did not notice that a_n is independent of n.

5. Murakami's method for predicting the maximum size

The Theorem of the previous section is the basis of our prediction. However, there are two drawbacks in the practical application of Corollary 4.1. First, the attraction coefficients (4.6)–(4.9) are rather complicated for our purpose. Second, the convergence to the Gumbel distribution in the corollary is not fast enough if $\gamma \neq 1$. See, the beginning of Section 6.

To overcome the drawbacks, we deal with $T_\omega = S_\omega^\gamma$, $\omega = V$, C and A, instead of S_ω, in Corollary 4.2. The effect of the transformation will be mentioned in the next Section 6. We suppose that the Gumbel approximation of the maximum of T_ω in Corollary 4.2 is good enough.

Let us repeat formally our prediction problem. Assume the area S_V of the great circle of inclusion sphere in steel to follow the generalized gamma distribution $\mathsf{Ga}(\alpha, \gamma, \xi^{1/\gamma})$ with the p.d.f.

$$(5.1)\qquad \frac{1}{\Gamma(\alpha)} \frac{\gamma}{\xi^\alpha} s^{\alpha\gamma - 1} e^{-s^\gamma/\xi} 1[0 < s < \infty],$$

with known $\alpha > 0$ and $\gamma > 0$. (The choice of the scale parameter is not essential as remarked in Proposition 2.2. The choice $\xi^{1/\gamma}$ is just to be cautious of the relationship between T_ω and S_ω.) The intensity of the sphere in steel is λ_V. The parameters ξ and λ_V are unknown. The observation is the maximum area W_A of the inclusion circles in a microscopic view-field of area A. The expected number of circles in the view-field is $\lambda_A A$, which should be large enough for the Gumbel approximation of W_A in the following Proposition.

There are two cases (V) and (C) in the problem.

(V) Predict the maximum area, W_V, of the great circles of the spheres in a part of volume V, where the stress of tension or compression of a test piece is maximum. The expected number of spheres in the part is $\lambda_V V$.

(C) Predict the maximum area, W_C, of the great circles of the spheres which intersect with the surface part of steel of area A_C, where the stress of bending and torsion is maximum. The expected number of spheres in the part is $\lambda_A A_C$. We are interested, in fact, in the inclusion spheres near the surface. We can modify, accordingly, the parameter $\lambda_A A_C$, slightly. However, the modification does not affect the results much.

PROPOSITION 5.1. *The distribution of the power transformation W_ω^γ, of the maximum area, is approximated by the Gumbel distribution $\Lambda((t-\eta_\omega)/\xi)$, $\omega = V$, C and A, where the scale parameter ξ is common and equal to that of $T_V = S_V^\gamma$, and the location parameter is determined as follows.*

$$\eta_V/\xi = \tau_V + (\alpha - 1)\log\tau_V - \log\Gamma(\alpha), \qquad \tau_V = \log(\lambda_V V), \tag{5.2}$$

$$\eta_C/\xi = \tau_C + \left(\alpha + \frac{1}{2\gamma} - 1\right)\log\tau_C - \log\Gamma\left(\alpha + \frac{1}{2\gamma}\right), \qquad \tau_C = \log(\lambda_A A_C), \tag{5.3}$$

and

$$\eta_A/\xi = \tau_A + \left(\alpha + \frac{1}{2\gamma} - \frac{3}{2}\right)\log\tau_A - \log\frac{2\Gamma\left(\alpha + \frac{1}{2\gamma}\right)}{\sqrt{\pi/\gamma}}, \qquad \tau_A = \log(\lambda_A A). \tag{5.4}$$

The two intensities are related by (2.1), *in which μ_0 is given by* (4.2) *or* (8.2) *with ξ replaced by $\xi^{1/\gamma}$.*

PROOF. Insert the expected number of spheres or circles into n of (4.11)–(4.13), and change ξ^γ in these expressions to ξ. □

Note that

$$\begin{aligned} \tau_V &= \tau_A + \log\frac{V}{A} - \delta, \\ \delta &= \log\left(\frac{2}{\sqrt{\pi}}E(\sqrt{S_V})\right) = \frac{1}{2\gamma}\log\xi + \log\frac{2\Gamma\left(\alpha + \frac{1}{2\gamma}\right)}{\sqrt{\pi}\,\Gamma(\alpha)}, \end{aligned} \tag{5.5}$$

and

$$\tau_C = \tau_A + \log(A_C/A). \tag{5.6}$$

The units of A, A_C, V, λ_A, λ_V and S_ω must be consistent. For example, area and volume must be measured in μ m^2 and μ m^3, respectively.

Now, we are ready to state a modified and extended version of Murakami's prediction method. His original version is explained in Subsection 8.4. For convenience, we drop the subscript A in W_A, η_A and τ_A.

Prediction procedure. The maximum areas $(W_1, \dots, W_k)$ in k view-fields are the available data. We fit $\Lambda((t-\eta)/\xi)$ to $(W_1^\gamma, \dots, W_k^\gamma)$ by the maximum likelihood method. Let $(\hat\eta, \hat\xi)$ denote the estimate, from which we estimate some functions of (η, ξ). Firstly,

$$\hat\delta = \frac{1}{2\gamma}\log\hat\xi + \log\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)}{\Gamma(\alpha)}. \tag{5.7}$$

Secondly, solve the equation

$$\hat\tau + \left(\alpha + \frac{1}{2\gamma} - \frac{3}{2}\right)\log\hat\tau = \frac{\hat\eta}{\hat\xi} + \log\frac{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)}{\sqrt{\pi/\gamma}}, \tag{5.8}$$

to obtain $\hat\tau$. The solution to the equation is discussed below, in the final paragraph of this section.

For the case (V),

$$\widehat{\tau_V} = \hat\tau + \log\frac{V}{A} - \hat\delta, \tag{5.9}$$

and

$$\widehat{\eta_V}/\hat\xi = \widehat{\tau_V} + (\alpha - 1)\log\widehat{\tau_V} - \log\Gamma(\alpha). \tag{5.10}$$

Finally, the mean and quantiles of W_V^γ are estimated by linear expressions $\widehat{\eta_V} + c\hat\xi$. For the mean $c = \gamma_E$, Euler's constant, and for the quantile $c = \omega_p$, $\Lambda(\omega_p) = p$ or $\omega_p = -\log(-\log p)$. We may need the corresponding values of W_V or $W_V^{1/2}$.

For the case (C), $\hat\delta$ is not necessary, and $\widehat{\tau_C}$ and $\widehat{\eta_C}$ are estimated by

$$\widehat{\tau_C} = \hat\tau + \log(A_C/A),$$

and

$$\widehat{\eta_C}/\hat\xi = \widehat{\tau_C} + \left(\alpha + \frac{1}{2\gamma} - 1\right)\log\widehat{\tau_C} - \log\Gamma\left(\alpha + \frac{1}{2\gamma}\right).$$

The last step for predicting W_C is the same as for W_V.

The estimation of the scale and location parameters of the Gumbel distribution has been studied by many authors. See, for example, Tiago de Oliveira (1983). The maximum likelihood method is good enough if the sample size is not small.

The equation (5.8) has a unique solution for any value of $\hat{\eta}/\hat{\xi}$ provided that $d = \alpha + 1/(2\gamma) - 3/2 > 0$. If $d < 0$, the l.h.s. of the equation is a convex function with the minimum at $\tau = -d$, which is very small compared with the ordinary value of $\tau = \log(\lambda_A A)$ or that of $\hat{\eta}/\hat{\xi}$. Near these ordinary values, the l.h.s. is almost linear in τ and the Newton process starting from $\hat{\tau} = \hat{\eta}/\hat{\xi}$ will converge fast.

6. Performance of Murakami's method

By simulations it turned out that the power transformation $T_\omega = S_\omega^\gamma$ is meaningful. The Gumbel approximation of $(W_1^\gamma, \ldots, W_k^\gamma)$ is justified by fitting the generalized extreme-value distribution $F((t-\eta)/\xi; c)$ defined by

$$F(x;c) = \begin{cases} \exp(-(1-cx)^{1/c})1[c^{-1} < x], & c < 0, \\ \exp(-\exp(-x)), & c = 0, \\ \exp(-(1-cx)^{1/c})1[x < c^{-1}], & c > 0, \end{cases} \tag{6.1}$$

and by comparing $\hat{c}$, under the assumption $c = 0$, to the normal approximation with the asymptotic variance. See Hosking *et al.* (1985) for the method of probability-weighted moments for the estimation and for testing the hypothesis $c = 0$. Without the power transformation, the Gumbel approximation of the maximum of S_A is unsatisfactory for the case, e.g. $(\alpha, \gamma) = (1, 1/2)$. We tried to justify the power transform, but were not successful. See the discussion in Subsection 8.3.

A problem is that the parameter (η, ξ) of (5.4) in the approximation $\Lambda((t - \eta)/\xi)$ of W_j^γ is not satisfactory. The main reason is the asymptotic approximation (8.3), which is shown to be an underestimate (or overestimate) of $1 - \Phi_A$ if $\alpha < 1$ and $\gamma < 1$ (or $\alpha > 1$ and $\gamma > 1$). Further, it is conjectured that the asymptotic expression underestimates (or overestimates) $1 - \Phi_A$ if $d = \alpha + 1/(2\gamma) - 3/2 > 0$ (or $d < 0$). This causes the positive (or negative) bias of both $\hat{\eta}$ and $\hat{\xi}$. The biases reflect the actual distribution. Since the Gumbel approximation of W_V is better, the positive biases of $\hat{\eta}$ and $\hat{\xi}$, which will occur in practice, cause the overestimation of the mean or quantiles of W_V, namely bias of the estimate to the safe side. The effect of the magnifying factor V/A or A_C/A on the final bias is discussed later.

The biases decrease if the area A of view-fields increases, but their analytic evaluation seems difficult. We shall discuss the improvement elsewhere.

If $(\alpha, \gamma) = (1, 1)$, F_V and F_A is the exponential distribution $\mathsf{Ga}(1, 1, \xi)$ (Proposition 2.3), the power transformation is unnecessary, and W_V and $(W_j)_{j=1}^k$ are lower-truncated Gumbel variables. The attraction coefficients (4.11)–(4.13) turn to

$$\begin{aligned} b_{Vn} &= b_{An} = \xi \log n, \\ b_{Cn} &= \xi\{\log n + 2^{-1} \log\log n - \log(\sqrt{\pi}/2)\}, \end{aligned}$$

and the prediction method is much simpler. This is the ideal case for Murakami's method. If (α,γ) is close to $(1,1)$ the prediction works very well.

The maximum likelihood estimate $(\hat\eta,\hat\xi)$ of the parameters of $\Lambda((x-\eta)/\xi)$, based on its random sample of size k, has the asymptotic variance

$$\text{(6.2)}\quad \operatorname{Var}(\hat\eta,\hat\xi)=\frac{\xi^2}{k}\begin{bmatrix}1+6(1-\gamma_E)^2/\pi^2 & 6(1-\gamma_E)/\pi^2\\ 6(1-\gamma_E)/\pi^2 & 6/\pi^2\end{bmatrix}+o\left(\frac{1}{k}\right),\qquad k\to\infty.$$

See, for example, Tiago de Oliveira (1983) for the Gumbel distribution. The estimate $f(\hat\eta,\hat\xi)$ of a parameter function $f(\eta,\xi)$ has

$$\operatorname{Var}(f(\hat\eta,\hat\xi))=(\nabla f)^t\operatorname{Var}(\hat\eta,\hat\xi)\nabla f+o(1/k),\qquad k\to\infty,$$

where $(\nabla f)^t=(f_\eta,f_\xi)$. For $f(\eta,\xi)=\eta_V+c\xi$,

$$\text{(6.3)}\quad \begin{aligned} f_\eta&=\xi\left(1+\frac{\alpha-1}{\tau_V}\right)\frac{\partial}{\partial\eta}\tau_V=\xi\left(1+\frac{\alpha-1}{\tau_V}\right)\frac{\partial}{\partial\eta}\tau,\\ f_\xi&=\frac{f}{\xi}+\xi\left(1+\frac{\alpha-1}{\tau_V}\right)\frac{\partial}{\partial\xi}\tau_V=\frac{f}{\xi}+\xi\left(1+\frac{\alpha-1}{\tau_V}\right)\left(\frac{\partial}{\partial\xi}\tau-\frac{1}{2\gamma\xi}\right),\\ \frac{\partial}{\partial\eta}\tau&=\frac{\tau}{\xi(\tau+d)},\qquad \frac{\partial}{\partial\xi}\tau=-\frac{\eta}{\xi}\frac{\partial}{\partial\eta}\tau,\qquad\text{and}\qquad d=\alpha+\frac{1}{2\gamma}-\frac{3}{2}.\end{aligned}$$

Within the range of (α,γ) and τ which are practically probable, $d=-0.5$–1.3, $\tau=2$–5, and $\tau/(\tau+d)=0.5$–1.3. In this range η/ξ is almost linear in τ, and $\eta/\xi=1$–7. See Tables 1(b) and 2 for typical parameter values. Conversely,

$$\hat\tau\doteqdot\hat\eta/\hat\xi+\log\frac{2\Gamma\left(\alpha+\dfrac{1}{2\gamma}\right)}{\sqrt{\pi/\gamma}},$$

and $\log V/A$ is dominant among the terms of τ_V. Hence $1+(\alpha-1)/\tau_V$ is close to one. Finally, roughly speaking,

$$\text{(6.4)}\qquad f_\eta\doteqdot 1\qquad\text{and}\qquad f_\xi\doteqdot f/\xi-\eta/\xi\doteqdot\tau_V\gg 1,$$

and $\operatorname{Var}(f(\hat\eta,\hat\xi))\doteqdot(\tau_V)^2\operatorname{Var}(\hat\xi)$.

Going back to the biases, if we find the biases $\Delta\eta$ and $\Delta\xi$ of $(\hat\eta,\hat\xi)$, the estimate $f(\hat\eta,\hat\xi)$ has the bias

$$f_\eta\Delta\eta+f_\xi\Delta\xi\doteqdot\tau_V\Delta\xi.$$

Simulations support the above evaluation of the variance and bias of $f(\hat\eta,\hat\xi)$.

Table 1(a). Data of the maximum square root (μm) of the area of inclusions within the steel (SUP12) in microscopic view-field of area 0.0309 mm^2.

2.015	2.387	2.651	3.045	3.205	3.206	3.303	3.371	3.374	3.426
3.459	3.494	3.627	3.711	3.745	3.870	3.993	4.096	4.141	4.191
4.237	4.237	4.510	4.545	4.658	4.663	4.733	4.774	4.889	5.018
5.024	5.093	5.122	5.317	5.385	5.533	5.758	6.507	6.707	6.794

Table 1(b). Estimate (estimate of its standard deviation) for the data of Table 1(a).

η	ξ	τ	mag. fac.	$E(\sqrt{W_V})$	Murakami
3.764(0.166)	0.996(0.123)	3.005(0.380)	$10^3 \times V/A$	18.27(1.78)	16.52
			$10^4 \times V/A$	20.56(2.06)	18.71
			$10^5 \times V/A$	22.85(2.34)	20.89

The parameter values are; $\alpha = 1$, $\gamma = 1/2$, $V = 89.8$ (mm^3), and $A = 0.0309$ (mm^2). mag. fac. and Murakami stand for the magnifying factor and the Murakami's original prediction method, respectively.

7. Analysis of a dataset

In Murakami (1993), Appendix, there are QQ-plots of 20 datasets. The original datasets of 18 of them are offered from Prof. Murakami, and were analyzed by the method of Section 5 for the case (V). The volume for prediction (see the beginning of Section 5) is $V = 89.8$ (mm^3), the volume of the central narrow part of a test piece, times the number of test pieces $N = 1$, 10, and 100. Table 1(a) shows one of the datasets, and Table 1(b) shows estimates and their error estimates. In Table 1(b) "mag. fac" is equal to $10^3 \times NV/A$. The predicted values are close to Murakami's, see Subsection 8.4. Figure 1 is the QQ-plot of the dataset and predicted values of Table 1(b).

Table 2 shows some results of simulations. The parameter value $\xi = 1$ is chosen close to Table 1(b). The values of $\tau = \log(\lambda_A A)$ is changed to see the effect of changing A, the area of microscopic view-fields. The value of η is determined by the asymptotic formula (5.4). The values of standard deviation of the estimates agree with the arguments of Section 6. The values of biases of estimates agree also with the arguments of Section 6 starting from the biases of $\hat{\eta}$ and $\hat{\xi}$. The standard deviation and bias of the final estimates are unsatisfactory.

To decrease the standard deviation and bias to a level, say, 5%, the sample size must be increased substantially. Alternatively, we may change the measurement at the cost of more works. One is to count the number of circles in each microscopic view-field. The mean number is an unbiased estimate of $\lambda_A A$. Another is to measure all the circles in each view-field. The estimation method using these will

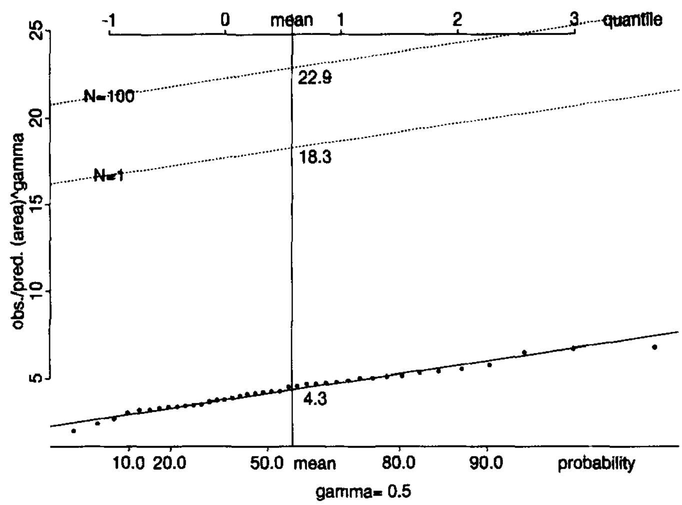

Fig. 1. Gumbel-QQ-plot. location = 3.764, scale = 0.996.

Table 2. Simulation results of estimation, assuming $\mathsf{Ga}(1, 1/2, 1)$, for $k = 40$, $V/A = 200{,}000$ and some values of $\tau = \log(\lambda_A A)$. 1,000 replications.

estimates	$\hat{\eta}$	$\hat{\xi}$	$\hat{\eta}/\hat{\xi}$	$\hat{\tau}$	$\widehat{\tau_V}$	$E(\widehat{\sqrt{W_V}})$
True	2.073	1.000	2.073	$\log 5$	13.695	14.272
Bias	0.330	0.153	0.039	0.034	−0.102	2.029
S. D.	0.196	0.142	0.294	0.227	0.336	1.673
True	2.945	1.000	2.945	$\log 10$	14.388	14.965
Bias	0.258	0.125	−0.057	−0.043	−0.153	1.645
S. D.	0.193	0.141	0.375	0.308	0.423	1.654
True	3.770	1.000	3.770	$\log 20$	15.081	15.658
Bias	0.231	0.098	−0.074	−0.060	−0.145	1.307
S. D.	0.191	0.136	0.459	0.393	0.511	1.592
True	4.239	1.000	4.239	$\log 30$	15.486	16.064
Bias	0.210	0.087	−0.086	−0.071	−0.147	1.161
S. D.	0.189	0.135	0.514	0.449	0.568	1.580

be discussed in a sequel paper.

A difficulty in Murakami's method is the assumption on the value of (α, γ). Among 18 datasets analyzed, the value $(\alpha, \gamma) = (1, 1/2)$ is sometimes doubtful, because of larger values of the estimate of $\tau = \log(\lambda_A A)$ which are due to the larger values of $\hat{\eta}/\hat{\xi}$. There are four cases such that $\hat{\tau} > 5$, and it is more reasonable to assume larger values of both α and γ for them.

To estimate the parameter (α, γ) we have to change the measurement as above mentioned. Since the development of more efficient prediction methods is firstly needed, we do not discuss here the effect of misspecification of (α, γ), but it is surely an important problem.

8. Supplements

8.1 *Engineering reports on the size distribution of inclusions in steel*

The following four papers report on the size distribution of nonmetallic inclusions in steel.

Asano *et al.* (1968): Measuring the inclusions directly, they observe that the weight distribution of inclusions is exponential in the clearer part of ingots. That is, F_V follows $\mathsf{Ga}(1, 2/3, \cdot)$.

Ishikawa and Fujimori (1985): Discussing the usage of ultrasonic detector, they show the area of inclusions on sectional plane to follow the exponential distribution, hence F_V to follow $\mathsf{Ga}(1, 1, \cdot)$.

Iwakura *et al.* (1988): They show a figure of the result of unfolding the distribution of diameters of sectional circles by Saltykov method. They don't mention what distribution fits to the figure, which looks like a p.d.f. of $\mathsf{Ga}(\alpha, \gamma, \cdot)$ with $\alpha\gamma > 1$ and $\gamma < 1$.

Chino *et al.* (1991): Measuring inclusions of specific substance, they fit log-normal to the distribution of diameter of spheres.

The QQ-plots in Murakami (1993) show the good fit of the Gumbel distribution for all of the datasets. The estimated values of $\lambda_A A$ are moderate in most cases. Hence, f_A can be assumed to belong to $\mathcal{D}(\Lambda)$, and not far away from $\mathsf{Ga}(1, 1/2, \cdot)$. This means, because of Proposition 3.1, f_V cannot have a longer or shorter tail.

The engineers believe that the 'shape parameter' (α, γ) may depend on the type of steel, and that for everyday control of the production process, the predicted value of the maximum size of inclusions is useful enough.

8.2 *Proofs of Theorem* 4.1 *and Corollary* 4.2

PROOF. (Theorem 4.1) We prove first the asymptotic expression (4.5) of f_A. From (2.5) and Proposition 4.1,

$$f_A(s) = \frac{\gamma}{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} \int_0^\infty \frac{1}{\sqrt{w}}(s+w)^{\alpha\gamma-1} e^{-(s+w)^\gamma} dw.$$

Put $u + s^\gamma = (s+w)^\gamma$ to find

$$\begin{aligned} f_A(s) &= \frac{e^{-s^\gamma}}{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} \int_0^\infty \frac{(u+s^\gamma)^{\alpha-1}}{\sqrt{(u+s^\gamma)^{1/\gamma} - s}} e^{-u} du \\ &= \frac{\sqrt{\gamma}}{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} s^{(\alpha-1/2)\gamma-1/2} e^{-s^\gamma} \int_0^\infty \frac{1}{\sqrt{u}} h(u,s) e^{-u} du, \end{aligned}$$

where

$$h(u,s) = \left(1+\frac{u}{s^\gamma}\right)^{\alpha-1}\left\{\frac{\frac{1}{\gamma}\frac{u}{s^\gamma}}{\left(1+\frac{u}{s^\gamma}\right)^{1/\gamma}-1}\right\}^{1/2} \to 1, \quad \text{as} \quad s\to\infty.$$

In general, for any $\beta > 0$, there exist β_1 and β_2 $(0<\beta_1<\beta_2)$ such that

$$1+\beta_1 z < (1+z)^\beta < 1+\beta_2 z, \qquad \text{for} \quad 0 < z \leq 2.$$

Hence, if $0 < u/s^\gamma \leq 2$, in the case $\alpha > 1$, there exists β_1 such that

$$h(u,s) \leq \frac{3^{\alpha-1}}{\sqrt{\gamma}}\left(\frac{u/s^\gamma}{\beta_1 u/s^\gamma}\right)^{1/2} = \frac{3^{\alpha-1}}{\sqrt{\beta_1\gamma}}.$$

We get also the same bound in the case $0 < \alpha \leq 1$. If $u/s^\gamma > 2$ and $s \gg 1$, there exist c and $n \in \mathbb{N}$ such that

$$h(u,s) \leq \frac{1}{\sqrt{\gamma}}\max\{(1+u)^{\alpha-1},1\}\frac{u^{1/2}}{(3^{1/\gamma}-1)^{1/2}} \leq cu^n.$$

Applying Lebesgue's dominated convergence theorem, we get

$$\int_0^\infty \frac{1}{\sqrt{u}}h(u,s)e^{-u}du \to \Gamma(1/2) = \sqrt{\pi}, \qquad \text{as} \quad s\to\infty.$$

Thus, the asymptotic expression of f_A is proved.

To prove the asymptotic expression (4.4) of $\bar{F}_A$, we find from (2.4),

$$\begin{aligned} 1-F_A(s) &= \frac{\gamma}{\Gamma\left(\alpha+\frac{1}{2\gamma}\right)}\int_0^\infty \sqrt{w}(s+w)^{\alpha\gamma-1}e^{-(s+w)^\gamma}dw \\ &= \frac{e^{-s^\gamma}}{\Gamma\left(\alpha+\frac{1}{2\gamma}\right)}\int_0^\infty ((u+s^\gamma)^{1/\gamma}-s)^{1/2}(u+s^\gamma)^{\alpha-1}e^{-u}du. \end{aligned}$$

We get the asymptotic expression corresponding to f_A by calculating upper bounds almost in the same way as above. □

PROOF. (Corollary 4.2) From Propositions 2.4 and the discussions at the beginning of Section 4, the distribution of T_V and T_C are $\mathsf{Ga}(\alpha,1,\xi^\gamma)$ and $\mathsf{Ga}(\alpha+\frac{1}{2\gamma},1,\xi^\gamma)$, respectively. The p.d.f. ϕ_A of T_A/ξ^γ and its asymptotic expression is given, from Proposition 2.4 and the Theorem 4.1, by

$$\text{(8.1)} \qquad \begin{aligned} \phi_A(t) &= \frac{t^{1/\gamma-1}}{2\mu_0\Gamma(\alpha)}\int_0^\infty \frac{1}{\sqrt{w}}(t^{1/\gamma}+w)^{\alpha\gamma-1}\exp(-(t^{1/\gamma}+w)^\gamma)dw \\ &\sim \frac{\sqrt{\pi/\gamma}}{2\,\Gamma\left(\alpha+\frac{1}{2\gamma}\right)}t^{\alpha-3/2+1/2\gamma}e^{-t}, \qquad t\to\infty, \end{aligned}$$

where

$$\mu_0 = E(T_V^{1/2\gamma}) = E(S_V^{1/2}) = \sqrt{\xi}\,\Gamma\left(\alpha + \frac{1}{2\gamma}\right) \Big/ \Gamma(\alpha). \tag{8.2}$$

The d.f. Φ_A of T_A/ξ^γ has the same asymptotic expression

$$1 - \Phi_A(t) \sim \frac{\sqrt{\pi/\gamma}}{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)} t^{\alpha-3/2+1/2\gamma} e^{-t}, \qquad t \to \infty. \tag{8.3}$$

The asymptotic expressions are obtained by comparing the integral form of ϕ_A or Φ_A with those of f_A or F_A, and by transforming the variables in (4.4) and (4.5). $\square$

8.3 *Power transformation for the Gumbel approximation*

A referee of this paper suggested that the observed improvement of the Gumbel approximation of W_V and W_A, in the beginning of Section 6, might be justified by theory on rate of convergence to the Gumbel distribution.

The discussions in Cohen (1982), Gomes (1984) and Resnick (1988) show that, for the d.f. F_V of $\mathsf{Ga}(\alpha, \gamma, \cdot)$, $\sup_{-\infty<x<\infty} |F_V^n(a_n x + b_n) - \Lambda(x)|$ is $O((\log n)^{-1})$ if $\gamma \neq 1$, $O((\log n)^{-2})$ if $\gamma = 1$, $\alpha \neq 1$ and $O(n^{-1})$ if $\alpha = \gamma = 1$.

However, this rate of convergence is achieved only if the attraction coefficients are 'optimal' in the sense of Theorem 4 in Cohen (1982). In case $\gamma = 1$, $\alpha \neq 1$, for example, this means that the attraction coefficients (a_n, b_n) should be specified up to $O((\log n)^{-2})$. Our attraction coefficients (4.6)–(4.13) are not optimal. The optimal ones are too complicated to be utilized for prediction, and the authors are not successful in finding the explicit form of the optimal ones for the Wicksell transform F_A.

8.4 *Murakami's original prediction method*

By metallurgic research, experiments and simulations, Murakami and his coauthors chose $(\alpha, \gamma) = (1, 1/2)$. That is, they assumed the distribution of $\sqrt{S_V}$ to be exponential. In our version this condition is generalized, although we still assume (α, γ) to be known.

At some points, the original version is different from our new version. Let $(X_n)_{n=1}^\infty$ be a sequence of i.i.d. random variables with the d.f. H. The expected number T of observations before the first occurrence of $X_n > x_0$ is $T = 1/(1 - H(x_0))$, which is called 'return period' in some applications. Conversely, $p = H(x_0) = 1 - 1/T$. For the standard Gumbel distribution, the quantile x_0 for the return period is $-\log(-\log(1 - 1/T)) \doteqdot \log T$, and for $\Lambda((x - \eta)/\xi)$ the quantile is $\eta + \xi \log T$.

Murakami and his coauthors anticipated that the difference of the locations of W_V, W_C and W_A are mainly due to the difference of the area or volume where we consider the maximum, and considered the magnifying factor, e.g. A_C/A as a return period. That is, the observation of the maximums are repeated $T = A_C/A$ times.

Table 3. Bias of Murakami's original predictor, given the value of $\hat{\eta}/\hat{\xi}$, for $\alpha = 1$, $\gamma = 1/2$ and some values of $\exp(\hat{\tau})$.

	$\hat{\eta}/\hat{\xi}$						
$\exp(\hat{\tau})$	2	2.5	3	3.5	4	4.5	5
10	0.301	0.238	0.189	0.150	0.118	0.094	0.074
20	0.349	0.284	0.232	0.189	0.154	0.126	0.102
50	0.398	0.333	0.278	0.233	0.195	0.163	0.136

In the case (C), their prediction value of W_C^γ is the quantile

$$\begin{aligned}\hat{\eta} + \hat{\xi}\log T &= \hat{\xi}(\hat{\eta}/\hat{\xi} + \tau_C - \tau) \\ &= \widehat{\eta_C} - \hat{\xi}\{(\widehat{\eta_C}/\hat{\xi} - \tau_C) - (\hat{\eta}/\hat{\xi} - \tau)\}.\end{aligned}$$

See (5.3) and (5.4) to find the second term to be much smaller than the first. Hence, Murakami's original method is close to the modified version in this case.

In the case (V), they adopted moreover the magnifying factor $V/(AE(W_A^\gamma))$ instead of $\sqrt{\pi}V/(2AE(\sqrt{S_V}))$ based on an intuitive reasoning. Hence, their prediction value of W_V^γ is the quantile

$$\hat{\eta} + \hat{\xi}\log T = \hat{\xi}(\hat{\eta}/\hat{\xi} + \log(V/A) - \log(\hat{\eta} + \hat{\xi}\gamma_E)). \tag{8.4}$$

Compare it with our estimate $\widehat{\eta_V} + \omega_p\hat{\xi}$, and find the value p of ω_p, corresponding to their prediction value. For simplicity, we discuss the case $\alpha = 1$, which was assumed by Murakami and others. From (5.9), (5.10) and (8.4),

$$\hat{\eta}/\hat{\xi} + \log(V/A) - \log(\hat{\eta} + \hat{\xi}\gamma_E) = \omega_p + \widehat{\tau_V} + \log(V/A) - \hat{\delta},$$

or

$$\omega_p = \left(\frac{1}{2\gamma} - \frac{1}{2}\right)\log\hat{\tau} - \log\sqrt{\gamma} + \frac{1}{2\gamma}\log\hat{\xi} - \log(\hat{\eta} + \hat{\xi}\gamma_E).$$

Since $\omega_p = -\log(-\log p)$,

$$p = \exp\{-\sqrt{\gamma}(\hat{\eta}/\hat{\xi} + \gamma_E)\hat{\xi}^{-1/2\gamma+1}\hat{\tau}^{-1/2\gamma+1/2}\}. \tag{8.5}$$

Some numerical values of p, with the estimates $\hat{\eta}$, $\hat{\xi}$ and $\hat{\tau}$ replaced by typical values are shown in Table 3. These values show that the original predictor is biased to lower, or risky, side, although these values support somehow the intuitive arguments by Murakami and others.

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