Ann. Inst. Statist. Math.
Vol. 47, No. 3, 435–446 (1995)

EXACT AND LIMITING DISTRIBUTIONS OF THE NUMBER OF SUCCESSIONS IN A RANDOM PERMUTATION*

JAMES C. FU

Institute of Applied Mathematics, National Donghwa University, Hualien, Taiwan, R.O.C.

(Received April 14, 1994; revised November 7, 1994)

Abstract. Traditionally the distributions of the number of patterns and successions in a random permutation of n integers $1, 2, \ldots,$ and n were studied by combinatorial analysis. In this short article, a simple way based on finite Markov chain imbedding technique is used to obtain the exact distribution of successions on a permutation. This approach also gives a direct proof that the limiting distribution of successions is a Poisson distribution with parameter $\lambda = 1$. Furthermore, a direct application of the main result, it also yields the waiting time distribution of a succession.

Key words and phrases: Permutation, succession, Markov chain imbedding, transition probabilities, Poisson convergence, waiting time.

1. Introduction

Let $F_n = \{\pi : \pi = (\pi(1), \ldots, \pi(n))\}$ be the set of $n!$ random permutations generated by n positive integers $1, 2, \ldots, (n-1)$, and n, where $\pi(i)$ stands for the integer at the i-th coordinate of a random permutation π. A succession (of size 2) in a random permutation π is any pair $\pi(i)$, $\pi(i+1)$ with $\pi(i+1) = \pi(i) + 1$, $i = 1, \ldots, n-1$. More general, for $2 \le k \le n$, we define the index functions on a random permutation π as, $i = 1, 2, \ldots, n-k+1$,

$$(1.1) \qquad 1_n(i, k, \pi) = \begin{cases} 1, & \text{if } \pi(i+k-1) = \pi(i+k-2) + 1, \ldots, \\ & \qquad\qquad\qquad = \pi(i) + k - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and define the random variable

$$(1.2) \qquad X_n(\pi, k) = \sum_{i=1}^{n-k+1} 1_n(i, k, \pi)$$

* This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grant NSERC A-9216, and National Science Council of Republic of China under Grant 85-2121-M-259-003.

being the total number of successions of size k in a random permutation π.

Further for $k = 2$, we define the random variable of the number of the circular successions of size 2 as

$$X_n^*(\pi, 2) = \sum_{i=1}^{n} 1_n(i, 2, \pi) \tag{1.3}$$

where $1_n(i, 2, \pi)$, $i = 1, \ldots, n-1$ are given by (1.1) and

$$1_n(n, 2, \pi) = \begin{cases} 1, & \text{if } \pi(n) + 1 = \pi(1), \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

For example, the random permutation $\pi = (51237864)$ has three successions of size 2, $X_8(\pi, 2) = 3$, four circular successions of size 2, $X_8^*(\pi, 2) = 4$, one succession of size 3, $X_8(\pi, 3) = 1$, and no succession of size $k > 3$, $X_8(\pi, k) = 0$.

Suppose we insert the integers $1, 2, \ldots$ randomly on a line one by one, starting with 1, then 2, and continue. We could define the random variable

(1.5) $W_k(S, 2) =$ *the smallest integer required that the k-th succession of size 2 occurred in the insertion process*,

as the waiting time (number of insertions) of k-th succession of size 2.

The number of successions of size 2 in a random permutation π has appeared often in mathematical statistics and combinatorial theory literature. It has a long and rich history. For the early history, one can find from the papers such as Whitworth (1943), Kaplansky (1944), Riordan (1945, 1958). The earliest results about successions are published in the Annals of Mathematical Statistics, for example, Kaplansky (1945), Riordan (1965) and Abramson and Moser (1967). Recently, the results appeared often in combinatorial theory journals, especially the Journal of Combinatorial Theory, for instance, Dwass (1973), Tanny (1976), Jackson and Aleliunas (1977), and Dymacek and Roselle (1978). This phenomenon is probably due to the fact that almost all the approaches, except the Dwass' (1973), are combinatorial in nature. Dwass (1973) provided a probabilistic argument which shows that the limiting distribution of the random variable $X_n^*(\pi, 2)$ of circular successions is a Poisson distribution with parameter $\lambda = 1$, i.e.;

$$\lim_{n \to \infty} P(X_n^*(\pi, 2) = x) = \frac{1}{x!} e^{-1}, \qquad x = 0, 1, \ldots, \tag{1.6}$$

and then proved that $X_n(\pi, 2)$ and $X_n^*(\pi, 2)$ have same limiting distribution.

For fix n, various formulae for the distribution of $X_n(\pi, 2)$ developed via combinatorial analysis and recursive equations are rather complex, for instance, Roselle (1968), Jackson and Reilly (1976), and Reilly and Tanny (1979). Currently, the exact distributions of many patterns and successions in a random permutation still remain unknown. The main difficulty for the traditional combinatorial approach is the long range inter-dependency among the n integers in a permutation. In this short manuscript, a simple approach based on the finite Markov chain imbedding technique introduced by Fu and Koutras (1994) and Fu (1994) is used to obtain the exact distribution of $X_n(\pi, 2)$. This approach is somewhat closer to Dwass' (1973) approach by nature but is more general. The main idea of our approach is to decompose the permutation π into a sequence of random sub-permutations so that a nice finite Markov chain would be established on the sequence of sub-permutations. This nice Markov chain not only gives a simple way to find the exact distribution of the successions $X_n(\pi, 2)$ but also provides a direct proof that its limiting distribution is a Poisson distribution with parameter $\lambda = 1$. For $k \geq 3$, the random variable $X_n(\pi, k)$ has a degenerating limiting distribution at zero. This means that if the size of a permutation π tends to infinity then there will be no succession of size greater than or equal to 3. Furthermore, this method also yields the waiting time distribution of a succession.

2. Main results

Within a random permutation π, the arrangements of integers are highly correlated. The forward and backward principle for finite Markov chain imbedding technique developed by Fu (1994) for studying the exact distribution of a specified pattern in a sequence of multi-state trials cannot be directly used in this correlated situation. With the following modifications, the finite Markov chain imbedding technique yields our main results.

For example, a random permutation $\pi = (31524687)$ of eight integers $1, 2, \dots,$ and 8, we decompose the permutation π into 8 sub-permutations by deleting the largest integer from the permutation one by one, hence, in this case they are $\pi_n = \pi$, $\pi_{n-1} = (3152467)$, $\pi_{n-2} = (315246)$, $\dots$, $\pi_3 = (312)$, $\pi_2 = (1, 2)$, and $\pi_1 = (1)$. In general, for each permutation π it has a unique decomposition $\{\pi_1, \pi_2, \dots, \pi_n\}$, where π_{t-1} equals to π_t with the largest integer t being deleted and $\pi_n = \pi$. This decomposition of π is equivalent to the method of inserting n integers $\{1, 2, \dots, n\}$ one by one. Let us consider a state space $\Omega = \{0, 1, 2, \dots, n-1\}$, an index set $\Gamma_n = \{0, 1, \dots, n\}$, and a sequence of transformations $Y_t : F_n \to \Omega$, $t = 1, 2, \dots, n$ as, for each $\pi \in F_n$, and $t = 1, 2, \dots, n$

$$Y_t(\pi) = X_t(\pi_t, 2) = \textit{the total number of successions of size } 2 \textit{ in the sub-permutation } \pi_t \textit{ generated by the random permutation } \pi. \tag{2.1}$$

For example, the realization of the Markov chain associated with the random permutation $\pi = (31524687)$ is $Y_1(\pi) = 0$, $Y_2(\pi) = 1$, $Y_3(\pi) = 1$, $Y_4(\pi) = 1$, $Y_5(\pi) = 0$, $Y_6(\pi) = 0$, $Y_7(\pi) = 1$, and $Y_8(\pi) = 0$.

From the method of insertion and above example, it is easy to see that if $Y_{t-1}(\pi) = k$ $(t = 2, \dots, n$ and $0 \leq k \leq t-2)$, then $Y_t(\pi)$ can only be at states $k-1$, k, and $k+1$. Since the integer "t" has equal probability being inserted into any one of t positions of the sub-permutation π_{t-1} then we have the following transition probability

$$
P(Y_t = x \mid Y_{t-1} = k) = \begin{cases} \dfrac{k}{t}, & \text{if } x = k-1, \\[2mm] \dfrac{t-k-1}{t}, & \text{if } x = k, \\[2mm] \dfrac{1}{t}, & \text{if } x = k+1. \end{cases} \tag{2.2}
$$

In view of (2.2), it is clear that the sequence of random variables $\{Y_t : t \in \Gamma_n\}$ form a non-homogeneous finite Markov chain on Ω with transition matrices given by

(2.3) $M_t(n)$

$$
= \begin{array}{c|ccccccc|ccc|}
 & 0 & 1 & 2 & 3 & \cdots & & t-2 & t-1 & \cdots & n-1 \\
0 & \frac{t-1}{t} & \frac{1}{t} & & & & & & & & \\
1 & \frac{1}{t} & \frac{t-2}{t} & \frac{1}{t} & & & 0 & & & & \\
2 & & \frac{2}{t} & \frac{t-3}{t} & \frac{1}{t} & & & & & & \\
\cdot & & & \cdot & \cdot & \cdot & & & & 0 & \\
\cdot & & & & \cdot & \cdot & \cdot & & & & \\
\cdot & & 0 & & & \cdot & \cdot & \cdot & & & \\
t-3 & & & & & \frac{t-3}{t} & \frac{2}{t} & \frac{1}{t} & & & \\
t-2 & & & & & & \frac{t-2}{t} & \frac{1}{t} & \frac{1}{t} & & \\
\hline
t-1 & & & & & & & & & & \\
\vdots & & & 0 & & & & & & I_{n-t+1} & \\
n-1 & & & & & & & & & &
\end{array}
$$

for $t = 1, 2, \ldots, n$, where $M_t(n)$ has entries $p_{ij}(t : n)$,

$$
\begin{aligned} p_{ij}(t : n) &= P(Y_t = j \mid Y_{t-1} = i) \\ &\quad \text{for} \quad i = 0, 1, \ldots, t-2 \text{ and } j = 0, 1, \ldots, t-2 \end{aligned} \tag{2.4}
$$

defined by (2.2), I_{n-t+1} is a $(n-t+1) \times (n-t+1)$ identity matrix, and 0 otherwise.

Let $a(0) = (1, 0, \ldots, 0)$ and $a(t) = (a_0(t), a_1(t), \ldots, a_{n-1}(t))$ for $t = 1, \ldots, n$, be n $1 \times n$ vectors with components

$$
a_i(t) = P(Y_t = i \mid Y_0 = 0), \tag{2.5}
$$

for $i = 0, 1, \ldots, n-1$, and $t = 1, \ldots, n$.

THEOREM 2.1. *Given $P(Y_0 = 0) = 1$ and $P(Y_0 = i) = 0$ for $i = 1, \dots, n-1$, then*

$$P(X_n(\pi,2) = i) = P(Y_n = i \mid Y_0 = 0) = a(0)\left(\prod_{t=1}^{n} M_t(n)\right) U'(i), \tag{2.6}$$

where $M_t(n)$ is given by (2.3) and $U'(i) = (0 \dots 0\ 1\ 0 \dots 0)'$ is a $n \times 1$ vector with one at the i-th coordinate and zero elsewhere.

PROOF. There are $n!$ random permutations of size n. For every permutation $\pi \in F_n$, it can be decomposed uniquely as $(\pi_n, \pi_{n-1}, \dots, \pi_2, \pi_1)$. The decomposition of π can be viewed as that we start with $\pi_1 \equiv (1)$ and insert the integer $2, 3, \dots,$ and n randomly one by one into the positions between the integers and the two end positions. Hence, the decomposition $(\pi_n, \dots, \pi_2, \pi_1)$ of π is one-to-one corresponding to a realization $(\pi_1, \pi_2, \dots, \pi_n)$ of the method of inserting n integers $1, 2, \dots,$ and n one by one. Therefore, it follows from the definition of Y_t, $t = 1, \dots, n$ that we have

$$P(X_n(\pi,2) = i) \equiv P(Y_n = i \mid Y_0 = 0) \tag{2.7}$$

for all $i = 0, 1, \dots, n-1$. Since the sequence $\{Y_t : t \in \Gamma_n\}$ is a non-homogeneous Markov chain with transition probability matrices $M_t(n)$, $t = 1, \dots, n$ given by (2.3), the results follows immediately from Chapman-Kolmogorov equation (see Kemeny and Snell (1960) and Feller (1968)), the definition of Y_t, and the equation (2.3).

The equations (2.3), (2.4) and (2.5) yield the following fundamental recursive equations for the Markov chain Y_t: for $0 \le x \le n-1$,

$$\begin{aligned} a_x(n) &= a(0)\left(\prod_{t=1}^{n} M_t(n)\right) U'(x) \\ &= (a_0(n-1), a_1(n-1), \dots, a_{n-1}(n-1)) M_n(n) U'(x) \\ &= \begin{cases} \dfrac{n-1}{n} a_0(n-1) + \dfrac{1}{n} a_1(n-1), & \text{if } x = 0, \\ \dfrac{1}{n} a_{x-1}(n-1) & \\ \quad + \dfrac{n-x-1}{n} a_x(n-1) + \dfrac{x}{n} a_{x+1}(n-1), & \text{if } 1 \le x \le n-3, \\ \dfrac{1}{n} a_{n-3}(n-1) + \dfrac{1}{n} a_{n-2}(n-1), & \text{if } x = n-2, \\ \dfrac{1}{n} a_{n-2}(n-1), & \text{if } x = n-1. \end{cases} \end{aligned} \tag{2.8}$$

The recursive equation (2.8) also provides a simple and direct way of approximating the probabilities $a_x(n) = P(X_n(\pi,2) = x)$, $x = 0, 1, 2, \dots, n-1$ as n

tends to infinity. This yields that the limiting distribution of the random variable $X_n(\pi, 2)$ is Poisson with parameter $\lambda = 1$.

THEOREM 2.2. *For given $x = 0, 1, \dots,$ it follows*

$$\lim_{n\to\infty} P(X_n(\pi, 2) = x) = \frac{1}{x!}e^{-1}, \qquad x = 0, 1, \dots. \tag{2.9}$$

In order to prove the above result we need the following lemma.

LEMMA 2.1. *Given n and $a(0) = (1, 0, \dots, 0)$, it follows*

$$\text{(i) } a_x(x+1) = \frac{1}{(x+1)!}, \qquad \textit{for} \quad x = 0, 1, 2, \dots, \ \textit{and} \ (n-1), \tag{2.10}$$

$$\text{(ii) } a_x(y) = 0, \qquad \textit{for} \quad y = 1, 2, \dots, n \ \textit{and} \ n-1 \geq x \geq y. \tag{2.11}$$

PROOF. To prove this result, we use method of induction. For given $y = 1$, it follows from (2.6) or (2.8) $a_0(1) = 1$, $a_1(1) = a_2(1) = \cdots = a_{n-1}(1) = 0$. Similarly, for $y = 2$, we have $a_1(2) = 1/2!$, and $a_2(2) = a_3(2) = \cdots = a_{n-1}(2) \equiv 0$. Now, let us assume that the results hold for $y = k$, it follows from last part of equation (2.8) that for $x = y = k + 1$, we have

$$a_k(k+1) = \frac{1}{k+1}a(k) = \frac{1}{(k+1)!},$$

and for $x > y = k + 1$

$$\begin{aligned} a_x(k+1) = a_{x-1}(k)P(Y_{k+1} = x \mid Y_k = x-1) + a_x(k)P(Y_{k+1} = x \mid Y_k = x) \\ + a_{x+1}(k)P(Y_{k+1} = x \mid Y_k = x+1) = 0, \end{aligned}$$

a direct consequence of $a_k(k) = a_{k+1}(k) = a_{k+2}(k) \cdots \equiv a_{n-1}(k) \equiv 0$.

Intuitively, this lemma is trivial for the following simple reasons; (i) for $Y_{x+1} = x$, the permutation π_{x+1} has to be the form $(1, 2, \dots, x, x+1)$ hence it has probability $1/(x+1)!$ and (ii) for $t \leq x$, $Y_t = x$ could not happen at all, hence, it has probability zero.

PROOF OF THEOREM 2.2. For $x = 0$, it follows from (2.8) and some simple algebra that

$$\begin{aligned} a_0(n) &= \left(1 - \frac{1}{n}\right) a_0(n-1) + \frac{1}{n} a_1(n-1) \\ &\cdots\cdots \\ &= \left(1 - \frac{1}{n}\right)^{n-1} [a_0(1) + o(1)]. \end{aligned} \tag{2.12}$$

Furthermore, for fixed x ($x \geq 1$), it follows for (2.8) again that

$$(2.13) \qquad a_x(n) = \frac{1}{n} a_{x-1}(n-1) + \frac{(n-x-1)}{n} a_x(n-1) + \frac{x}{n} a_{x+1}(n-1)$$
$$\cdots\cdots$$
$$= \left(1 - \frac{1}{n}\right)^{n-x-1} [a_x(x+1) + o(1)].$$

Taking $n \to \infty$, the result (2.9) follows immediately from equations (2.12) and (2.13) and Lemma 2.1.

THEOREM 2.3. *For $k \geq 3$,*

$$(2.14) \qquad \lim_{n \to \infty} P(X_n(\pi, k) = x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \geq 1. \end{cases}$$

The random variable $X_n(\pi, k)$, for $k \geq 3$, has a degenerate limiting distribution at zero.

PROOF. Let, for $i = 1, \ldots, n-k+1$, $E_i =$ be the event that $\pi(i) + k - 1 = \pi(i+1) + k - 2 = \cdots = \pi(i+k-1)$. It follows from the definition of E_i that $P(E_i) = 1/n^{k-1} + o(1/n^{k-1})$, for each $i = 1, \ldots, n-1$. By Bonferroni inequality, it gives

$$(2.15) \qquad P(X_n(\pi, k) \geq 1) = P\left(\bigcup_{i=1}^{n-k+1} E_i\right) \leq \sum_{i=1}^{n-k+1} P(E_i)$$
$$= \sum_{i=1}^{n-k+1} \left(\frac{1}{n^{k-1}} + o\left(\frac{1}{n^{k-1}}\right)\right)$$
$$= \frac{1}{n^{k-2}} + o\left(\frac{1}{n^{k-2}}\right).$$

For $k \geq 3$, the result (2.14) is an immediate consequence of the above inequality.

With the following modifications of the Markov chain $\{Y_t\}$, the exact distribution of the waiting time random variable $W_k(S, 2)$ can also be obtained by using Markov chain imbedding technique. Given k, let $\{Y_t(k), t = 1, 2, \ldots\}$ be a Markov chain defined on the state space $\Omega_k = \{0, 1, \ldots, k-1, k\}$ with transition probability matrices

(2.16) $M_t(k)$

$$
= \begin{array}{c|cccccc|ccc|}
 & 0 & 1 & 2 & 3 & \cdots & t-2 & t-1 & \cdots & k \\
\hline
0 & \frac{t-1}{t} & \frac{1}{t} & & & & & & & \\
1 & \frac{1}{t} & \frac{t-2}{t} & \frac{1}{t} & & 0 & & & & \\
2 & & \frac{2}{t} & \frac{t-3}{t} & \frac{1}{t} & & & & 0 & \\
\vdots & & & \ddots & \ddots & \ddots & & & & \\
 & & 0 & & \ddots & \ddots & \ddots & & & \\
t-3 & & & & \frac{t-3}{t} & \frac{2}{t} & \frac{1}{t} & & & \\
t-2 & & & & & \frac{t-2}{t} & \frac{1}{t} & \frac{1}{t} & & \\
\hline
t-1 & & & & & & & & & \\
\vdots & & & 0 & & & & & I_{k-t+2} & \\
k & & & & & & & & & \\
\end{array}
$$

for $t \leq k+1$,

$$
= \begin{array}{c|ccccccc|}
 & 0 & 1 & 2 & 3 \cdots & k-2 & k-1 & k \\
0 & \frac{t-1}{t} & \frac{1}{t} & 0 & \cdot & \cdot & 0 & 0 \\
1 & \frac{1}{t} & \frac{t-2}{t} & \frac{1}{t} & \cdot\ \cdot & 0 & & \\
2 & 0 & \frac{2}{t} & \frac{t-3}{t} & \frac{1}{t}\ \cdot & & & \\
\vdots & & \ddots & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & \ddots & \ddots & \ddots & \\
k-1 & & & & & \frac{k-1}{t} & \frac{t-k}{t} & \frac{1}{t} \\
k & 0 & 0 & \cdot & \cdot\ \cdot & 0 & 0 & 1 \\
\end{array}
$$

for $t \geq k+2$.

The two Markov chains $\{Y_t\}$ and $\{Y_t(k)\}$ are very similar. The major difference is that the absorbing state "k" in the Markov chain $Y_t(k)$ corresponds to the states k, $k+1, \ldots,$ and $n-1$ in the Markov chain Y_t lumped together. It follows from the definition of $Y_t(k)$ and (2.16) that

(i) if $Y_t(k) \leq k-1$ it implies $Y_i(k) \leq k$ for all $i \leq t$, and

(ii) if $Y_t(k) = k$ it implies $Y_i(k) = k$, for all $i \geq t$.

Furthermore, the random variables $Y_t(k)$ and $W_k(S,2)$ are one-to-one related in the following way:

(2.17) $$W_k(S,2) \geq n \quad \text{if and only if} \quad Y_{n-1}(k) \leq k-1.$$

This yields the following theorem:

THEOREM 2.4. *For* $n \geq k+1$, $k = 1, 2, \ldots$

$$P(W_k(S,2) = n) = a(0)\left(\prod_{t=1}^{n-1} M_t(k)\right)(I - M_n(k))U'(k) \tag{2.18}$$

where $M_t(k)$ *is defined by* (2.16) *and* $U(k) = (1, \ldots, 1, 0)$.

PROOF. It follows from (2.16) and (2.17) that

$$\begin{aligned} P(W_k(S,2) = n) &= P(W_k(S,2) \geq n) - P(W_k(S,2) \geq n+1) \\ &= P(Y_{n-1}(k) \leq k-1) - P(Y_n(k) \leq k-1) \\ &= a(0)\left(\prod_{t=1}^{n-1} M_t(k)\right)(I - M_n(k))U'(k). \end{aligned}$$

This completes our proof.

For $k = 1$, the exact distribution of the waiting time of the first succession of size 2, $W_1(S,2)$, can also be determined by the following recursive equation:

$$P(W_1(S,2) = n) = \frac{1}{n}\left(1 - \sum_{i=1}^{n-1} P(W_1(S,2) = i)\right), \qquad n > 1 \tag{2.19}$$

with $P(W_1(S,2) = 1) \equiv 0$. This can be proved in the following way. By the definition of $W_1(S,2)$ it is clear that $P(W_1(S,2) = 1) = 0$ and $P(W_1(S,2) = 2) = 1/2$. If $W_1(S,2) = n$, it means that no succession of size 2 has occurred until the stage $(n-1)$ and at stage n, that the "n" is inserted to the right of "$n-1$" in the insertion process (with probability $1/n$), thus equation (2.19) holds.

For $k = 1$, the matrices $M_t(k)$ given by (2.16) reduce to $M_1(1) = I$ and

$$M_t(1) = M_t(1) = \begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} \frac{t-1}{t} & \frac{1}{t} \\ 0 & 1 \end{bmatrix} \quad \text{for} \quad t \geq 2. \tag{2.20}$$

Since, by Theorem 4.2,

$$\begin{aligned} \sum_{i=1}^{n-1} P(W_1(S,2) = i) &= \sum_{i=1}^{n-1} (1,0)\left(\prod_{t=1}^{i-1} M_t(1)\right)(I - M_i(1))(1,0)', \\ &= 1 - (1,0)\left(\prod_{t=1}^{n-1} M_t(1)\right)(1,0)', \end{aligned} \tag{2.21}$$

then the recursive equation (2.19) follows immediately from Theorem 4.2, (2.21) and

$$P(W_1(S,2)=n) = (1,0)\left(\prod_{t=1}^{n-1} M_t(1)\right)(I - M_n(1))(1,0)'$$
$$= \frac{1}{n}(1,0)\left(\prod_{t=1}^{n-1} M_t(1)\right)(1,0)'.$$

It follows, for example, $P(W_1(S,2)=2)=1/2$, $P(W_1(S,2)=3)=1/6$ and $P(W_1(S,2)=4)=1/12$ ((3421) and (2134) are the only two permutations with $W_1(S,2)=4$ among 24 permutations of four integers 1, 2, 3 and 4).

3. Numerical results and discussions

The Markov chain approach of finding the exact distribution of successions differs greatly with the combinatorial approach. This method is rather simple and direct, both in concept and computation. It also provides a potential to handle other patterns on a permutation. To illustrate the Theorems 2.1 and 2.2, we give the following numerical results.

Table 1. The exact distribution of $X_n(\pi, 2)$.

$x\backslash n$	3	5	7	10	15	20
0	.500	.44167	.420437	.40466700	.3924050	.38627300
1	.333	.36667	.367857	.36787900	.3678790	.36787900
2	.167	.15000	.157738	.16554600	.1716770	.17474300
3		.03333	.043651	.04905090	.0531381	.05518179
4		.00833	.008929	.01072920	.0122626	.01302910
5			.001190	.00184028	.0022482	.00245253
6			.000198	.00025463	.0003406	.00038321
7				.00002976	.0000438	.00051093
8				$.248\times10^{-5}$	$.4867\times10^{-5}$	$.59306\times10^{-5}$
9				$.276\times16^{-6}$	$.4731\times10^{-6}$	$.60826\times10^{-6}$
10					$.4057\times10^{-7}$	$.55758\times10^{-7}$
11					$.3062\times10^{-8}$	$.46081\times10^{-8}$
12					$.2088\times10^{-9}$	$.34560\times10^{-9}$
13					$.1071\times10^{-10}$	$.2363\times10^{-10}$
14					$.7647\times10^{-12}$	$.1477\times10^{-11}$
15						$.8444\times10^{-13}$
16						$.4381\times10^{-14}$
17						$.2109\times10^{-15}$
18						$.7810\times10^{-17}$
19						$.4110\times10^{-18}$

The numerical evaluation of the exact distribution of $X_n(\pi, 2)$ is rather straightforward. The computation was done on a PC computer with Mathematica. The CPU time for each case takes only a few seconds. The numerical results manifest that the rate of $X_n(\pi, 2)$ converges to the Poisson random variable with parameter $\lambda = 1$ rather fast. The results of the cases $n = 3$ and $n = 5$ could be checked out easily by hand computation. For instance the case of $n = 3$, $P(X_5(\pi, 2) = 0) = 0.5$ which is the case $\{(1, 3, 2), (2, 1, 3)$ and $(3, 2, 1)\}$, $P(X_5(\pi, 2) = 1) = .333$ which is the case $\{(3, 1, 2)$ and $(2, 3, 1)\}$, and $P(X_5(\pi, 2) = 2) = 0.1167$ which is the case $\{(1, 2, 3)\}$.

The main two intuitive reasons for the random variable $X_n(\pi, 2)$ convergence to a Poisson random variable with $\lambda = 1$ are (a) the stochastic dependency of any two index functions disappears gradually as $n \to \infty$ and (b) the probability of the event $1_n(i, \pi, 2) = 1$ is approximately $1/n$. Since the number of successions of size 2 on a random permutation π, $X_n(\pi, 2) = \sum_{i=1}^{n-1} 1_n(i, \pi, 2)$, is a sum of $(n - 1)$ dependent index functions $1_n(i, \pi, 2)$, $i = 1, \ldots, n - 1$, we believe, in view of the facts (a) and (b), that the standard method such as Stem-Chan method could also be used to prove the Poisson convergence.

For the number of circular successions of size 2 $X_n^*(\pi, 2)$, it follows from the definitions of $X_n^*(\pi, 2)$ and $X_n(\pi, 2)$ that the relationship

$$\begin{aligned} P(X_n^*(\pi, 2) = x) &= P(X_n(\pi, 2) = x - 1, 1_n(n, 2, \pi) = 1) \\ &\quad + P(X_n(\pi, 2) = x, 1_n(n, 2, \pi) = 0) \\ &\doteq P(X_n(\pi, 2) = x - 1)\frac{1}{n-1} + P(X_n(\pi, 2) = x)\frac{n-2}{n-1} \end{aligned}$$

holds for all x. Taking $n \to \infty$, it shows that the circular successions $X_n^*(\pi, 2)$ also has a same limiting distribution as $X_n(\pi, 2)$, a Poisson distribution with parameter $\lambda = 1$.

Acknowledgements

The author wishes to thank Dr. C. C. Chao for suggesting this problem and his helpful comments throughout the preparation of this manuscript. The author also wishes to thank the Associate Editor and referee for their suggestions and comments, especially Theorem 2.4.

References

Abramson, M. and Moser, W. O. J. (1967). Permutations without rising of falling w-sequences, *Ann. Math. Statist.*, **38**, 1245–1254.

Dwass, M. (1973). The number of increases in a random permutation, *J. Combin. Theory Ser. A*, **15**, 192–199.

Dymacek, W. M. and Roselle, D. P. (1978). Circular permutations by numbers of rises and successions, *J. Combin. Theory Ser. A*, **25**, 196–201.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Wiley, New York.

Fu, J. C. (1994). *Distribution Theory of Runs and Patterns Associated with a Sequence of Multi-State Trials*, Tech. Report, No. 152, University of Manitoba, Canada.

Fu, J. C. and Koutras, M. V. (1994). Distribution theory of runs: a Markov chain approach, *J. Amer. Statist. Assoc.*, **89**, 1050–1057.

Jackson, D. M. and Aleliunas, R. (1977). Decomposition based generating functions for sequences, *Canada. J. Math.*, **29**, 971–1009.

Jackson, D. M. and Reilly, J. W. (1976). Permutations with a prescribed number of p-runs, *Ars Combin.*, **1**, 297–305.

Kaplansky, I. (1944). Symbolic solutions of certain problems in permutations, *Bull. Amer. Math. Soc.*, **50**, 906–914.

Kaplansky, I. (1945). The asymptotic distribution of runs of consecutive elements, *Ann. Math. Statist.*, **16**, 200–203.

Kemeny, J. G. and Snell, J. L. (1960). *Finite Markov Chains*, Van Nostrand, Princeton, New Jersey.

Reilly, J. W. and Tanny, S. M. (1979). Counting successions in permutations, *Stud. Appl. Math.*, **61**, 73–81.

Riordan, J. (1945). Permutations without 3-sequences, *Bull. Amer. Math. Soc.*, **51**, 745–748.

Riordan, J. (1958). *An Introduction to Combinatorial Analysis*, Wiley, New York.

Riordan, J. (1965). A recurrence for permutations without rising or falling successions, *Ann. Math. Statist.*, **36**, 745–748.

Roselle, D. P. (1968). Permutations by number of rises and successions, *Proc. Amer. Math. Soc.*, **19**, 8–16.

Tanny, S. M. (1976). Permutations and successions, *J. Combin. Theory Ser. A*, **21**, 196–202.

Whitworth, W. A. (1943). *Choice and Chance*, Strechert, New York.