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ESTIMATION OF SECOND-ORDER PROPERTIES FROM JITTERED TIME SERIES

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Abstract. This paper considers spectral and autocovariance estimation for a zero-mean, band-limited, stationary process that has been sampled at time points jittered from a regular, equi-interval, sampling scheme. The case of interest is where the sampling scheme is near regular so that the jitter standard deviation is small compared to the sampling interval. Such situations occur with many time series collected in the physical sciences including, in particular, oceanographic profiles.

Spectral estimation procedures are developed for the case of independent jitter and autocovariance estimation procedures for both independent and dependent jitter. These are typically modifications of general estimation procedures proposed elsewhere, but tailored to the particular jittered sampling scheme considered. The theoretical properties of these estimators are developed and their relative efficiencies compared.

The properties of the jittered sampling point process are also developed. These lead to a better understanding, in this situation, of more general techniques available for processes sampled by stationary point processes.

Key words and phrases: Jittered sampling, stationary processes, spectral estimation, autocovariance estimation, kernel density estimation.

1. Introduction

Consider a zero-mean, continuous-time, covariance stationary process $X(t)$ that has been sampled at times T_n perturbed from a regular equi-interval time scale or equi-spaced grid with interval Δ. We assume that $X(t)$ has autocovariance function $\gamma(t) = E\{X(s)X(s+t)\}$ which is continuous and absolutely integrable so that $X(t)$ has bounded spectral density

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\gamma(t)dt. \tag{1.1}$$

To avoid aliasing effects we further assume that $f(\omega)$ is band-limited with $f(\omega)$ zero outside $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$. Furthermore, the jittered sampling times T_n are assumed to satisfy

$$T_n = \alpha + n\Delta + \epsilon_n \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{1.2}$$

where the "jitters" ϵ_n form a zero-mean, strictly stationary process with finite variance that is independent of the $X(t)$ process. Of interest is the case where the sampling scheme is near regular so that σ, the standard deviation of ϵ_n, is small by comparison to Δ. This paper focusses on ways of estimating $\gamma(t)$ and $f(\omega)$ from the jittered time series $Y_n = X(T_n)$.

The assumptions given in the previous paragraph will be referred to as Assumptions 1. The requirement that $X(t)$ have zero mean has been chosen for expositional simplicity. We believe that, under the conditions of this paper, mean correction by the sample mean will not alter the asymptotic results given in the following sections. Note that α is a fixed quantity which allows for an arbitrary initial reference point for the sampling scheme and the aliasing assumption, although highly restrictive, will be approximately true in practice provided Δ is sufficiently small.

For many time series, especially those collected in the physical sciences, jittered sampling schemes can arise in many ways. For example, they can arise as a result of noise in the clock signal used to time the samples, as a result of perturbations in the medium through which remote sensing (radar or sonar) pulses are propagated, or through unexpected or unavoidable residual motion of a sampling probe about its desired path. The jitters may be independent or correlated, depending on the situation.

Jitter models of this sort were first considered in Akaike (1960) and also independently by Balakrishnan (1962). A discussion of the effects of jitter on spectra in practice and in theory is given in Moore *et al.* (1988) and Moore and Thomson (1991) respectively. In the latter it is shown that the effect of jitter is to redistribute spectral mass without destroying it. Even in the case where the jitter standard deviation is relatively small, significant spectral damage can result, especially at the higher frequencies.

If the sampling process is free of jitter so that $\sigma = 0$ and $T_n = \alpha + n\Delta$, then $\gamma(n\Delta)$ and $f(\omega)$ can be estimated in the conventional manner. The general case where $\sigma \neq 0$ is considerably more complicated and an analysis of irregularly observed data must be undertaken. See Parzen (1983) for a useful collection of papers on the latter topic. Also note the distinction between our problem, entailing a continuous time model, and that of a discrete time model with missing observations (see Akaike and Ishiguro (1980), Jones (1971) and Robinson (1984) for example).

General methods due originally to Brillinger (1972) can be used here (see Moore *et al.* (1988) for a practical application). These have been further developed in Brillinger (1983) and in a series of papers by Masry. See, in particular, Masry (1983*a*, 1983*b*), Lii and Masry (1992) and the references contained therein. These general methods consider the recovery and estimation of the second-order properties of a continuous-time stationary process $X(t)$ that has been sampled at times generated by a stationary point process. However, as we show later, the point process specified by T_n can have lines in its spectrum at frequency $\frac{2\pi}{\Delta}$ and its harmonics so that spectral methods must be used with care. Moreover, in terms of the definition of alias-free spectra proposed by Masry (1978), T_n need not necessarily be alias-free. Indeed, for σ small, as is the situation here, the jittered sampling process T_n will typically not be alias-free in practice.

These circumstances have led us to propose estimators which are variants of the general Brillinger and Masry estimation procedures more closely tailored to the band-limited processes and jittered sampling scheme considered here. In Section 2 we consider spectral estimation in the presence of independent jitter and, in Section 4, the much more difficult case of dependent jitter. Section 3 develops the second-order properties of the jittered sampling process T_n. These properties are needed for Section 4, but are also of independent interest (see Lewis (1961) and Lawrance (1972) for example).

2. Spectral estimation for independent jitter

When the jitter ϵ_n is a sequence of independent random variables, $Y_n = X(T_n)$ has autocovariance function

$$\gamma_Y(n) = \begin{cases} \gamma(0) & (n=0) \\ \displaystyle\int_{-\infty}^{\infty} \gamma(x+n\Delta) g^{(2)}(x)dx & (n = \pm1, \pm2, \ldots). \end{cases} \tag{2.1}$$

Here

$$g^{(2)}(x) = \int_{-\infty}^{\infty} g(y)g(x+y)dy \tag{2.2}$$

is the probability density function of $\epsilon_{m+n} - \epsilon_m$ for all $n \neq 0$, and it has been assumed that ϵ_n is absolutely continuous with probability density function $g(x)$. Replacing $\gamma(t)$ by its spectral representation in (2.1) we have

$$\gamma_Y(n) = \int_{-\pi/\Delta}^{\pi/\Delta} e^{in\Delta\omega}|\phi(\omega)|^2 f(\omega)d\omega \qquad (n \neq 0) \tag{2.3}$$

where $\phi(\omega)$ is the characteristic function of ϵ_n, and so Y_n has spectral density

$$f_Y(\omega) = |\phi(\omega)|^2 f(\omega) + c \qquad \left(|\omega| \le \frac{\pi}{\Delta}\right) \tag{2.4}$$

where

$$c = \frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} (1 - |\phi(\lambda)|^2) f(\lambda)d\lambda. \tag{2.5}$$

Note that $\gamma_Y(0) = \gamma(0)$ yields the spectral mass preservation property that

$$\int_{-\pi/\Delta}^{\pi/\Delta} f_Y(\omega)d\omega = \int_{-\infty}^{\infty} f(\omega)d\omega. \tag{2.6}$$

These results have been established in Akaike (1960), Balakrishnan (1962) and Moore and Thomson (1991). It can be seen that $f_Y(\omega)$ and $\gamma_Y(n)$ approach $f(\omega)$ and $\gamma(n\Delta)$ respectively as σ approaches zero.

We now consider solutions of (2.4) for $f(\omega)$ in terms of $f_Y(\omega)$ and $|\phi(\omega)|^2$. These are then used to derive simple moment estimators of $f(\omega)$ from estimators of $f_Y(\omega)$ and $|\phi(\omega)|^2$. This approach yields simple non-parametric estimators which, in many cases, can also be derived from a frequency-domain approximation to a quasi-likelihood based on the (almost certainly incorrect) assumption that the Y_n are unconditionally Gaussian, averaging out over the T_n (see also Robinson (1980)). The method of estimation developed in this section and Section 4 assumes that α and Δ are known so that the ϵ_n are completely determined. The situation where α and Δ are not known and must be estimated is briefly discussed in Section 5.

Consider first the case where $\phi(\omega) \neq 0$ on $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$ as would pertain if σ is sufficiently small. Then (2.4) and (2.6) yield

$$c = \frac{\int_{-\pi/\Delta}^{\pi/\Delta} (|\phi(\lambda)|^{-2} - 1) f_Y(\lambda) d\lambda}{\int_{-\pi/\Delta}^{\pi/\Delta} |\phi(\lambda)|^{-2} d\lambda} \tag{2.7}$$

and the (unique) solution of (2.4) over $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$ is

$$f(\omega) = \frac{f_Y(\omega) - c}{|\phi(\omega)|^2} \tag{2.8}$$

with c determined by (2.7).

If $\phi(\omega)$ or $f(\omega)$ are zero over an interval within $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$ then a solution to (2.4) for $f(\omega)$ over the entire frequency range may not be possible. Suppose, for example, that

$$f_Y(\omega) = \begin{cases} |\phi(\omega)|^2 f(\omega) + c & \left(|\omega| \leq \omega_0 < \dfrac{\pi}{\Delta}\right) \\ c & \left(\omega_0 < |\omega| \leq \dfrac{\pi}{\Delta}\right) \end{cases} \tag{2.9}$$

where $\phi(\omega)$ is non-zero on $[-\omega_0, \omega_0]$. In practice this situation might occur if the underlying sampling rate is sufficiently rapid for $f(\omega)$ to be assumed to be zero over a non-degenerate interval that includes the Nyquist frequency $\frac{\pi}{\Delta}$. In this case a simple and not necessarily unique solution to (2.9) for $f(\omega)$ over $[-\omega_0, \omega_0]$ is (2.8) with

$$c = \frac{1}{\frac{\pi}{\Delta} - \omega_0} \int_{\omega_0}^{\pi/\Delta} f_Y(\omega) d\omega. \tag{2.10}$$

More needs to be known about $f(\omega)$ and $\phi(\omega)$ in order that $f(\omega)$ can be recovered over the remainder of the frequency range.

Using these relationships, simple non-parametric estimators of $f(\omega)$ can be obtained from conventional spectral estimators of $f_Y(\omega)$. Suppose

$$\hat{f}_Y(\omega) = \frac{2\pi}{N\Delta}\sum_{j=1}^{N} k_N(\omega - \lambda_j) I_Y(\lambda_j) \qquad \left(\lambda_j = \frac{2\pi j}{N\Delta}\right) \tag{2.11}$$

where $k_N(x) = \frac{1}{b_N}k(\frac{x}{b_N})$ is a standard spectral window, $\omega \in [0, \frac{\pi}{\Delta})$ and

$$I_Y(\omega) = \frac{\Delta}{2\pi N}\left|\sum_{n=1}^{N} Y(n)e^{in\Delta\omega}\right|^2 \tag{2.12}$$

denotes the periodogram of the observed time series Y_n. The kernel density function $k(x)$ is assumed to be an even, continuous, square integrable function with uniformly bounded second derivatives and $\int_{-\infty}^{\infty} k(x)dx = 1$. Moreover, the bandwidth parameter b_N satisfies $b_N \to 0$ and $Nb_N \to \infty$ as $N \to \infty$. *The assumptions given in this paragraph will be referred to as* Assumptions 2.

In addition to assuming that $\gamma(t)$ is continuous and absolutely integrable we further assume that $X(t)$ is fourth order stationary with an absolutely integrable fourth cumulant function $q(t_1, t_2, t_3)$. This, in turn, is assumed to satisfy

$$q(t_1, t_2, t_3) = \iiint e^{it_1\nu_1 + it_2\nu_2 + it_3\nu_3} p(\nu_1, \nu_2, \nu_3) d\nu_1 d\nu_2 d\nu_3 \tag{2.13}$$

where $p(\nu_1, \nu_2, \nu_3)$, the fourth cumulant spectrum of $X(t)$, is absolutely integrable. These fourth cumulant assumptions are trivially satisfied when $X(t)$ is Gaussian.

The assumptions given in the previous paragraph will be referred to as Assumptions 3. They imply that Y_n is fourth order stationary and, since the T_n are independent, has cumulant function

$$\iiint e^{i\Delta\sum_{i=1}^{3} t_i\nu_i} E\{e^{i\sum_{i=1}^{3}\nu_i(\epsilon_{s+t_i} - \epsilon_s)}\} p(\nu_1, \nu_2, \nu_3) d\nu_1 d\nu_2 d\nu_3 \tag{2.14}$$

so that the fourth cumulant spectrum of Y_n is also absolutely integrable. Given Assumptions 1–3 it is known (see Parzen (1957) for example) that $\hat{f}_Y(\omega)$ is a consistent estimator of $f_Y(\omega)$ and that the asymptotic variance of $\sqrt{Nb_N}\hat{f}_Y(\omega)$ is $\frac{2\pi}{\Delta}f_Y^2(\omega)\int_{-\infty}^{\infty} k^2(x)dx$ $(\omega \neq 0)$ and $\frac{4\pi}{\Delta}f_Y^2(0)\int_{-\infty}^{\infty} k^2(x)dx$ $(\omega = 0)$. Moreover the $\sqrt{Nb_N}\hat{f}_Y(\omega)$ are asymptotically uncorrelated at distinct, fixed frequencies ω in $[0, \frac{\pi}{\Delta})$.

If $\phi(\omega) \neq 0$ on $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$, then $f(\omega)$ can be estimated over this range as

$$\hat{f}(\omega) = \frac{\hat{f}_Y(\omega) - \hat{c}}{|\hat{\phi}(\omega)|^2} \tag{2.15}$$

where c is estimated from (2.7) as

$$\hat{c} = \frac{\sum_{j=1}^{N}(|\hat{\phi}(\lambda_j)|^{-2} - 1)I_Y(\lambda_j)}{\sum_{j=1}^{N}|\hat{\phi}(\lambda_j)|^{-2}} \qquad \left(\lambda_j = \frac{2\pi j}{N\Delta}\right) \tag{2.16}$$

and a simple non-parametric estimator of $\phi(\omega)$ is given by the sample characteristic function

$$\hat{\phi}(\omega) = \frac{1}{N}\sum_{n=1}^{N} e^{-i\epsilon_n \omega}. \tag{2.17}$$

Now assume that the ϵ_n are independent random variables each with a common absolutely continuous distribution, probability density $g(x)$ and a characteristic function $\phi(\omega)$ which is bounded away from zero on $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$. *These assumptions will be referred to as* Assumptions 4. Then, since $\hat{\phi}(\omega)$ is a sample mean of N independent and identically distributed random variables each with mean $\phi(\omega)$ and variance $1 - |\phi(\omega)|^2$, $\hat{\phi}(\omega)$ is a $\sqrt{N}$-consistent estimator of $\phi(\omega)$ uniformly in ω so that $\hat{\phi}(\omega) - \phi(\omega)$ is $O(1/\sqrt{N})$ in probability, uniformly in ω. This, together with standard properties of the $I_Y(\lambda_j)$, implies that $\hat{c}$ is a $\sqrt{N}$-consistent estimator of c and so

$$\begin{aligned}&\sqrt{Nb_N}(\hat{f}_Y(\omega) - |\phi(\omega)|^2\hat{f}(\omega) - c)\\ &\quad = \sqrt{Nb_N}\{(|\hat{\phi}(\omega)|^2 - |\phi(\omega)|^2)\hat{f}(\omega) + \hat{c} - c\}\end{aligned}$$

is $O(\sqrt{b_N})$ in probability. Thus $\hat{f}_Y(\omega)$ and $|\phi(\omega)|^2\hat{f}(\omega) - c$ are asymptotically equivalent in the sense that their difference is $o(1/\sqrt{Nb_N})$ in probability as $N \to \infty$. These observations lead to the following result.

THEOREM 2.1. *Given independent jitter, Assumptions* 1–4 *and* $\omega \in [0, \frac{\pi}{\Delta})$, $\hat{f}(\omega)$ *is a consistent estimator of* $f(\omega)$ *with*

$$\lim_{N\to\infty} Nb_N \operatorname{var}(\hat{f}(\omega)) = \begin{cases} \dfrac{2\pi}{\Delta}\left(f(\omega) + \dfrac{c}{|\phi(\omega)|^2}\right)^2 \displaystyle\int_{-\infty}^{\infty} k^2(x)dx & (\omega \neq 0) \\ \dfrac{4\pi}{\Delta}(f(0) + c)^2 \displaystyle\int_{-\infty}^{\infty} k^2(x)dx & (\omega = 0). \end{cases}$$

Estimates at distinct frequencies in $[0, \frac{\pi}{\Delta})$ *are asymptotically uncorrelated.*

When $\sigma = 0$ and the process is free of jitter the standard results for regular sampling are recovered. However if $\sigma > 0$ the relative efficiency of $\hat{f}(\omega)$ with respect to the conventional regular sampling estimator is

$$\left(1 - \frac{c}{|\phi(\omega)|^2 f(\omega) + c}\right)^2 = \left(1 - \frac{c}{f_Y(\omega)}\right)^2. \tag{2.18}$$

Recalling that c is given by (2.5), considerable loss of efficiency can result when $|\phi(\omega)|^2 f(\omega)$ is small relative to c at frequency ω, a situation that can occur even for relatively small values of σ. By contrast to the standard case of regular sampling, note also that both the efficiency (2.18) and $\operatorname{var}(\hat{f}(\omega))$ depend on c which is a function of $\phi(\lambda)$ and $f(\lambda)$ over all frequencies λ and not just the particular frequency ω chosen. Although these results are for the independent jitter case and involve simpler estimation procedures, these results and observations are consistent with the numerical studies of Moore *et al.* (1987) who investigated the sampling variability of the Brillinger estimators by simulation.

In the case of (1.2) where $|\phi(\omega)|^2$ or $f(\omega)$ are zero for $\omega > \omega_0$, Theorem 2.1 continues to hold for ω in $[-\omega_0, \omega_0]$. However now $\hat{c}$ can be estimated from (2.10) by

$$\frac{\sum_{\lambda_j} \hat{f}_Y(\lambda_j)}{\frac{\pi}{\Delta} - \omega_0} \qquad \left(\lambda_j = \frac{2\pi j}{N\Delta}\right)$$

where the summation is over those frequencies λ_j in $(\omega_0, \frac{\pi}{\Delta})$. Note in general that alternative estimators for c and $\phi(\omega)$ could be used. All that is required is that they be $\sqrt{N}$-consistent (uniformly in ω in the case of estimators of $\hat{\phi}(\omega)$).

3. Properties of the jittered sampling process

The point process properties of T_n are needed in order to take advantage of the general results of Masry (1983*a*, 1983*b*) which give general estimation procedures for recovering the second-order properties of a continuous-time stationary process from irregularly sampled data. However these properties are also of independent interest. In the point process literature the model (1.2) with independent ϵ_n is referred to as the process of displaced regular events (see Lewis (1961), Lawrance (1972) and Cox and Lewis (1966)). It is a specific type of cluster process and has been used as an arrival process for certain queueing and inventory systems. However the case of dependent ϵ_n does not appear to have been treated.

As it stands T_n given by (1.2) is not stationary. However, as we shall see, it is stationary if we add the seemingly harmless assumption that α is uniformly distributed on $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$. We also require that the process ϵ_n be strictly stationary with finite moments of all orders and have absolutely continuous finite-dimensional joint distributions of all orders with joint densities of bounded variation. (The latter is implied if the densities are boundedly differentiable.) *The assumptions given in this paragraph will be referred to as* Assumptions 5.

Let $N(s,t]$ denote the number of sampling points T_n in $(s,t]$ and $N(t) = N(0,t]$. Denote the probability density function of $\epsilon_{m+n} - \epsilon_m$ by $g_n(x)$ and the joint probability density function of ϵ_m, ϵ_n by $g_{m,n}(x,y) = g_{0,n-m}(x,y)$. Then we have the following result.

THEOREM 3.1. *Given Assumptions* 5, *$N(t)$ is a stationary, orderly, point process with finite moments of all orders, mean intensity $\frac{1}{\Delta}$ and covariance density*

$$r_N(t) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} g_n(t - n\Delta) - \frac{1}{\Delta^2} \tag{3.1}$$

where $g_0(t)$ is defined to be the Dirac delta function $\delta(t)$.

PROOF. To establish the stationarity of $N(t)$ we consider the characteristic functional

$$\Phi(h) = E\{e^{i\int_{-\infty}^{\infty} h(s)dN(s)}\} \tag{3.2}$$

and show that this is invariant under any arbitrary time translation of h from $h(s)$ to $h(s+t)$. Here $h(s)$ belongs to the class of bounded measurable functions with compact support so that expressions such as $\int_{-\infty}^{\infty} h(s)dN(s)$ are well defined (see Daley and Vere-Jones (1988) for further details). Now write $t = k\Delta + \delta$ where t is arbitrary, $0 \leq \delta < \Delta$ and k is an integer. Then

$$\begin{aligned}
\Phi(h) &= E\left\{\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^{i\sum_{-\infty}^{\infty} h(\alpha+n\Delta+\epsilon_n)}d\alpha\right\} \\
&= E\left\{\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2-\delta} e^{i\sum_{-\infty}^{\infty} h(\alpha+t+(n-k)\Delta+\epsilon_n)}d\alpha\right\} \\
&\quad + E\left\{\frac{1}{\Delta}\int_{\Delta/2-\delta}^{\Delta/2} e^{i\sum_{-\infty}^{\infty} h(\alpha+t+(n-k-1)\Delta+\epsilon_n)}d\alpha\right\}.
\end{aligned}$$

Replacing n by $n+k$ in the first integral, n by $n+k+1$ in the second, and using the stationarity of the ϵ_n we obtain

$$\begin{aligned}
\Phi(h) &= E\left\{\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^{i\sum_{-\infty}^{\infty} h(\alpha+t+n\Delta+\epsilon_n)}d\alpha\right\} \\
&= E\{e^{i\int_{-\infty}^{\infty} h(s+t)dN(s)}\}
\end{aligned}$$

as required.

Let A, B denote arbitrary finite intervals. To show that $N(A)$ has finite moments of finite order k it is sufficient to consider the quantities

$$M_k = \sum_{n_1\neq 0}\sum_{n_2\neq 0}\cdots\sum_{n_k\neq 0} P(T_{n_1}, T_{n_2}, \ldots, T_{n_k} \in A).$$

Because of the monotonicity and stationarity of $N(t)$ we can restrict consideration to intervals $A = [-a, a]$ with length less than Δ. Then

$$M_k \leq \sum_{n_1\neq 0}\sum_{n_2\neq 0}\cdots\sum_{n_k\neq 0} \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} P\left(\bigcap_{i=1}^{k}\{|\epsilon_{n_i}| > |n_i|\Delta - a - |\alpha|\}\right) d\alpha$$

which, by Chebyshev's inequality, is bounded by

$$\sum_{n_1\neq 0}\sum_{n_2\neq 0}\cdots\sum_{n_k\neq 0} \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \frac{E(\Pi_i \epsilon_{n_i}^2)}{\Pi_i(|n_i|\Delta - a - |\alpha|)^2} d\alpha < \infty$$

as required.

Now

$$E\{N(A)\} = \sum_{-\infty}^{\infty} P(T_n \in A) \tag{3.3}$$
$$= \frac{1}{\Delta}\int_A \sum_{-\infty}^{\infty} \int_{-\Delta/2+n\Delta}^{\Delta/2+n\Delta} g(x-\alpha)\,d\alpha\,dx$$
$$= \frac{1}{\Delta}\int_A \int_{-\infty}^{\infty} g(\alpha)\,d\alpha\,dx = \frac{\mu(A)}{\Delta}$$

where $\mu(\cdot)$ denotes Lebesgue measure. Moreover

$$E\{N(A)N(B)\} = \sum_m \sum_{n\neq 0} P(T_m \in A, T_{m+n} \in B) + E\{N(A\cap B)\}$$

where the first term on the right-hand side of the above is

$$\frac{1}{\Delta}\int_A \int_B \sum_m \int_{-\Delta/2+m\Delta}^{\Delta/2+m\Delta} \sum_{n\neq 0} g_{0,n}(x-\alpha, y-n\Delta-\alpha)\,d\alpha\,dx\,dy.$$

Completing the integral for α in much the same way as was done for (3.3) we obtain

$$E\{N(A)N(B)\} = \frac{1}{\Delta}\int_A \int_B \sum_{n\neq 0} g_n(y-x-n\Delta)\,dx\,dy + \frac{\mu(A\cap B)}{\Delta}. \tag{3.4}$$

Note that the exchange of summation and integration needed to establish (3.3) and (3.4) is justified by Fubini's theorem and the fact that $N(A)$ has finite moments. In particular $\sum_{n\neq 0} g_n(x-n\Delta)$ is a positive function which is integrable over any finite interval and thus finite almost everywhere. This establishes the results for the mean intensity and the covariance density of $N(t)$.

Now, from (3.4),

$$P(N(t) \geq 2) \leq \sum_m \sum_{n\neq 0} P(T_m, T_{m+n} \in (0,t]) \tag{3.5}$$
$$= \frac{1}{\Delta}\int_{-t}^{t} (t-|x|) \sum_{n\neq 0} g_n(x-n\Delta)\,dx.$$

Since $\sum_{n\neq 0} g_n(x-n\Delta)$ is integrable over any finite interval we conclude that (3.5) is $o(t)$ which implies that $N(t)$ is orderly. This completes the proof. □

Now $g_n(t)$ converges to $g^{(2)}(t)$ given by (2.2) as $|n|$ increases, with $g_n(t) = g^{(2)}(t)$ for all $n \neq 0$ in the case of independent jitter. If we require

$$\sum_{n\neq 0} \iint |g_{0,n}(s,t) - g(s)g(t)|\,ds\,dt < \infty \tag{3.6}$$

then

$$\Delta \sum_{n=-\infty}^{\infty} (g_n(t-n\Delta) - g^{(2)}(t-n\Delta))$$

has Fourier transform $h(\omega)$ where

$$h(\omega) = \frac{\Delta}{2\pi} \sum_{n=-\infty}^{\infty} h_n(\omega) e^{-in\Delta\omega} \tag{3.7}$$

and $h_n(\omega)$ is the difference between the characteristic functions of $g_n(t)$ and $g^{(2)}(t)$. Note that $h_n(\omega)$ is just the autocovariance function of the (complex) stationary process $\exp(-i\epsilon_n\omega)$ and (3.6) is of the form of a mixing condition for ϵ_n. From the Cauchy-Schwarz inequality and Lemma 5, Chapter 4, of Ibragimov and Rozanov (1978) it follows that (3.6) is satisfied by Gaussian processes with absolutely summable autocovariance functions.

Moreover, since $g^{(2)}(x)$ is of bounded variation and integrable, $\sum g^{(2)}(t-n\Delta)$ is periodic with period Δ and

$$\sum_{n=-\infty}^{\infty} g^{(2)}(t-n\Delta) = \sum_{j=-\infty}^{\infty} c_j e^{it(2\pi j/\Delta)} \tag{3.8}$$

where

$$\begin{aligned} c_j &= \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \sum_{n=-\infty}^{\infty} g^{(2)}(t+n\Delta) e^{-it(2\pi j/\Delta)} dt \\ &= \frac{1}{\Delta} \int_{-\infty}^{\infty} e^{-it(2\pi j/\Delta)} g^{(2)}(t) dt = \frac{1}{\Delta} \left| \phi\left(\frac{2\pi j}{\Delta}\right) \right|^2 . \end{aligned}$$

The previous discussion is summarised in the following corollary.

COROLLARY 3.1. *Subject to* (3.6) *and Assumptions* 5, *the jittered point process* $N(t)$ *has spectral density*

$$f_N(\omega) = \frac{1}{\Delta^2} \sum_{j \neq 0} \left| \phi\left(\frac{2\pi j}{\Delta}\right) \right|^2 \delta\left(\omega - \frac{2\pi j}{\Delta}\right) + \frac{1}{\Delta^2} h(\omega) \tag{3.9}$$

where $h(\omega)$ *is given by* (3.7).

Note, in particular, that $f_N(\omega)$ has spectral lines at all frequencies $\frac{2\pi j}{\Delta}$ $(j \neq 0)$ where $\phi(\frac{2\pi j}{\Delta}) \neq 0$. If $\phi(\omega)$ is zero or approximately zero for $|\omega| \geq \frac{2\pi}{\Delta}$ as would typically be the case when σ is large in relation to Δ, then the right-hand side of (3.8) becomes $\frac{1}{\Delta}$. In the Gaussian case this approximation is accurate to four decimal places for $\frac{\sigma}{\Delta} > 0.5$ and accurate to two decimal places for $\frac{\sigma}{\Delta} > 0.4$ (see Moran (1950) for further details concerning this approximation). In this case $f_N(\omega)$ reduces to the simple form $\frac{1}{\Delta^2} h(\omega)$.

4. Estimation for dependent jitter

The case for dependent jitter is more difficult. Now the autocovariance function of Y_n is given by

$$\gamma_Y(n) = \int_{-\infty}^{\infty} \gamma(x + n\Delta) g_n(x) dx \tag{4.1}$$

and, if $\gamma(t)$ is replaced by its spectral representation, this becomes

$$\gamma_Y(n) = \int_{-\pi/\Delta}^{\pi/\Delta} e^{in\Delta\omega}(|\phi(\omega)|^2 + h_n(\omega)) f(\omega) d\omega. \tag{4.2}$$

Note that $h_n(\omega)$ is a non-negative definite function of n since it is the autocovariance function of the (complex) stationary process $\exp -i\epsilon_n\omega$. Thus, provided $\sum_{n=-\infty}^{\infty} |h_n(\omega)|$ is finite for ω in $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$,

$$h_n(\omega) = \int_{-\pi/\Delta}^{\pi/\Delta} e^{in\Delta\lambda} H(\lambda, \omega) d\lambda \tag{4.3}$$

where $H(\lambda, \omega)$ is given by

$$H(\lambda, \omega) = \frac{\Delta}{2\pi} \sum_{n=-\infty}^{\infty} h_n(\omega) e^{-in\Delta\lambda}. \tag{4.4}$$

Replacing $h_n(\omega)$ in (4.2) by its spectral representation (4.3) we see that the spectral density of Y_n is now given by

$$f_Y(\omega) = |\phi(\omega)|^2 f(\omega) + \int_{-\pi/\Delta}^{\pi/\Delta} H(\omega - \lambda, \lambda) f(\lambda) d\lambda \qquad \left(|\omega| \leq \frac{\pi}{\Delta}\right) \tag{4.5}$$

and the spectral mass preservation property (2.6) is again satisfied. This result was given in Moore and Thomson (1991).

The various spectral quantities in (4.5) could be estimated and then the sample analogue of (4.5) solved for $f(\omega)$. Alternatively one could proceed in the time domain and utilise (4.1) to estimate $\gamma(n\Delta)$. This is the approach we adopt.

Before proceeding further we briefly discuss the implications of dependent jitter for the general point process approach given in Brillinger (1972) and Masry (1983*b*). Here the starting point is the marked point process $Z(t)$ where

$$dZ(t) = X(t) dN(t) \tag{4.6}$$

and $N(t)$ is the jittered sampling point process whose properties have been explored in Section 3. Following Brillinger (1972), $Z(t)$ is a zero-mean stationary interval process with autocovariance function

$$r_Z(t) = \gamma(t)\left(r_N(t) + \frac{1}{\Delta^2}\right) + \frac{\gamma(0)}{\Delta}\delta(t) \tag{4.7}$$

from which $\gamma(t)$ can be obtained for $t \neq 0$ provided $r_N(t) + \frac{1}{\Delta^2} > 0$. Point processes satisfying the latter condition are termed alias-free by Masry (1978). The jittered sampling point process $N(t)$ is not alias-free when $\frac{\sigma}{\Delta}$ is very small, but will typically be so if $\frac{\sigma}{\Delta}$ is large enough. Consider, for example, the case of independent Gaussian jitter when time is measured in units of the interval Δ so that $\Delta = 1$. Then a simple graphical analysis shows that $r_N(t) + \frac{1}{\Delta^2}$ is zero to 6 decimal places when $\sigma = 0.1$, $0 < |t| \leq .2$ and zero to 1 decimal place when $\sigma = 0.2$, $0 < |t| \leq .25$. Thus, in this particular situation, $N(t)$ can be regarded as alias-free for $\frac{\sigma}{\Delta} > 0.3$ say, but will typically not be alias-free in practice when $\frac{\sigma}{\Delta} < 0.2$. In general the Brillinger and Masry procedures will work only for point processes that are alias-free.

From (4.7) and Corollary 3.1 the spectral density of $Z(t)$ is given by

$$(4.8) \qquad f_Z(\omega) = \frac{1}{\Delta^2} \sum_{-\infty}^{\infty} \left| \phi\left(\frac{2\pi j}{\Delta}\right) \right|^2 f\left(\omega - \frac{2\pi j}{\Delta}\right) + \frac{1}{\Delta^2} \int_{-\infty}^{\infty} f(\omega - \lambda) h(\lambda) d\lambda$$

which can be solved for $f(\omega)$ provided $N(t)$ is alias-free. A modification of the general solution given in Brillinger (1972) and Masry (1978) yields

$$(4.9) \qquad f(\omega) = \Delta^2 \sum_{-\infty}^{\infty} b_j \left\{ \tilde{f}_z\left(\omega - \frac{2\pi j}{\Delta}\right) - \int_{-\infty}^{\infty} \tilde{f}_z\left(\omega - \frac{2\pi j}{\Delta} - \lambda\right) \beta(\lambda) d\lambda \right\}$$

where $\tilde{f}_z(\omega) = f_Z(\omega) - \frac{\gamma(0)}{2\pi\Delta}$ and $\beta(\omega)$ is the Fourier transform of

$$\frac{\int_{-\infty}^{\infty} e^{it\omega} \left(h(\omega) - \frac{\Delta}{2\pi}\right) d\omega}{\sum_{-\infty}^{\infty} \left|\phi\left(\frac{2\pi j}{\Delta}\right)\right|^2 e^{it(2\pi j/\Delta)} + \int_{-\infty}^{\infty} e^{it\omega} \left(h(\omega) - \frac{\Delta}{2\pi}\right) d\omega}.$$

The Fourier coefficients b_j satisfy $\sum |\phi(\frac{2\pi j}{\Delta})|^2 b_{j+k} = \delta_k$ with δ_k denoting Kronecker's delta. If $N(t)$ is alias-free then the b_j will typically converge to zero rapidly as $|j|$ increases. For Gaussian jitter with $\frac{\sigma}{\Delta} > 0.5$ the b_j are all effectively zero save b_0 which is unity. Thus (4.9) can be used as a basis for forming estimates of $f(\omega)$ in much the same way as was done by Brillinger and Masry. However this line of development has not been pursued and instead we have concentrated on the simpler problem of estimating the autocovariances $\gamma(n\Delta)$.

We now consider estimating $\gamma(n\Delta)$ for $n > 0$ by

$$(4.10) \quad \hat{\gamma}(n\Delta) = \frac{\sum_{m=1}^{N-n} X(T_m) X(T_{m+n}) k\left(\frac{T_m - \alpha - m\Delta}{b_N}\right) k\left(\frac{T_{m+n} - \alpha - (m+n)\Delta}{b_N}\right)}{\sum_{m=1}^{N-n} k\left(\frac{T_m - \alpha - m\Delta}{b_N}\right) k\left(\frac{T_{m+n} - \alpha - (m+n)\Delta}{b_N}\right)}$$

where the kernel density function $k(x)$ satisfies Assumptions 2, but now b_N satisfies $b_N \to 0$ and $Nb_N^2 \to \infty$ as $N \to \infty$. As in Section 2, α and Δ are assumed known so that the $\epsilon_n = T_n - \alpha - n\Delta$ are completely determined (see Section 5 for a discussion of the case where α and Δ are estimated from the data). For $n = 0$ we choose to estimate the variance of $X(t)$ by

$$\hat{\gamma}(0) = \frac{\sum_{n=1}^{N} X(T_n)^2}{N}. \tag{4.11}$$

To establish the consistency and the asymptotic variances of these estimators we need to assume that the underlying process $X(t)$ is stationary to fourth order with $\gamma(t)$, $q(t_1, t_2, t_3)$ and $p(\nu_1, \nu_2, \nu_3)$ being continuous, absolutely integrable and of bounded variation. In addition to Assumptions 5, the joint densities of the ϵ_n are required to be uniformly bounded and equicontinuous at zero with $g_{0,n}(0,0)$ bounded uniformly away from zero. In place of (3.6), we require

$$\begin{aligned}\sum_v \iiiint |g_{0,m,v,v+n}(x_1, x_2, x_3, x_4) \\ - g_{0,m}(x_1, x_2) g_{0,n}(x_3, x_4)| dx_1 dx_2 dx_3 dx_4 < \infty\end{aligned} \tag{4.12}$$

uniformly in m and n with $g_{s,t,u,v}(x_1, x_2, x_3, x_4)$ denoting the joint density of $\epsilon_s, \epsilon_t, \epsilon_u, \epsilon_v$. Moreover

$$\sum_v (g_{0,m,v,v+n}(x_1, x_2, x_3, x_4) - g_{0,m}(x_1, x_2) g_{0,n}(x_3, x_4)) \tag{4.13}$$

is assumed to be equicontinuous at zero. Since (4.13) is absolutely integrable by virtue of (4.12), a sufficient condition for (4.13) and the joint densities of the ϵ_n to be equicontinuous is that their Fourier transforms be uniformly bounded by absolutely integrable functions. Like (3.6) the mixing condition (4.12) specified above is satisfied by Gaussian processes with absolutely summable autocovariance functions. *The assumptions given in this paragraph will be referred to as* Assumptions 6.

THEOREM 4.1. *For all fixed, finite $n > 0$ and subject to Assumptions* 1 *and* 6, *$\hat{\gamma}(n\Delta)$ is a mean-squared consistent estimator of $\gamma(n\Delta)$ with*

$$\lim_{N\to\infty} N b_N^2 \operatorname{var}(\hat{\gamma}(n\Delta)) = \frac{(\gamma^2(0) + \gamma^2(n\Delta) + q(n\Delta, 0, n\Delta))(\int_{-\infty}^{\infty} k^2(x)dx)^2}{g_{0,n}(0,0)}$$

and $\hat{\gamma}(0)$ is an unbiased estimator of $\gamma(0)$ with

$$\lim_{N\to\infty} N \operatorname{var}(\hat{\gamma}(0)) = \Delta \int_{-\infty}^{\infty} (2\gamma^2(x) + q(0, x, x)) \left(r_N(x) + \frac{1}{\Delta^2} \right) dx.$$

Here $r_N(x)$ is given by (3.1). *Moreover estimates at distinct lags are asymptotically uncorrelated.*

PROOF. We first consider $\hat{\gamma}(n\Delta)$ for $n > 0$ and write it as $\hat{c}(n\Delta)/\hat{d}(n\Delta)$ where

$$\hat{c}(n\Delta) = \frac{1}{Nb^2} \sum_{m=1}^{N-n} X(T_m)X(T_{m+n})k\left(\frac{\epsilon_m}{b}\right)k\left(\frac{\epsilon_{m+n}}{b}\right)$$
$$\hat{d}(n\Delta) = \frac{1}{Nb^2} \sum_{m=1}^{N-n} k\left(\frac{\epsilon_m}{b}\right)k\left(\frac{\epsilon_{m+n}}{b}\right).$$

Here the suffix on b_N has been suppressed for notational convenience. Now

$$\begin{aligned} E\{\hat{c}(n\Delta)\} &= \left(1 - \frac{n}{N}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x)k(y)\gamma(b(y-x)+n\Delta)g_{0,n}(bx,by)dxdy \\ &= \left(1 - \frac{n}{N}\right)(\gamma(n\Delta)g_{0,n}(0,0) + o(1)) \end{aligned}$$

uniformly in n as $N \to \infty$, since $\gamma(t)$ is uniformly continuous, $g_{0,n}(x,y)$ is equicontinuous at zero, and $b \to 0$. Similarly

$$E\{\hat{d}(n\Delta)\} = \left(1 - \frac{n}{N}\right)(g_{0,n}(0,0) + o(1))$$

uniformly in n.

Turning to the covariances we note that $\operatorname{cov}\{\hat{c}(m\Delta), \hat{c}(n\Delta)\}$ can be written as

$$E[\operatorname{cov}\{\hat{c}(m\Delta), \hat{c}(n\Delta)\} \mid \boldsymbol{\epsilon}] + \operatorname{cov}\{E[\hat{c}(m\Delta) \mid \boldsymbol{\epsilon}], E[\hat{c}(n\Delta) \mid \boldsymbol{\epsilon}]\} \tag{4.14}$$

where $\boldsymbol{\epsilon}$ denotes the N-dimensional vector with typical element ϵ_i. The first term of (4.14) is

$$\begin{aligned} \frac{1}{N^2b^4} \sum_{u=1}^{N-m} \sum_{v=1}^{N-n} E\Big\{&\tilde{q}(T_{u+m} - T_u, T_v - T_u, T_{v+n} - T_u) \\ &\cdot k\left(\frac{\epsilon_u}{b}\right)k\left(\frac{\epsilon_{u+m}}{b}\right)k\left(\frac{\epsilon_v}{b}\right)k\left(\frac{\epsilon_{v+n}}{b}\right)\Big\} \end{aligned} \tag{4.15}$$

where $\tilde{q}(t,u,v)$ denotes $\operatorname{cov}\{X(s)X(s+t), X(s+u)X(s+v)\}$ and

$$\tilde{q}(t,u,v) = q(t,u,v) + \gamma(u)\gamma(v-t) + \gamma(v)\gamma(u-t).$$

The summation in (4.15) can now be broken up into separate summations over distinct subsets of the indices u, $u+m$, v, $v+n$ and evaluated over each of these. In the case where $m < n$ and the indices are all distinct, for example, (4.15) becomes

$$\begin{aligned} \frac{1}{N}\sum a_v^{(N)} \int k(\boldsymbol{x})\tilde{q}(b(x_2 - x_1) + m\Delta,& \\ b(x_3 - x_1) + v\Delta, b(x_4 - x_1) &+ (v+n)\Delta) \\ \times g_{0,m,v,v+n}(b\boldsymbol{x})d\boldsymbol{x}& \end{aligned} \tag{4.16}$$

where the summation is over distinct sets of indices 0, m, v, $v+n$, the vector $\boldsymbol{x}$ has typical element x_i, $k(\boldsymbol{x})$ is the product of the $k(x_i)$ and

$$a_v^{(N)} = \begin{cases} 1 - \dfrac{m+|v|}{N} & (-N+m+1 \le v \le m-n) \\ 1 - \dfrac{n}{N} & (m-n \le v \le 0) \\ 1 - \dfrac{n+|v|}{N} & (0 \le v \le N-n-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Since the $g_{0,m,v,v+n}(b\boldsymbol{x})$ are uniformly bounded we note that (4.16) is bounded above by a term proportional to

$$\frac{1}{N}\int k(\boldsymbol{x})\Big\{\sum_v |\tilde{q}(b(x_2-x_1)+m\Delta, b(x_3-x_1)+v\Delta,$$
$$b(x_4-x_1)+(v+n)\Delta)|\Big\}d\boldsymbol{x}$$

which is $O(\frac{1}{N})$ uniformly in m and n. The latter follows from the fact that $\tilde{q}(t,u,v)$ comprises absolutely integrable functions of bounded variation and the bracketted factor is an approximating sum to an integral. The case where $m < n$ and the indices are not all distinct follows similarly and these terms are $O(\frac{1}{Nb})$ uniformly in m and n.

When $m = n$ the terms for $u \neq v$ are dealt with in the same way and we find that (4.15) differs from

$$\frac{1}{Nb^4}\left(1-\frac{n}{N}\right)E\left\{\tilde{q}(\epsilon_n-\epsilon_0+n\Delta, 0, \epsilon_n-\epsilon_0+n\Delta)k^2\left(\tfrac{\epsilon_0}{b}\right)k^2\left(\tfrac{\epsilon_n}{b}\right)\right\}$$

by terms of $O(\frac{1}{Nb})$ uniformly in n. The latter is

$$\frac{1}{Nb^2}\left(1-\frac{n}{N}\right)\iint k^2(x)k^2(y)\tilde{q}(b(y-x)+n\Delta, 0, b(y-x)+n\Delta)g_{0,n}(bx,by)dxdy$$

which equals

$$(4.17)\qquad \frac{1}{Nb^2}\left(1-\frac{n}{N}\right)\left\{\tilde{q}(n\Delta,0,n\Delta)g_{0,n}(0,0)\left(\int k^2(x)dx\right)^2 + o(1)\right\}$$

uniformly in n since $\tilde{q}(t,u,v)$ is uniformly continuous, $g_{0,n}(x,y)$ is equicontinuous at zero and $\int k^2(x)dx$ is finite.

Now consider the second term of (4.14) which is

$$(4.18)\qquad \frac{1}{Nb^4}\sum_{u=1}^{N-m}\sum_{v=1}^{N-n}\operatorname{cov}\Big\{\gamma(T_{u+m}-T_u)k\left(\tfrac{\epsilon_u}{b}\right)k\left(\tfrac{\epsilon_{u+m}}{b}\right),$$
$$\gamma(T_{v+n}-T_v)k\left(\tfrac{\epsilon_v}{b}\right)k\left(\tfrac{\epsilon_{v+n}}{b}\right)\Big\}.$$

The summation is again broken up into distinct subsets of indices and evaluated over each of these. In the case where $m < n$ and the indices are distinct, (4.18) becomes

$$(4.19)\quad \frac{1}{Nb^4}\int k\left(\frac{\boldsymbol{x}}{b}\right)\Big\{\gamma(m\Delta + x_2 - x_1)\gamma(n\Delta + x_4 - x_3) \times \sum a_v^{(N)}(g_{0,m,v,v+n}(\boldsymbol{x}) - g_{0,m}(x_1,x_2)g_{0,n}(x_3,x_4))\Big\}d\boldsymbol{x}$$

where, as before, the summation is over distinct sets of indices 0, m, v, $v+n$. However, from (4.12), the bracketted factor is absolutely integrable and continuous at zero so that (4.19) is $O(\frac{1}{N})$ uniformly in m and n. The case where $m < n$ and the indices are not all distinct follows similarly. For $m = n$ the various terms are treated as before and we find that (4.18) differs from

$$(4.20)\quad \frac{1}{Nb^2}\left(1-\frac{n}{N}\right)\iint k^2(x)k^2(y)\gamma^2(n\Delta + b(y-x))g_{0,n}(bx,by)dxdy$$

by terms of $O(\frac{1}{Nb})$ uniformly in n and so (4.20) is

$$\frac{1}{Nb^2}\left(1-\frac{n}{N}\right)\left\{\gamma^2(n\Delta)g_{0,n}(0,0)\left(\int k^2(x)dx\right)^2 + o(1)\right\}$$

uniformly in n.

Thus $\lim_{N\to\infty} Nb^2 \operatorname{cov}\{\hat{c}(m\Delta), \hat{c}(n\Delta)\}$ is now given by

$$(\gamma^2(0) + 2\gamma^2(n\Delta) + q(n\Delta,0,n\Delta))g_{0,n}(0,0)\left(\int_{-\infty}^{\infty} k^2(x)dx\right)^2$$

for $m = n$ and zero for $m \neq n$. The covariance properties of the $\hat{d}(n\Delta)$ are established in exactly the same way as for the second term of (4.14) yielding

$$(4.21)\quad \lim_{N\to\infty} Nb^2 \operatorname{cov}\{\hat{d}(m\Delta), \hat{d}(n\Delta)\} = \begin{cases} g_{0,n}(0,0)\left(\int_{-\infty}^{\infty} k^2(x)dx\right)^2 & (m = n) \\ 0 & (m \neq n) \end{cases}$$

and

$$\lim_{N\to\infty} Nb^2 \operatorname{cov}\{\hat{c}(m\Delta), \hat{d}(n\Delta)\} = \begin{cases} \gamma(n\Delta)g_{0,n}(0,0)\left(\int_{-\infty}^{\infty} k^2(x)dx\right)^2 & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Since $\hat{d}(n\Delta)$ converges in probability to $g_{0,n}(0,0) > 0$, standard variational arguments applied to the ratio $\hat{\gamma}(n\Delta) = \hat{c}(n\Delta)/\hat{d}(n\Delta)$ yield the required results for the limit of $Nb^2 \operatorname{cov}\{\hat{\gamma}(m\Delta), \hat{\gamma}(n\Delta)\}$.

Finally observe that $\hat{\gamma}(0)$ is an unbiased estimator of $\gamma(0)$ with variance

$$\frac{1}{N}\left\{\tilde{q}(0,0,0) + \int \tilde{q}(0,x,x)\sum_{n\neq 0}\left(1-\frac{|n|}{N}\right)g_n(x-n\Delta)dx\right\}$$

which equals

$$\frac{\Delta}{N}\left\{\int \tilde{q}(0,x,x)\left(r_N(x)+\frac{1}{\Delta^2}\right)dx + o(1)\right\}. \tag{4.22}$$

Similarly $Nb\operatorname{cov}\{\hat{\gamma}(0), \hat{c}(n\Delta)\}$ converges to zero uniformly in n. This completes the proof. □

Note that Theorem 4.1 also applies to the case of independent jitter. Then the denominator of the limiting variance for $\sqrt{N}b_N\hat{\gamma}(n\Delta)$ becomes $g(0)^2$ which is independent of n.

An alternative estimator of $\gamma(n\Delta)$ is provided for $n > 0$ by

$$\tilde{\gamma}(n\Delta) = \frac{\sum_{u=1}^N \sum_{v=1}^N X(T_u)X(T_v)k\left(\dfrac{T_v - T_u - n\Delta}{b_N}\right)}{\sum_{u=1}^N \sum_{v=1}^N k\left(\dfrac{T_v - T_u - n\Delta}{b_N}\right)} \tag{4.23}$$

where now $k(x)$ and b_N satisfy Assumption 2. This estimator has much the same form as that considered by Masry (1983*a*). However the denominator of Masry's estimator involves $r_N(n\Delta) + 1/\Delta^2$ rather than an estimator of this quantity as in (4.23) above. Adapting the results of Masry (1983*a*) or following a similar development to that given in Theorem 4.1, it can be shown that $\tilde{\gamma}(n\Delta)$ is also a mean-squared consistent estimator of $\gamma(n\Delta)$ with $\lim_{N\to\infty} Nb_N \operatorname{cov}\{\tilde{\gamma}(m\Delta), \tilde{\gamma}(n\Delta)\}$ given by

$$\frac{(\gamma^2(0) + \gamma^2(n\Delta) + q(n\Delta, 0, n\Delta))\int_{-\infty}^{\infty} k^2(x)dx}{\Delta(r_N(n\Delta) + 1/\Delta^2)}$$

for $m = n$ and zero for $m \neq n$.

It is of interest to compare the two estimators $\hat{\gamma}(n\Delta)$ and $\tilde{\gamma}(n\Delta)$. Clearly $\hat{\gamma}(n\Delta)$ enjoys significant computational advantages since it is a ratio of two conventional autocovariance estimates. However, in terms of efficiency of estimation, $\tilde{\gamma}(n\Delta)$ will typically be more efficient for moderate to large values of $\frac{\sigma}{\Delta}$ and $\hat{\gamma}(n\Delta)$ more efficient for smaller values of $\frac{\sigma}{\Delta}$.

This is illustrated in Fig. 1 where the efficiency of $\hat{\gamma}(n\Delta)$ relative to $\tilde{\gamma}(n\Delta)$ for Gaussian jitter is plotted as a function of lag for four values of $\frac{\sigma}{\Delta}$. The kernel functions of both estimators have been assumed to make the same contribution to the asymptotic variance. The Gaussian jitter models chosen correspond to the

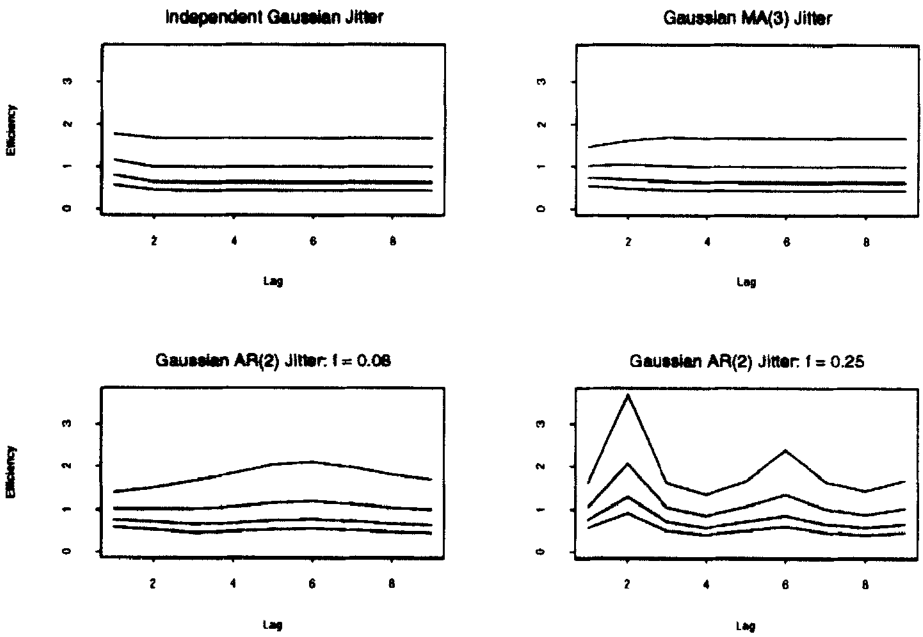

Fig. 1. Plots of the efficiency of $\hat{\gamma}(n\Delta)$ relative to $\tilde{\gamma}(n\Delta)$ at lag n for $\frac{\sigma}{\Delta}$ = 0.3, 0.4, 0.5, 0.6 and Gaussian jitter. The plots, from highest to lowest, are in increasing order of $\frac{\sigma}{\Delta}$. Four cases are considered: independent jitter; MA(3) jitter with non-zero autocorrelations 1, 0.578, 0.223, 0.025 respectively; AR(2) jitter with autocorrelation functions $0.9^{|n\Delta|}\cos(2\pi n\Delta f)$ for frequencies $f = 0.08$ and $f = 0.25$ respectively.

cases of independent jitter, moving average jitter and two cases of autoregressive jitter with peaked spectra. These models were used in Moore and Thomson (1991) and are used here for the purposes of comparison. If these results are at all indicative of what might occur in practice, then it would seem that one should use $\hat{\gamma}(n\Delta)$ when $\frac{\sigma}{\Delta} < 0.5$ and $\tilde{\gamma}(n\Delta)$ when $\frac{\sigma}{\Delta} > 0.4$.

In practice these estimators will be difficult to make operational because of the need to estimate $q(n\Delta, 0, n\Delta)$ in order to compute standard errors. However, if the latter are small as would be the case if $X(t)$ is near Gaussian, then this may not be so important. Of course there are no problems in the important case where $X(t)$ is Gaussian since then the $q(n\Delta, 0, n\Delta)$ are identically zero.

5. Further comments

If α and Δ are unknown then they will need to be estimated from the T_n. Least-squares regression, for example, will yield estimators $\hat{\alpha}$ and $\hat{\Delta}$ that are $N^{1/2}$ and $N^{3/2}$ consistent estimators of α and Δ respectively in this situation. Thus, since $\hat{f}(\omega)$ in Section 2 and $\hat{\gamma}(n\Delta)$ in Section 4 typically involve α and Δ only through the kernel function $k(x)$, Theorems 2.1 and 4.1 should continue to hold with α and Δ replaced by $\hat{\alpha}$ and $\hat{\Delta}$ provided stronger conditions are made concerning b_N and the smoothness of $k(x)$. Similarly, if stronger conditions are made concerning the mixing properties of both the $X(t)$ and ϵ_n processes then these theorems can no doubt also be extended to include the asymptotic normality of the estimators.

For the case of independent jitter, the results of Sections 2 and 4 go some way towards obtaining spectral and autocovariance estimators with known sampling properties. However the results for dependent jitter are mixed. Satisfactory estimators with prescribed sampling properties are only given for the autocovariance function, although a better understanding of the spectral estimators proposed by Brillinger (1972) and Masry (1983*a*) has been achieved in this case. Even when these latter techniques are applicable, as in the case when $\frac{\sigma}{\Delta}$ is large and $f_N(\omega)$ is approximately $h_N(\omega)/\Delta^2$, sampling properties are not yet available.

Further development could be undertaken along the lines of Section 4 using $\gamma(n\Delta)$ as the basis for a weighted covariance spectral estimator. At first sight this would seem to involve two kernel functions, one to estimate $\gamma(n\Delta)$ and the other to act as a convergence factor for the estimator. The interaction between the two bandwidth parameters would need to be handled with care. In the case where ϵ_n is Gaussian and follows a parametric model such as an ARMA process, nonparametric spectral estimators and their sampling properties may more readily be determined. However these and other possibilities remain the subject of further research.

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