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BARTLETT'S FORMULAE—CLOSED FORMS AND RECURRENT EQUATIONS*

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Abstract. We show that the entries of the asymptotic covariance matrix of the sample autocovariances and autocorrelations of a stationary process can be expressed in terms of the square of its spectral density. This leads to closed form expressions and fast computational algorithms.

Key words and phrases: Bartlett's formula, ARMA, sample autocovariances, sample autocorrelations.

1. Introduction

Let $\{X_t\}$ be a stationary process with mean μ, autocovariance function $R_k = E(X_t-\mu)(X_{t-k}-\mu)$, autocorrelation function $r_k = R_k/R_0$, and spectral density $f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_k \cos(wk)$. The sample autocovariances $\hat{R}_k$, $k = 0, 1, \dots,$ and the sample autocorrelations $\hat{r}_k$, $k = 1, 2, \dots,$ from a stretch $(X_1, \dots, X_N)$ from $\{X_t\}$ of length N are defined as

$$\hat{R}_k = C_{N,k}\sum_{i=k+1}^{N}(X_i - m)(X_{i-k} - m), \qquad \hat{r}_k = \hat{R}_k/\hat{R}_0,$$

where $C_{N,k}$ is usually equal to $1/N$ or $1/(N-k)$, while m is equal to the mean of the process or to the sample mean according to whether the mean is known or unknown.

Because of the important role of the sample autocovariances and autocorrelations in time series modelling their statistical properties are subject to much research. One way to describe such properties is through the asymptotic covariances of $\hat{R}_k$ and $\hat{r}_k$, defined as

$$\Gamma_{k,l} = \lim_{N\to\infty} N\operatorname{Cov}(\hat{R}_k, \hat{R}_l),$$
$$\gamma_{k,l} = \lim_{N\to\infty} N\operatorname{Cov}(\hat{r}_k, \hat{r}_l).$$

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Under suitable conditions these limits exist and are given by Bartlett's formulae (Bartlett (1955), Anderson (1971))

$$\Gamma_{l,k} = \sum_{i=-\infty}^{\infty} (R_{i+l}R_{i+k} + R_{i+l}R_{i-k}) + A_{\kappa}(k,l), \tag{1.1}$$

$$\gamma_{k,l} = \sum_{i=-\infty}^{\infty} (r_{i+l}r_{i+k} + r_{i+l}r_{i-k} - 2r_l r_i r_{i+k} - 2r_k r_i r_{i+l} + 2r_k r_l r_i^2), \tag{1.2}$$

where $A_{\kappa}(k,l)$ depend on the fourth order cumulants and $A_{\kappa}(k,l) = 0$ when the process $\{X_t\}$ is Gaussian. Note that the formula for the autocorrelations does not involve higher order characteristics of the process.

Theorems for joint asymptotic normality of any finite number n of sample autocovariances $\hat{R}_k$, $k = 0, 1, \ldots, n$ or sample autocorrelations $\hat{r}_k$, $k = 1, 2, \ldots, n$ are also available (see Anderson (1971)). The entries of the covariance matrices of the limiting distributions are given by $\Gamma_{l,k}$ and $\gamma_{l,k}$ respectively.

The infinite sums in these formulae make them not sufficiently convenient for "exact" computations. It is reasonable to expect that for some important classes of models finite algorithms should exist. This is indeed the case. Bruzzone and Kaveh (1984) obtained closed form formulae for $\Gamma_{k,l}$ in the ARMA case under some restrictions on the roots of the ARMA polynomials (they should be complex and simple). Their solution is in terms of the roots of the ARMA polynomials. It is useful in simulation and in some theoretical considerations, but its value as computational tool is limited not only because of the restrictions on the roots, but because usually the coefficients of the polynomials are available, not their roots.

Recently computationally feasible expressions and recurrence relations for the pure autoregression have been obtained by Cavazos-Cadena (1994).

A general solution to this problem has been announced in Boshnakov (1989). The solution given there covers completely the ARMA case without any restrictions on the autoregressive and moving average polynomials. Conditions on the distribution of the innovation process are necessary only to ensure the validity of Bartlett's formulae. The aim of this paper is to represent in some length this solution. Namely, we will show that

$$\Gamma_{k,l} = R_g(l-k) + R_g(l+k) + A_{\kappa}(k,l), \tag{1.3}$$

and

$$\gamma_{k,l} = \frac{1}{R_0^2}[R_g(l-k) + R_g(l+k) - 2R_g(k)r_l - 2R_g(l)r_k + 2r_k r_l R_g(0)], \tag{1.4}$$

where $R_g(k)$ is the autocovariance function corresponding to the spectral density $g(\omega) = 2\pi f^2(\omega)$.

This result reduces the computation of the asymptotic covariances of the sample autocovariances and sample autocorrelations to the computation of the autocovariance sequence $R_g(k)$.

Conditions when these results hold are discussed in Section 3. From computational point of view the most important case is when $\{X_t\}$ is an ARMA process for which we have the following corollary.

COROLLARY 1.1. *Let $\{X_t\}$ be an $ARMA(p,q)$ process,*

$$\phi(B)X_t = \theta(B)\varepsilon_t,$$

where ε_t is white noise, the polynomials $\phi(z)$ and $\theta(z)$ have no common factors and $\phi(z)$ has no roots with $|z| = 1$. Then, if (1.1) (respectively (1.2)) holds then (1.3) (respectively (1.4)) holds with $R_g(k)$ being the autocovariance sequence of an $ARMA(2p,2q)$ process

$$\phi^2(B)Y_t = \theta^2(B)a_t,$$

where the variances of the white noises obey the condition $\sigma_\varepsilon^4 = \sigma_a^2$.

PROOF. It is well known that the spectral density $f_x(\omega)$ of the process X is given by the formula (see, for example, Brockwell and Davis ((1991), Theorem 4.4.2))

$$f_x(\omega) = \frac{\sigma_\epsilon^2}{2\pi}\left|\frac{\theta(e^{-i\omega})}{\phi(e^{-i\omega})}\right|^2.$$

Hence,

$$2\pi f_x^2(\omega) = \frac{2\pi\sigma_\epsilon^4}{4\pi^2}\left|\frac{\theta(e^{-i\omega})}{\phi(e^{-i\omega})}\right|^4 = \frac{\sigma_\epsilon^4}{2\pi}\left|\frac{\theta^2(e^{-i\omega})}{\phi^2(e^{-i\omega})}\right|^2 = f_y(\omega). \qquad \square$$

Various efficient algorithms for the computation of the autocovariance sequence of an ARMA process exist, e.g. Wilson (1979), Kay (1985). They can be used for the computation of $R_g(k)$, and therefore of $\Gamma_{k,l}$ and $\gamma_{k,l}$.

It is important to note that only the probabilistic structure of the white noise sequence of the ARMA model may preclude the validity of Bartlett's formulae and the above formulae. This is so because the coefficients in the infinite moving average representations of the ARMA models decrease sufficiently fast to ensure the validity of the conditions on them in all known results concerning Bartlett's formulae (see Anderson (1971) and Section 3 below).

Furthermore, causality conditions on the model are not necessary. This is of some importance in the non-Gaussian case since then the innovations sequences of the different representations of the ARMA model have different probabilistic properties. For example, if an ARMA process is non-Gaussian and the "forward" residuals are independent identically distributed, then the "backward" ones are only uncorrelated. Hence, the conditions for the validity of Bartlett's formulae may turn out to be fulfilled for some of the ARMA representations of a process, and not for others.

Bartlett's formulae for the sample autocorrelations and sample autocovariances look similar but there exist important differences. The conditions under which the former hold are weaker than these for the latter. Moreover, the formulae for the autocovariances involve fourth-order cumulants, except for the Gaussian case when these are zero. The asymptotic normality is easier for the sample autocorrelations as well. Detailed presentation of these and related issues can be found in Anderson (1971).

2. Closed form of Bartlett's formulae

Since the Fourier transform of a convolution is simply the product of the Fourier transforms of its arguments (Fuller (1976), Corollary 3.4.1.1) and the autocovariance function is an even function, we have the following lemma.

LEMMA 2.1. *Suppose that* $\sum_{i=-\infty}^{\infty}|R_i| < \infty$. *Then*

$$\sum_{i=-\infty}^{\infty} R_{i+l}R_{i+k} = 2\pi\int_{-\pi}^{\pi}\cos(\omega(k-l))f^2(\omega)d\omega. \tag{2.1}$$

We use this lemma in our proofs. They could be equally well based on the integral representations, given in Anderson (1971). For absolutely summable autocovariance functions both approaches are essentially the same.

THEOREM 2.1. *Suppose that formulae* (1.2) *hold and that* $\sum_{i=-\infty}^{\infty}|R_i| < \infty$. *Then formulae* (1.4) *hold.*

PROOF. Multiplying and dividing the righthand side of (1.2) by R_0^2, substituting (2.1) into (1.2), and bearing in mind that $r_k = R_k/R_0$, we obtain

$$\begin{aligned}
\gamma_{k,l} &= \sum_{i=-\infty}^{\infty}(r_{i+l}r_{i+k} + r_{i+l}r_{i-k} - 2r_lr_ir_{i+k} - 2r_kr_ir_{i+l} + 2r_kr_lr_i^2)\\
&= \frac{1}{R_0^2}\sum_{i=-\infty}^{\infty}(R_{i+l}R_{i+k} + R_{i+l}R_{i-k} - 2r_lR_iR_{i+k} - 2r_kR_iR_{i+l} + 2r_kr_lR_i^2)\\
&= \frac{2\pi}{R_0^2}\Bigg[\int_{-\pi}^{\pi}\cos(\omega(k-l))f^2(\omega)d\omega + \int_{-\pi}^{\pi}\cos(\omega(k+l))f^2(\omega)d\omega\\
&\qquad - 2r_l\int_{-\pi}^{\pi}\cos(\omega k)f^2(\omega)d\omega - 2r_k\int_{-\pi}^{\pi}\cos(\omega l)f^2(\omega)d\omega\\
&\qquad + 2r_kr_l\int_{-\pi}^{\pi}f^2(\omega)d\omega\Bigg]\\
&= \frac{1}{R_0^2}[R_g(l-k) + R_g(l+k) - 2R_g(k)r_l - 2R_g(l)r_k + 2r_kr_lR_g(0)]. \qquad \square
\end{aligned}$$

Similar arguments lead to the corresponding result for the autocovariances.

THEOREM 2.2. *Suppose that formulae* (1.1) *hold and that* $\sum_{i=-\infty}^{\infty}|R_i| < \infty$. *Then formulae* (1.3) *hold.*

A closer look at equation (1.4) reveals that $\gamma_{k,l}$ can be written in terms of $\Gamma_{i,j}$ as (assuming $A_\kappa(k,l) = 0$)

$$\gamma_{k,l} = \frac{1}{R_0^2}(\Gamma_{k,l} - r_l\Gamma_{k,0} - r_k\Gamma_{l,0} + r_kr_l\Gamma_{0,0}), \tag{2.2}$$

since $2R_k = R_{k-0} + R_{k+0}$.

The equation (2.2) can be obtained also directly from equation (1.1). The function $g(x_0, x_k, x_l)$ defined as

$$g(x_0, x_k, x_l) = \left(\frac{x_k}{x_0}, \frac{x_l}{x_0}\right)',$$

transforms $(R_0, R_k, R_l)'$ into $(r_k, r_l)'$. The matrix D of its first derivatives at (R_0, R_k, R_l) is given by

$$\begin{aligned} D \equiv \frac{\partial g}{\partial x}\bigg|_{(R_0,R_k,R_l)} &= \begin{pmatrix} -x_k/x_0^2 & 1/x_0 & 0 \\ -x_l/x_0^2 & 0 & 1/x_0 \end{pmatrix}\bigg|_{(R_0,R_k,R_l)} \\ &= \frac{1}{R_0}\begin{pmatrix} -r_k & 1 & 0 \\ -r_l & 0 & 1 \end{pmatrix}. \end{aligned}$$

Assuming that the sample autocovariances are asymptotically normal, it can be verified easily that the conditions of Brockwell and Davis ((1991), Proposition 6.4.3) are fulfilled. Therefore, the sample autocorrelations are also asymptotically normal with asymptotic covariance matrix equal to $D\Sigma D'$, where

$$\Sigma = \begin{pmatrix} \Gamma_{0,0} & \Gamma_{0,k} & \Gamma_{0,l} \\ \Gamma_{k,0} & \Gamma_{k,k} & \Gamma_{k,l} \\ \Gamma_{l,0} & \Gamma_{l,k} & \Gamma_{l,l} \end{pmatrix}.$$

Direct calculations show that

$$D\Sigma D' = \frac{1}{R_0^2}\begin{pmatrix} r_k^2\Gamma_{0,0} - 2r_k\Gamma_{0,k} + \Gamma_{k,k} & \cdots \\ r_kr_l\Gamma_{0,0} - r_k\Gamma_{0,l} - r_l\Gamma_{0,k} + \Gamma_{l,k} & r_l^2\Gamma_{0,0} - 2r_l\Gamma_{0,l} + \Gamma_{l,l} \end{pmatrix}$$

which, as expected, coincides with (2.2).

This derivation shows also that $\gamma_{k,l}$ does not depend on higher order cumulants if and only if

$$r_kr_lA_\kappa(0,0) - r_kA_\kappa(0,l) - r_lA_\kappa(0,k) + A_\kappa(l,k) = 0.$$

3. Some sufficient conditions

The sample autocorrelations have "better" asymptotic behaviour than the sample autocovariances—higher order cumulants do not enter Bartlett's formulae; when the sample autocovariances are asymptotically normal, so are the sample autocorrelations; asymptotic normality has been proved without any conditions on the higher order moments (a result which is due to Anderson and Walker (1964), see also Anderson ((1971), Theorem 8.4.6)).

In this section we give some sufficient conditions under which formulae (1.3) and (1.4) hold. We state the conditions as in Anderson (1971).

DEFINITION 1. A process $\{X_t\}$ is said to be linear process if it admits a representation as

$$X_t = \sum_{i=-\infty}^{\infty} h_i \varepsilon_{t-i}, \tag{3.1}$$

where $\sum_{i=-\infty}^{\infty} |h_i| < \infty$ and the process $\{\varepsilon_t\}$ is such that $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2 < \infty$, $E\varepsilon_t\varepsilon_s = 0$ when $t \neq s$.

To say it another way, $\{X_t\}$ is a linear process if there exist white noise $\{\varepsilon_t\}$ and absolutely summable sequence of constants $\{h_i\}$ such that equation (3.1) holds.

THEOREM 3.1. *Let the process $\{X_t\}$ be linear with representation* (3.1), *where*

(i) $\{\varepsilon_t\}$ *is a sequence of independent identically distributed random variables, and*

(ii) $\sum_{i=-\infty}^{\infty} |i| h_i^2 < \infty$.

Then formulae (1.4) *hold and the joint distribution of any fixed number of sample autocorrelations is asymptotically normal with elements of the asymptotic covariance matrix given by* (1.4).

PROOF. The validity of Bartlett's formulae and the asymptotic normality follow from Anderson ((1971), Equation (47), Theorem 8.4.6). Then, by Theorem 2.2, the formulae (1.4) also hold. □

The fourth-order cumulants of $\{X_t\}$ are denoted below by $\kappa(k, l, m)$ (Anderson (1971), §8.2.2, Equations (34)–(36)).

THEOREM 3.2. *If* $\left|\sum_{i=-\infty}^{\infty} \kappa(k, -i, l-i)\right| < \infty$ *and the spectral density* $f(w)$ *of the process* $\{X_t\}$ *is continuous, then*

$$\Gamma_{k,l} = R_g(l-k) + R_g(l+k) + \sum_{i=-\infty}^{\infty} \kappa(k, -i, l-i), \tag{3.2}$$

where $R_g(k)$ *is the autocovariance function corresponding to* $g(\omega) = 2\pi f^2(\omega)$.

PROOF. Under the imposed conditions we have from the first part of Theorem 8.3.3 in Anderson (1971) that

$$\Gamma_{k,l} = 4\pi \int_{-\pi}^{\pi} \cos(\omega k)\cos(\omega l) f^2(\omega) d\omega + \sum_{i=-\infty}^{\infty} \kappa(k, -i, l-i). \tag{3.3}$$

As any continuous function on $[-\pi, \pi]$ is square integrable we can split the integral into two integrals, using the formula for the product of cosines,

$$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

to get the desired result. □

Note that the spectral density of a process with absolutely convergent autocovariance function is continuous, while the converse is not true. Difficulties may arise in the reconstruction of a convolution by inverting the product of the Fourier transforms of its arguments, when the arguments are not absolutely convergent. This explains why we do not use the second part of the Anderson's theorem which establishes Bartlett's formulae (1.1) under the weaker condition that the squared autocorrelations form a convergent series.

For linear processes the infinite sum in (3.3) simplifies to a single term, under some distributional assumptions about the innovation process, as described in the following corollary.

COROLLARY 3.1. *Let the process* $\{X_t\}$ *be linear with representation* (3.1), *where*

(i) $E\varepsilon_t\varepsilon_s\varepsilon_r\varepsilon_q = 0$, *when* $t \neq s$ *and* $t \neq r$ *and* $t \neq q$,
(ii) $E\varepsilon_t^4 < \infty$, $\kappa_4 = E\varepsilon_t^4 - 3\sigma^4$,
(iii) $E\varepsilon_t^2\varepsilon_s^2 = \sigma^4$, *when* $t \neq s$.

Then

$$\Gamma_{k,l} = R_g(l-k) + R_g(l+k) + \frac{\kappa_4}{\sigma^4} R_k R_l. \tag{3.4}$$

PROOF. The result follows from Theorem 2.1 and from Anderson ((1971), Corollary 8.3.1). □

If the innovations sequence is strictly stationary then asymptotic normality can be obtained.

COROLLARY 3.2. *Let the process* $\{X_t\}$ *be linear with representation* (3.1), *where*

(i) $\{\varepsilon_t\}$ *is a sequence of independent identically distributed random variables*,
(ii) $E\varepsilon_t^4 < \infty$, $\kappa_4 = E\varepsilon_t^4 - 3\sigma^4$.

Then for any fixed n the vector $(\hat{R}_0, \dots, \hat{R}_n)'$ is asymptotically normal with elements of the asymptotic covariance matrix given by equation (3.4).

PROOF. The result follows from Theorem 2.1 and from Anderson ((1971), Theorem 8.4.2). □

We have given the main result (see Corollary 1.1) in the Introduction. In view of the above results to prove it it remains to note that the infinite moving average representation $X_t = \psi(B)\varepsilon_t$ of the process $\phi(B)X_t = \theta(B)\varepsilon_t$ exists and its coefficients form an absolutely convergent series (recall that $\phi(z) \neq 0$ when $|z| \neq 1$).

The following results show that $\Gamma_{k,l}$ and $\gamma_{k,l}$ satisfy difference equations, which can be used for further simplification of the computations.

COROLLARY 3.3. *Suppose that*

$$\Gamma_{k,l} = R_g(l-k) + R_g(l+k) \tag{3.5}$$

and let $k > 0$, $l > 0$, $l - k > \max(2q, 2p)$. *Then*

$$\phi^2(B_l)\Gamma_{k,l} = 0,$$

where the shift operator B_l operates on l, i.e. $B_l\Gamma_{k,l} = \Gamma_{k,l-1}$.

Note nonetheless that when $R_g(l-k)$ and $R_g(l+k)$ are already available there is no need to use recurrences. More valuable appears to be the corresponding result for the autocorrelations.

COROLLARY 3.4. *Suppose that* (1.4) *holds. Then,*
(i) *if* $l \geq q+1$ *then*

$$\phi(B_l)\gamma_{k,l} = \frac{1}{R_0^2}[\phi(B_l)\Gamma_{k,l} - r_k\phi(B_l)\Gamma_{l,0}], \tag{3.6}$$

(ii) *if* $l \geq \max(2q+1, 2p+1)$ *then*

$$\phi^2(B_l)\gamma_{k,l} = 0.$$

PROOF. From (2.2) it follows that

$$\gamma_{k,l} = \frac{1}{R_0^2}(\Gamma_{k,l} - \Gamma_{k,0}r_l - \Gamma_{l,0}r_k + \Gamma_{0,0}r_k r_l).$$

Applying the operator B_l to the both sides of this equality, we obtain

$$\phi(B_l)\gamma_{k,l} = \frac{1}{R_0^2}(\phi(B_l)\Gamma_{k,l} - \Gamma_{k,0}\phi(B_l)r_l - r_k\phi(B_l)\Gamma_{l,0} + \Gamma_{0,0}r_k\phi(B_l)r_l),$$

which proves (i) since $\phi(B_l)r_l = 0$ when $l \geq q+1$. Applying the operator $\phi(B_l)$ to equation (3.6) and using the previous corollary we obtain (ii). □

We end this section with a generalization of the Bruzzone and Kaveh's result (see Bruzzone and Kaveh (1984)). Although Corollary 1.1 shows that $R_g(k)$ can be obtained as the solution of the difference equation $\phi^2(B)R_g(k) = 0$, for $k \geq 2q+1$, subject to the initial conditions given by the even property of $R_g(k)$, we state the result in the form obtained in Bruzzone and Kaveh (1984).

COROLLARY 3.5. *Suppose that $\phi(z)$ can be written in the form*

$$\phi(z) = \prod_{i=1}^{p}(1 - P_i z^{-1}),$$

where P_i are distinct and formulae (1.1) *hold. Then,*

(i) *$R_g(j)$ is given by the following formulae*

- *for $j = 0$*

$$2\left[\sum_{i=0}^{q-1} R_i^2 + \sum_{r=1}^{p}\sum_{s=1}^{p} \frac{v_r v_s}{1 - P_r P_s}\right] - R_0^2$$

- *for j odd*

$$2\left[\sum_{i=0}^{q-1} R_i R_{i+j} + \sum_{i=1}^{(j-1)/2} R_i R_{j-i} + \sum_{r=1}^{p}\sum_{s=1}^{p} \frac{v_r v_s P_s^j}{1 - P_r P_s}\right]$$

- *for j even*

$$2\left[\sum_{i=0}^{q-1} R_i R_{i+j} + \sum_{i=1}^{j/2-1} R_i R_{j-i} + \sum_{r=1}^{p}\sum_{s=1}^{p} \frac{v_r v_s P_s^j}{1 - P_r P_s}\right] + R_{j/2}^2,$$

where v_j, $j = 1, \ldots, p$ are the solution of the system

$$R_k = \sum_{i=1}^{p} v_i P_i^{k-q}, \qquad k = q, q+1, \ldots, q+p-1.$$

(ii) *$\phi^2(B)R_g(k) = 0$ when $k \geq 2q+1$.*

PROOF. The first part of the corollary has been proved by Bruzzone and Kaveh (1984) under the additional assumptions that the roots of $\phi(z)$ are complex, and those of $\theta(z)$ are complex and distinct. It can be seen that their proof can be carried out without these additional assumptions as well. The second part of the corollary follows from the previous results. □

4. An example

Let $\{X_t\}$ be an autoregression of order 1, i.e.

$$(1 - \phi B)X_t = \varepsilon_t.$$

The zero lag autocovariance of $\{X_t\}$ in this case is $R_0 = \sigma_\varepsilon^2/(1-\phi^2)$. The autocorrelation function is given by $r_k = \phi^k$. The function $R_g(k)$ is the autocovariance function of the AR(2) process

$$(1 - \phi B)^2 Y_t = a_t,$$

with $\sigma_a^2 = \sigma_\varepsilon^4$. Solving the Yule-Walker system

$$\begin{aligned}
&R_g(2) - 2\phi R_g(1) + \phi^2 R_g(0) = 0\\
&(1+\phi^2)R_g(1) - 2\phi R_g(0) = 0\\
&\phi^2 R_g(2) - 2\phi R_g(1) + R_g(0) = \sigma_a^2,
\end{aligned}$$

we obtain

$$R_g(0) = \frac{1+\phi^2}{(1-\phi^2)^3}\sigma_a^2, \qquad R_g(1) = \frac{2\phi}{(1-\phi^2)^3}\sigma_a^2, \qquad R_g(2) = \frac{3\phi^2 - \phi^4}{(1-\phi^2)^3}\sigma_a^2.$$

Now, equation (1.4) shows that, for example, the variance of r_1 is

$$\begin{aligned}
\gamma_{1,1} &= \frac{1}{R_0^2}((1+2r_1^2)R_g(0) + R_g(2) - 4r_1 R_g(1))\\
&= \frac{(1+2\phi^2)(1+\phi^2) + \phi^2(3-\phi^2) - 8\phi^2}{1-\phi^2}\left(\frac{\sigma_a^2}{\sigma_\varepsilon^4}\right)\\
&= \frac{\phi^4 - 2\phi^2 + 1}{1-\phi^2}\\
&= 1 - \phi^2,
\end{aligned}$$

which is a well known result.

5. Conclusion

We have shown that the infinite sums in Bartlett's formulae, under quite general conditions, can be written in closed form in terms of the autocovariance sequence of a model, closely related to the model of the process under consideration. In the ARMA case this reduces to the computation of the autocovariances of the "squared" model, which is also an ARMA model. Efficient algorithms exist for this task. We also presented a closed form expression which may be useful occasionally. Conditions under which Bartlett's formulae can be written in our form have been given as well.

The recurrent expressions of this paper can be used for efficient computation of the asymptotic covariance matrix of the sample autocovariances and autocorrelations.

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