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COMPLETE CLASSES OF TESTS FOR REGULARLY VARYING DISTRIBUTIONS

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Abstract. This paper establishes essentially complete class theorems and gives conditions for admissibility of tests for parametric families of distributions having some kinds of regular variation properties. A complete treatment of topologically contiguous one-dimensional bounded and unbounded hypotheses is given. Examples of applications to well known families of distributions are presented.

Key words and phrases: Essentially complete class, admissibility, Bayes test, regular variation.

1. Introduction

Most of the research in the field of complete classes of tests has been devoted to testing problems involving exponential families of distributions and there are few results for non-exponential cases. In this paper we consider problems of testing one-dimensional hypotheses for families of distributions whose densities vary asymptotically at a lower rate than exponential families in the sense given by Assumption 2.3, which is a generalisation of Karamata's (1930) definition of regular variation. Well known examples of such families are the location families of the double exponential, logistic and Cauchy distributions and the scale family of the Cauchy distributions. Other examples are provided by the location families of Rider's (1957) generalised Cauchy distribution, Gumbel's (1944) generalised logistic distribution, Talacko's (1956) hyperbolic secant distribution and the scale family of generalised Cauchy distributions. The generalised logistic and hyperbolic secant distributions belong to Perks' (1932) family of distributions. For the definitions and basic properties of the distributions mentioned above, see Johnson and Kotz (1970). If a null or alternative parameter space is unbounded or if these spaces are topologically contiguous an explicit characterisation of a proper essentially complete class is usually difficult. Lehmann (1947), Birnbaum (1955), Matthes and Truax (1967), Farrell (1968), Ghia (1976), Eaton (1970) and Marden (1982) present certain complete class theorems for a simple null hypothesis for cases such as exponential families of distributions, separated null and alternative parameter spaces or alternative spaces contained in a cone. A quite general case of a simple null hypothesis versus an arbitrary alternative is considered in Kudô (1961) but explicit forms of tests constituting a complete class are not presented. Only the result by Brown and Marden (1989) yields a satisfactory characterisation of such tests.

We assume a sample space $\mathcal{X}$, a σ-field $\mathcal{F}$ of subsets of $\mathcal{X}$, and a family of distributions $\{P_\theta : \theta \in \Theta\}$ on $\mathcal{F}$, where the parameter space Θ is a subset of the real line. We also assume that every P_θ has a probability density function (PDF) f_θ with respect to a certain σ-finite measure μ on $\mathcal{F}$ and all f_θ have the same support. A test of a null hypothesis $H_0 : \theta \in \Theta_0 \subseteq \boldsymbol{R}$ versus an alternative $H_A : \theta \in \Theta_A \subseteq \boldsymbol{R}$ is represented by a measurable function $\phi : \mathcal{X} \to [0,1]$ and its risk function is defined by

$$r_\theta(\phi) = \begin{cases} E_\theta(\phi), & \text{if } \theta \in \Theta_0 \\ 1 - E_\theta(\phi), & \text{if } \theta \in \Theta_A. \end{cases} \tag{1.1}$$

In Sections 2, 3 and 4 we give complete class theorems for simple null versus two-sided alternative, non-simple null versus two-sided alternative and one-sided null versus one-sided alternative hypothesis testing problems, respectively. Section 5 is devoted to applications of the notions of regular variation and (in our present terminology) additive regular variation to problems of testing scale or location parameters. In Section 6 we give some conditions for admissibility of tests from the essentially complete classes. In Section 7, we introduce the notion of locally best at infinity tests which are the most sensitive to big departures from a null hypothesis. Section 8 contains examples of test procedures from the complete classes obtained and characterisations of their properties.

2. Simple null hypothesis

In this section we deal with simple null hypotheses and two-sided alternatives under an asymptotic assumption that admits the two-point compactification of the real line. An essentially complete class theorem, proven here, could be obtained as a corollary of the result of Brown and Marden (1989) by an appropriate reparameterisation of the parameter space as a bounded subset of the real line. However, using similar methods we give a simpler proof of our theorem which sets the stage for the next two sections where we deal with non-simple null hypotheses. Before formulating the main theorem of this section, we state some assumptions and definitions similar to those of Brown and Marden (1989).

For a Borel set $\Omega \subseteq \boldsymbol{R}$ denote by $\mathcal{P}(\Omega)$ and $\mathcal{F}(\Omega)$ the sets of all probability and finite Borel measures on Ω, respectively.

Consider testing the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_A : \theta \in \Theta_A \subseteq \{\theta : \theta \neq \theta_0\}, \tag{2.1}$$

where θ_0 is a left and right limit point of Θ_A. Take real numbers θ_1', θ_2' such that $\theta_1' < \theta_0 < \theta_2'$ and define sets Θ_1 and Θ_2 by

$$\Theta_1 = [\theta_1', \theta_2'] \cap \Theta_A \quad \text{and} \quad \Theta_2 = [\theta_1', \theta_2']^c \cap \Theta_A.$$

ASSUMPTION 2.1. A family of PDFs $\{f_\theta(x)\}$ is such that $f_\theta(x)$ is a continuous function of the parameter θ for μ-almost all $x \in \mathcal{X}$. There exists a positive function $\alpha(\theta)$ such that $R_\theta(x) \equiv \alpha(\theta) f_\theta(x)$ can be extended to a positive function of θ on the closure $\bar{\Theta}$ of the parameter space Θ, which is continuous except possibly at θ_1' and θ_2', and which is right-continuous at θ_1' and left-continuous at θ_2', and the limits $\lim_{\theta \to \theta_1' -} R_\theta(x)$ and $\lim_{\theta \to \theta_2' +} R_\theta$ exist and are finite for μ-almost all x.

From now on we use the convention that whenever we integrate $R_\theta(x)$ over the set $\bar{\Theta}_2$ we put

$$R_{\theta_1'}(x) = \lim_{\theta \to \theta_1' -} R_\theta(x) \quad \text{and} \quad R_{\theta_2'}(x) = \lim_{\theta \to \theta_2' +} R_\theta(x).$$

ASSUMPTION 2.2. For μ-almost all x there exists a neighbourhood U_x of θ_0 such that $R_\theta(x)$ has continuous first and second partial derivatives with respect to θ at every point of U_x.

ASSUMPTION 2.3. There exist positive functions $R_+(x)$ and $R_-(x)$ such that

$$\lim_{\theta \to +\infty} R_\theta(x) = R_+(x) \quad \text{and} \quad \lim_{\theta \to -\infty} R_\theta(x) = R_-(x)$$

for μ-almost all $x \in \mathcal{X}$.

Assumption 2.2 is used to deal with the problem of topological contiguity of Θ_0 and Θ_A. The purpose of Assumption 2.3 is to treat unbounded alternative hypotheses. If Assumptions 2.1–2.3 are satisfied, for a certain function $\alpha(\theta)$, then by considering $R_\theta(x)/R_\theta(x_0)$ we can see that these assumptions hold with $\alpha(\theta)$ defined by $f_\theta(x_0)^{-1}$ for μ-almost all x_0. For convenience, in the case of testing problems involving location families, we usually take functions $\alpha(\theta)$ defined by

$$\alpha(\theta) = \begin{cases} f_\theta(0)^{-1}, & \text{if } \theta \in \Theta_2, \\ 1, & \text{otherwise.} \end{cases} \tag{2.2}$$

Define, for μ-almost all $x \in \mathcal{X}$, the following function of θ on $\bar{\Theta}_1$:

$$D_\theta(x) = \begin{cases} \dfrac{R_\theta(x) - R_{\theta_0}(x) - \left.\dfrac{\partial R_\theta(x)}{\partial \theta}\right|_{\theta=\theta_0} (\theta - \theta_0)}{(\theta - \theta_0)^2}, & \text{if } \theta \neq \theta_0, \\ \dfrac{1}{2} \left.\dfrac{\partial^2 R_\theta(x)}{\partial \theta^2}\right|_{\theta=\theta_0}, & \text{if } \theta = \theta_0. \end{cases} \tag{2.3}$$

Let $\Gamma = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_3 \geq 0, |\alpha_1| + |\alpha_2| + \alpha_3 = 1\}$. For $F \in \mathcal{P}(\Theta_1)$, $G \in \mathcal{F}(\Theta_2)$, $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, set

$$\begin{aligned} d(x) = {} & \alpha_1 R_{\theta_0}(x) + \alpha_2 \frac{\partial}{\partial \theta} R_\theta(x)\bigg|_{\theta=\theta_0} + \alpha_3 \int_{\bar{\Theta}_1} D_\theta(x) F(d\theta) \\ & + \int_{\bar{\Theta}_2} R_\theta(x) G(d\theta) + \beta_1 R_+(x) + \beta_2 R_-(x). \end{aligned} \tag{2.4}$$

Under Assumptions 2.1–2.3, we define Φ to be the class of all tests of the form

$$\phi(x) = \begin{cases} 1, & \text{if } d(x) > 0, \\ 0, & \text{if } d(x) < 0, \ \mu\text{-a.e.} \end{cases} \tag{2.5}$$

THEOREM 2.1. *If Assumptions* 2.1–2.3 *are satisfied, then* Φ *is an essentially complete class of tests for testing hypotheses* (2.1).

PROOF. Let $\{\pi_i\} \subseteq \mathcal{P}(\Theta)$ be a sequence of prior probabilities concentrated on finite sets and let

$$\phi_i(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{as} \quad \int_{\Theta_A} f_\theta(x)\pi_i(d\theta) - f_{\theta_0}(x)\pi_i(\{\theta_0\}) \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \tag{2.6}$$

be the corresponding Bayes tests. By the complete class theorem of Wald (1950), all limits in the regular sense (which is equivalent to weak* convergence in $L^\infty(\mathcal{X}, \mathcal{F}, \mu)$) of convergent sequences of tests (2.6) constitute an essentially complete class. Assume that a sequence (2.6) converges in the weak* sense to a test ϕ. We shall prove that $\phi \in \Phi$ by showing that, there exists a subsequence $\{i_s\} \subseteq \{i\}$, such that the sequence $\{\phi_{i_s}\}$ converges pointwise to ϕ on the set $\{d(x) \neq 0\}$ for a certain function $d(x)$ of the form (2.4).

We can assume that $\pi_i(\{\theta_0\}) > 0$ for almost all i. Otherwise, it is possible to extract a subsequence $\{i_s\}$ of $\{i\}$ such that $\phi_{i_s}(x) = 1$ μ-a.e., and therefore the limit test $\phi(x) = 1$ μ-a.e. belongs to Φ.

If we set $\pi_i'(d\theta) = \alpha(\theta)^{-1}\pi_i(d\theta)$, and use $D_\theta(x)$ defined by (2.3), then

$$\begin{aligned} \int_{\Theta_A} f_\theta(x)\pi_i(d\theta) - f_{\theta_0}(x)\pi_i(\{\theta_0\}) &= R_{\theta_0}(x)[\pi_i'(\Theta_1) - \pi_i'(\{\theta_0\})] \\ &\quad + \int_{\Theta_1} \frac{\partial}{\partial\theta}R_\theta(x)\bigg|_{\theta=\theta_0} (\theta - \theta_0)\pi_i'(d\theta) \\ &\quad + \int_{\Theta_1} D_\theta(x)(\theta - \theta_0)^2\pi_i'(d\theta) \\ &\quad + \int_{\Theta_2} R_\theta(x)\pi_i'(d\theta). \end{aligned} \tag{2.7}$$

For $i = 1, \ldots$ define probability measures $F_i \in \mathcal{P}(\Theta_1)$ by

$$F_i(d\theta) = \begin{cases} \dfrac{I_{\Theta_1}(\theta)(\theta - \theta_0)^2\pi_i'(d\theta)}{\int_{\Theta_1}(\theta - \theta_0)^2\pi_i'(d\theta)}, & \text{if } \pi_i'(\Theta_1) > 0, \\ \pi(d\theta), & \text{otherwise,} \end{cases} \tag{2.8}$$

where I_A denotes the indicator function of a set A and π is an arbitrary element of $\mathcal{P}(\Theta_1)$. Using the probability measures (2.8), we can rewrite the right-hand side of (2.7) as

$$\begin{aligned} &[\pi_i'(\Theta_1) - \pi_i'(\{\theta_0\})]R_{\theta_0}(x) + \int_{\Theta_1} (\theta - \theta_0)\pi_i'(d\theta) \frac{\partial}{\partial\theta}R_\theta(x)\bigg|_{\theta=\theta_0} \\ &\quad + \int_{\Theta_1} (\theta - \theta_0)^2\pi_i'(d\theta) \int_{\Theta_1} D_\theta(x)F_i(d\theta) + \int_{\Theta_2} R_\theta(x)\pi_i'(d\theta). \end{aligned} \tag{2.9}$$

Set

$$S_i = |\pi_i'(\Theta_1) - \pi_i'(\{\theta_0\})| + \left| \int_{\Theta_1} (\theta - \theta_0)\pi_i'(d\theta) \right| + \int_{\Theta_1} (\theta - \theta_0)^2 \pi_i'(d\theta). \tag{2.10}$$

Since $\pi_i(\{\theta_0\}) > 0$ for almost all i, $S_i > 0$ for almost all i.

Define measures $G_i \in \mathcal{F}(\Theta_2)$ by $G_i(d\theta) = I_{\Theta_2}(\theta) S_i^{-1} \pi_i'(d\theta)$. Dividing the expression (2.9) by S_i for each i such that $S_i > 0$ yields

$$\alpha_1^{(i)} R_{\theta_0}(x) + \alpha_2^{(i)} \frac{\partial}{\partial\theta} R_\theta(x) \Big|_{\theta=\theta_0} + \alpha_3^{(i)} \int_{\Theta_1} D_\theta(x) F_i(d\theta) + \int_{\Theta_2} R_\theta(x) G_i(d\theta) \tag{2.11}$$

for a certain triple $(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}) \in \Gamma$.

We now make the convention that whenever in the further course of the proof we assume that a sequence is convergent we mean that it is possible to pass to a convergent subsequence which can be relabeled as the original sequence.

Each F_i is a probability concentrated on $\bar{\Theta}_1$ so that the sequence $\{F_i\}$ is tight. Hence, by Prohorov's theorem, we may assume that $F_i \to F$ weakly. Further the set Γ is compact, hence, again we may assume that $(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}) \to (\alpha_1, \alpha_2, \alpha_3) \in \Gamma$. We treat each G_i as a measure on the two-point compactification $\bar{\Theta}_2 \cup \{-\infty, +\infty\}$ of $\bar{\Theta}_2$, which we denote by $c(\Theta_2)$. If $\limsup_{i\to\infty} G_i(\Theta_2) = +\infty$, then we may assume that $\lim_{i\to\infty} G_i(\Theta_2) = +\infty$. Since $R_\theta(x)$ is separated from 0, the expression (2.11) diverges to infinity and $\phi_i(x) \to 1$ for μ-almost all $x \in \mathcal{X}$. In the case when $\limsup_{i\to\infty} G_i(\Theta_2) < +\infty$, we can assume that for some measure $\bar{G} \in \mathcal{F}(c(\Theta_2))$, $G_i \to$ weakly. By Assumption 2.3, we can extend $R_\theta(x)$ to a function $\bar{R}_\theta(x)$ of θ on $c(\Theta_2)$ by putting $\bar{R}_{-\infty}(x) = R_-(x)$ and $\bar{R}_{+\infty}(x) = R_+(x)$. Since $\bar{R}_\theta(x)$ is a bounded and continuous function of θ on $c(\Theta_2)$, the sequence (2.11) converges μ-a.e. to (2.4) and the sequence of tests corresponding to (2.10) converges μ-a.e. to the given test ϕ on the set $\{d(x) \neq 0\}$. Coefficients β_1, β_2 in (2.4) satisfy $\beta_1 = \bar{G}(+\infty)$, $\beta_2 = \bar{G}(-\infty)$ and the measure G in (2.4) is the restriction of $\bar{G}$ to $\bar{\Theta}_2$. □

The set Φ is independent of a particular choice of θ_1' and θ_2'.

Note that it is not clear whether all measures $F \in \mathcal{P}(\bar{\Theta}_1)$, $G \in \mathcal{F}(\bar{\Theta}_2)$, triples $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$ and numbers $\beta_1 \geq 0$, $\beta_2 \geq 0$ which are used in the definition of the class Φ, can be attained by the limiting process considered in the proof of Theorem 2.1. The following theorem gives a solution to this problem:

THEOREM 2.2. *Under Assumptions* 2.1–2.3, *for each* $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $F \in \mathcal{P}(\bar{\Theta}_1)$ *and* $G \in \mathcal{F}(\bar{\Theta}_2)$, *the corresponding test* (2.5) *is a weak* limit of a certain sequence of Bayes tests with respect to prior probabilities concentrated on finite sets.*

PROOF. Note that, the expression (2.11) is invariant with respect to rescaling measures π_i. Hence, we may assume that π_i' can be any arbitrary finite measure concentrated on finite points.

Let $F \in \mathcal{P}(\bar{\Theta}_1)$. There exists a sequence $\{H_i\} \subseteq \mathcal{P}(\Theta_1)$ of probabilities concentrated on finite sets of points, that converges weakly to the measure F, see Billingsley (1968). For $i = 1, \ldots$ define probabilities $F_i = p_i H_i + \frac{q_i}{2}(\delta_{\xi_i} + \delta_{\vartheta_i})$, where δ_γ denotes the probability concentrated in the point γ, $\xi_i \in [\theta_1', \theta_0)$, $\vartheta_i \in (\theta_0, \theta_2']$ and p_i, q_i are non-negative numbers such that $p_i + q_i = 1$ and $q_i \to 0$.

Denote by T_i the support of F_i and assume that the number of elements in T_i is n_i. We can think of (2.8) as a system of n_i linear equations with respect to variables $\pi_i'(\theta)$ for $\theta \in T_i$. Since for $\theta \in T_i$, $F_i(\theta) > 0$ and $\sum_{\theta \in T_i} F_i(\theta) = 1$ this system has positive solutions and therefore there exists a measure π_i' such that (2.8) holds.

Extend now every measure π_i' by putting a certain mass $\pi_i'(\{\theta_0\})$ at θ_0. We can rewrite the first two coefficients in (2.9) in the following way:

$$\pi_i'(\Theta_1) - \pi_i'(\{\theta_0\}) = \int_{\Theta_1} \frac{F_i(d\theta)}{(\theta-\theta_0)^2} \int_{\Theta_1} (\theta-\theta_0)^2 \pi_i'(d\theta) - \pi_i'(\{\theta_0\}),$$

$$\int_{\Theta_1} (\theta-\theta_0)\pi_i'(d\theta) = \int_{\Theta_1} \frac{F_i(d\theta)}{(\theta-\theta_0)} \int_{\Theta_1} (\theta-\theta_0)^2 \pi_i'(d\theta).$$

Set

$$x_i = \int_{\Theta_1} \frac{F_i(d\theta)}{(\theta-\theta_0)}, \qquad y_i = \int_{\Theta_1} \frac{F_i(d\theta)}{(\theta-\theta_0)^2} - \frac{\pi_i'(\{\theta_0\})}{\int_{\Theta_1}(\theta-\theta_0)^2 \pi_i'(d\theta)}$$

and rewrite the coefficients $\alpha_k^{(i)}$ in (2.11) as $\alpha_1^{(i)} = y_i z_i^{-1}$, $\alpha_2^{(i)} = x_i z_i^{-1}$, $\alpha_3^{(i)} = z_i^{-1}$, where $z_i = |y_i| + |x_i| + 1$. It is easy to see that, by a suitable choice of ξ_i, ϑ_i and the proportion between $\pi_i'(\{\theta_0\})$ and $\pi_i'(\Theta_1)$, x_i and y_i can be any real numbers and therefore any triple from the set $\Gamma' = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_3 > 0, |\alpha_1| + |\alpha_2| + \alpha_3 = 1\}$, which is a dense subset of Γ, is attainable. On the other hand, since $q_i \to 0$ as $i \to \infty$, we can see that the sequence $\{F_i\}$ converges weakly to F. Therefore any triple $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$ and any probability measure $F \in \mathcal{P}(\bar{\Theta}_1)$ can be attained in the limit as $i \to \infty$.

To complete the proof, we extend each measure π_i' concentrated on $\Theta_1 \cup \{\theta_0\}$ to Θ by setting

$$\pi_i'(d\theta) = S_i[G_i(d\theta) + \beta_1 \delta_{(\theta_2'+i)}(d\theta) + \beta_2 \delta_{(\theta_1'-i)}(d\theta)],$$

where $\{G_i\}$ is a sequence of finite measures, concentrated on finite subsets of Θ_2, that converges weakly to the measure $G \in \mathcal{F}(\bar{\Theta}_2)$ and S_i is defined by (2.10). □

Remark 2.1. It may happen that

$$\left. \frac{\partial}{\partial\theta} R_\theta(x) \right|_{\theta_0} = 0 \qquad \mu\text{-a.e.,} \tag{2.12}$$

see Remark 2.2 in Brown and Marden (1989). Examples of such cases shall be given in Remark 2.3. If (2.12) holds, then by letting $\alpha_2 = 1$, $\beta_1 = \beta_2 = 0$ and letting a measure G be the zero measure in (2.4), we have that every test belongs to class Φ, and therefore Theorem 2.1 does not yield a proper essentially complete class. However, if we follow the proof until (2.9), then we can see that the second term containing the first partial derivative is zero μ-a.e. Eliminating the term $|\int_{\Theta_1}(\theta-\theta_0)\pi_i'(d\theta)|$ from (2.10) and proceeding further with the proof, we find that a new essentially complete class Φ consists of tests of the form (2.5) with $d(x)$ defined by (2.4) such that $\alpha_2 = 0$.

It is easy to see that the counterpart of Theorem 2.2 is true for this case.

Remark 2.2. If H_A is a one-sided hypothesis, then Theorems 2.1 and 2.2 can be easily adapted to this case by the following alterations: measures F and G in (2.4) are concentrated on the right side of θ_0 and $\Gamma = \{(\alpha_1,\alpha_2,\alpha_3) : \alpha_3 \geq 0, \alpha_2 \geq 0, |\alpha_1| + \alpha_2 + \alpha_3 = 1\}$ if H_A is right-sided, and measures F and G in (2.4) are concentrated on the left side of θ_0 and $\Gamma = \{(\alpha_1,\alpha_2,\alpha_3) : \alpha_3 \geq 0, \alpha_2 \leq 0, |\alpha_1| - \alpha_2 + \alpha_3 = 1\}$ if H_A is left-sided.

Remark 2.3. Consider testing the simple null hypothesis $H_0 : \theta = 0$ versus $H_A : \theta \neq 0$ for the family of PDFs of the form

$$f_\theta(x) = f(x_1 - \theta)\cdots f(x_n - \theta), \tag{2.13}$$

where $x = (x_1,\ldots,x_n)$, $\theta \in \boldsymbol{R}$ and f is an even positive differentiable almost everywhere PDF such that for every $s \in \boldsymbol{R}\lim_{\theta\to+\infty} f(s-\theta)/f(\theta) = R(s)$, for some function $R : \boldsymbol{R} \to \boldsymbol{R}$.

This testing problem is invariant under the two-element group $\mathcal{G}$ of transformations consisting of the identity mapping and the symmetry about the origin.

Let

$$g_\theta(x) = \frac{f_\theta(x) + f_\theta(-x)}{2}. \tag{2.14}$$

Consider testing of $H_0 : \theta = 0$ versus $H_A : \theta > 0$ for the family defined by (2.14), i.e. the reduction by invariance of the original two-sided testing problem. If we take the function $\alpha(\theta)$ defined by (2.2) with $\theta_1' = -a$ and $\theta_2' = a$ for some $a > 0$, then Assumptions 2.1–2.3 are satisfied.

Note that, since $f_\theta(-x) = f_{-\theta}(x)$, we have that

$$\left.\frac{\partial g_\theta(x)}{\partial\theta}\right|_{\theta=0} = 0 \quad\text{and}\quad \left.\frac{\partial^2 g_\theta(x)}{\partial\theta_2}\right|_{\theta=0} = \left.\frac{\partial^2 f_\theta(x)}{\partial\theta_2}\right|_{\theta=0}.$$

To determine an essentially complete class Φ_{inv} for invariant tests, we follow Remarks 2.1 and 2.2 and find that

$$\begin{aligned} d(x) = {} & \alpha_1 f_0(x) + \alpha_3\int_0^a D_\theta(x)F(d\theta) \\ & + \int_a^{+\infty}\frac{g_\theta(x)}{f_\theta(0)}G(d\theta) \\ & + \beta[R(x_1)\cdots R(x_n) + R(-x_1)\cdots R(-x_n)], \end{aligned} \tag{2.15}$$

where

$$D_\theta(x) = \begin{cases} \dfrac{g_\theta(x) - g_0(x)}{\theta^2}, & \text{if } \theta \neq 0, \\ \dfrac{1}{2} \dfrac{\partial^2 f_\theta(x)}{\partial \theta^2}\bigg|_{\theta=0}, & \text{if } \theta = 0, \end{cases} \tag{2.16}$$

and $\alpha_3 \geq 0$, $|\alpha_1| + \alpha_3 = 1$, $\beta \geq 0$, $F \in \mathcal{P}([0,a])$, $G \in \mathcal{F}([a,+\infty))$.

Certain examples of PDFs of the form (2.13) are given closer attention in Section 5.

3. Bounded non-simple null hypotheses

In this section we are interested in testing the hypothesis

$$H_0 : \theta \in \Theta_0 \subseteq [\theta_1, \theta_2] \quad \text{versus} \quad H_A : \theta \in \Theta_A \subseteq [\theta_1, \theta_2]^c \tag{3.1}$$

where $\theta_1 < \theta_2$ and θ_1, θ_2 are limit points of Θ_0 and Θ_A. For real numbers θ_0, θ_1', θ_2' such that $\theta_1' < \theta_1 < \theta_0 < \theta_2 < \theta_2'$ we define sets:

$$\begin{aligned} &\Theta_1 = [\theta_1', \theta_1) \cap \Theta_A, \qquad \Theta_2 = [\theta_1, \theta_0] \cap \Theta_0, \qquad \Theta_3 = (\theta_0, \theta_2] \cap \Theta_0, \\ &\Theta_4 = (\theta_2, \theta_2'] \cap \Theta_A \quad \text{and} \quad \Theta_5 = [\theta_1', \theta_2']^c \cap \Theta_A. \end{aligned} \tag{3.2}$$

Two of the assumptions of this section are identical to the assumptions of the previous section but we repeat them to simplify cross-referencing.

Assumption 3.1. This assumption is identical to Assumption 2.1.

Assumption 3.2. $R_\theta(x)$ has first partial derivatives with respect to θ at θ_1 and θ_2 for μ-almost all $x \in \mathcal{X}$.

Assumption 3.3. This assumption is identical to Assumption 2.3.

From now on we use the convention that whenever we integrate $R_\theta(x)$ over the set $\bar{\Theta}_5$ we put

$$R_{\theta_1}(x) = \lim_{\theta \to \theta_1' -} R_\theta(x) \quad \text{and} \quad R_{\theta_2}(x) = \lim_{\theta \to \theta_2' +} R_\theta(x).$$

For μ-almost all $x \in \mathcal{X}$ and $k = 1, 2$ we define the following functions of θ on $\bar{\Theta}_{2k-1} \cup \bar{\Theta}_{2k}$:

$$D_\theta^{(k)}(x) = \begin{cases} \dfrac{R_\theta(x) - R_{\theta_k}(x)}{\theta - \theta_k}, & \text{if } \theta \neq \theta_k, \\ \dfrac{\partial R_\theta(x)}{\partial \theta}\bigg|_{\theta=\theta_k}, & \text{if } \theta = \theta_k. \end{cases} \tag{3.3}$$

Let $\Gamma = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_3 \geq 0, \alpha_4 \geq 0, |\alpha_1| + |\alpha_2| + \alpha_3 + \alpha_4 = 1\}$. For $F_1 \in \mathcal{P}(\bar\Theta_1 \cup \bar\Theta_2)$, $F_2 \in \mathcal{P}(\bar\Theta_3 \cup \bar\Theta_4)$, $G \in \mathcal{F}(\bar\Theta_5)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Gamma$ and $\beta_1, \beta_2 \geq 0$, define

$$\begin{aligned} d(x) = {} & \alpha_1 R_{\theta_1}(x) + \alpha_2 R_{\theta_2}(x) - \alpha_3 \int_{\bar\Theta_1 \cup \bar\Theta_2} D^{(1)}_\theta(x) F_1(d\theta) \\ & + \alpha_4 \int_{\bar\Theta_3 \cup \bar\Theta_4} D^{(2)}_\theta(x) F_2(d\theta) + \int_{\bar\Theta_5} R_\theta(x) G(d\theta) \\ & + \beta_1 R_+(x) + \beta_2 R_-(x). \end{aligned} \tag{3.4}$$

Under Assumptions 3.1–3.3, we denote by Φ' the class of all tests of the form (2.5) with $d(x)$ defined by (3.4).

Before we prove the main result of this section we establish the following:

LEMMA 3.1. *Let $\{f_\theta(x) : \theta \in \Theta \subseteq \boldsymbol{R}^n\}$ be a family of PDFs with respect to a certain measure μ on $(\mathcal{X}, \mathcal{F})$ and for μ-almost all $x \in \mathcal{X}$, let $f_\theta(x)$ be a continuous function of θ. Assume that the sets $\bar\Theta_0, \Theta_A \subseteq \Theta$ are disjoint. Then the partial order generated in the set of all tests by the risk function* (1.1) *for testing the hypothesis $H_0 : \theta \in \Theta_0$ versus $H_A : \theta \in \Theta_A$ is identical to the partial order generated by* (1.1) *for $H_0 : \theta \in \bar\Theta_0$ versus $H_A : \theta \in \Theta_A$.*

PROOF. By Scheffe's theorem (see e.g. Billingsley (1968)) the mapping $\Theta \ni \theta \to f_\theta \in L^1(\mathcal{X}, \mathcal{F}, \mu)$ is norm-continuous. Hence $E_\theta(\phi)$ is a continuous function of θ for an arbitrary test ϕ. Therefore for tests ϕ and ϕ', we have that $E_\theta(\phi) \leq E_\theta(\phi')$ for all $\theta \in \Theta_0$ iff $E_\theta(\phi) \leq E_\theta(\phi')$ for all $\theta \in \bar\Theta_0$. □

THEOREM 3.2. *Under Assumptions* 3.1–3.3, *Φ' is an essentially complete class of tests for testing hypotheses* (3.1).

PROOF. The methods of the proof are similar to those of the proof of Theorem 2.1 and therefore some technical details are omitted. We concentrate our attention on differences resulting from the presence of two boundary points. By Lemma 3.1, we can assume without loss of generality that $\Theta_0 \subseteq (\theta_1, \theta_2)$.

Let

$$\phi_i(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{as} \quad \int_{\Theta_A} f_\theta(x)\pi_i(d\theta) - \int_{\Theta_0} f_\theta(x)\pi_i(d\theta) \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \tag{3.5}$$

be a sequence of Bayes tests weak* convergent to a test ϕ. We may assume that $\pi_i(\Theta_0) > 0$ for almost all i. Set $\pi'_i(d\theta) = \alpha(\theta)^{-1}\pi_i(d\theta)$, and note that $\int_{\Theta_A} f_\theta(x)\pi_i(d\theta) - \int_{\Theta_0} f_\theta(x)\pi_i(d\theta)$ can be rewritten as

$$\begin{aligned} & [\pi'_i(\Theta_1) - \pi'_i(\Theta_2)]R_{\theta_1}(x) + [\pi'_i(\Theta_4) - \pi'_i(\Theta_3]R_{\theta_2}(x) \\ & + \int_{\Theta_1} [R_\theta(x) - R_{\theta_1}(x)]\pi'_i(d\theta) + \int_{\Theta_4} [R_\theta(x) - R_{\theta_2}(x)]\pi'_i(d\theta) \\ & - \int_{\Theta_2} [R_\theta(x) - R_{\theta_1}(x)]\pi'_i(d\theta) - \int_{\Theta_3} [R_\theta(x) - R_{\theta_2}(x)]\pi'_i(d\theta) \\ & + \int_{\Theta_5} R_\theta(x)\pi'_i(d\theta). \end{aligned} \tag{3.6}$$

Define probability measures $F_{ki} \in \mathcal{P}(\Theta_{2k-1} \cup \Theta_{2k})$ for $k = 1, 2$ by

$$F_{ki}(d\theta) = \begin{cases} \dfrac{I_{\Theta_{2k-1}\cup\Theta_{2k}}(\theta)|\theta - \theta_k|\pi_i'(d\theta)}{\int_{\Theta_{2k-1}\cup\Theta_{2k}} I_{\Theta_{2k}}(\theta)|\theta - \theta_k|\pi_i'(d\theta)}, & \\ \qquad \text{if } \displaystyle\int_{\Theta_{2k-1}\cup\Theta_{2k}} |\theta - \theta_k|\pi_i'(d\theta) > 0, & \\ \pi(d\theta), \qquad \text{otherwise,} & \end{cases} \tag{3.7}$$

where π is an arbitrary element of $\mathcal{P}(\Theta_{2k-1} \cup \Theta_{2k})$.

Using (3.3) and (3.7) we rewrite (3.6) as

$$\begin{aligned} &[\pi_i'(\Theta_1) - \pi_i'(\Theta_2)]R_{\theta_1}(x) + [\pi_i'(\Theta_4) - \pi_i'(\Theta_3]R_{\theta_2}(x) \\ &\quad - \int_{\Theta_1\cup\Theta_2} |\theta - \theta_1|\pi_i'(d\theta) \int_{\Theta_1\cup\Theta_2} D_\theta^{(1)}(x)F_{1i}(d\theta) \\ &\quad + \int_{\Theta_3\cup\Theta_4} |\theta - \theta_2|\pi_i'(d\theta) \int_{\Theta_3\cup\Theta_4} D_\theta^{(2)}(x)F_{2i}(d\theta) + \int_{\Theta_5} R_\theta(x)\pi_i'(d\theta). \end{aligned} \tag{3.8}$$

Set

$$\begin{aligned} S_i = &|\pi_i'(\Theta_1) - \pi_i'(\Theta_2)| + |\pi_i'(\Theta_4) - \pi_i'(\Theta_3)| \\ &+ \int_{\Theta_1\cup\Theta_2} |\theta - \theta_1|\pi_i'(d\theta) + \int_{\Theta_3\cup\Theta_4} |\theta - \theta_2|\pi_i'(d\theta). \end{aligned}$$

Since $\pi_i(\Theta_2) + \pi_i(\Theta_3)$ is positive for almost all i, S_i is positive for almost all i.

Define $G_i \in \mathcal{F}(\Theta_5)$ by

$$G_i(d\theta) = I_{\Theta_5}(\theta)S_i^{-1}\pi_i'(d\theta).$$

Dividing the expression (3.8) by S_i, for each i such that S_i is positive, yields

$$\begin{aligned} &\alpha_1^{(i)}R_{\theta_1}(x) + \alpha_2^{(i)}R_{\theta_2}(x) - \alpha_3^{(i)} \int_{\Theta_1\cup\Theta_2} D_\theta^{(1)}(x)F_{1i}(d\theta) \\ &\quad + \alpha_4^{(i)} \int_{\Theta_3\cup\Theta_4} D_\theta^{(2)}(x)F_{2i}(d\theta) + \int_{\Theta_5} R_\theta(x)G_i(d\theta). \end{aligned} \tag{3.9}$$

for some $(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}) \in \Gamma$.

The remainder of the proof is analogous to the last part of the proof of Theorem 2.1. □

THEOREM 3.3. *Under Assumptions 3.1–3.3, for every $F_1 \in \mathcal{P}(\bar\Theta_1 \cup \bar\Theta_2)$, $F_2 \in \mathcal{P}(\bar\Theta_3 \cup \bar\Theta_4)$, $G \in \mathcal{F}(\bar\Theta_5)$, $\beta_1, \beta_2 \geq 0$, the corresponding test (2.5) with $d(x)$ of the form (3.4) is a weak* limit of a certain sequence of Bayes tests with respect to prior probabilities concentrated on finite sets.*

PROOF. As in the proof of Theorem 2.2 we can assume that π_i' can be any finite measures concentrated on finite sets.

Let $\{H_{1i}\} \subseteq \mathcal{P}(\Theta_1 \cup \Theta_2 - \{\theta_1\})$ and $\{H_{2i}\} \subseteq \mathcal{P}(\Theta_3 \cup \Theta_4 - \{\theta_2\})$ be sequences of probability measures, concentrated on finite sets, converging weakly to F_1 and F_2, respectively, and let $\xi_{li} \in \Theta_l$, for $l = 1, \ldots, 4$ and $i = 1, \ldots$ be points that are different from θ_1 and θ_2. Define probability measures

$$F_{1i} = p_i H_{1i} + \frac{q_i}{2}(\delta_{\xi_{1i}} + \delta_{\xi_{2i}}) \quad \text{and} \quad F_{2i} = p_i H_{2i} + \frac{q_i}{2}(\delta_{\xi_{3i}} + \delta_{\xi_{4i}}),$$

where $p_i \geq 0$, $q_i \geq 0$, $p_i + q_i = 1$ and $q_i \to 0$.

Let $\pi'_i \in \mathcal{F}(\Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Theta_4)$ be measures such that (3.7) is satisfied for all $i = 1, \ldots$. Note that the first two coefficients in (3.8) can be expressed in the following way:

$$\pi'_i(\Theta_1) - \pi'_i(\Theta_2) = \left(\int_{\Theta_1} \frac{F_{1i}(d\theta)}{|\theta - \theta_1|} - \int_{\Theta_2} \frac{F_{1i}(d\theta)}{|\theta - \theta_1|} \right) \int_{\Theta_1 \cup \Theta_2} |\theta - \theta_1| \pi'_i(d\theta)$$

and

$$\pi'_i(\Theta_4) - \pi'_i(\Theta_3) = \left(\int_{\Theta_4} \frac{F_{2i}(d\theta)}{|\theta - \theta_2|} - \int_{\Theta_3} \frac{F_{2i}(d\theta)}{|\theta - \theta_2|} \right) \int_{\Theta_3 \cup \Theta_4} |\theta - \theta_2| \pi'_i(d\theta).$$

Set

$$x_{1,i} = \int_{\Theta_1} \frac{F_{1i}(d\theta)}{|\theta - \theta_1|} - \int_{\Theta_2} \frac{F_{1i}(d\theta)}{|\theta - \theta_1|}, \qquad x_{2,i} = \int_{\Theta_4} \frac{F_{2i}(d\theta)}{|\theta - \theta_2|} - \int_{\Theta_3} \frac{F_{2i}(d\theta)}{|\theta - \theta_2|},$$

$$x_{3,i} = \int_{\Theta_1 \cup \Theta_2} |\theta - \theta_1| \pi'_i(d\theta) \quad \text{and} \quad x_{4,i} = \int_{\Theta_3 \cup \Theta_4} |\theta - \theta_2| \pi'_i(d\theta).$$

The coefficients in the expression (3.9) can now be rewritten as

$$\alpha_r^{(i)} = \frac{x_{r,i} x_{r+2,i}}{|x_{1,i}| x_{3,i} + |x_{2,i}| x_{4,i} + x_{3,i} + x_{4,i}} \quad \text{and}$$
$$\alpha_s^{(i)} = \frac{x_{s,i}}{|x_{1,i}| x_{3,i} + |x_{2,i}| x_{4,i} + x_{3,i} + x_{4,i}},$$

where $r = 1, 2$ and $s = 3, 4$. By a suitable choice of ξ_{li}, for $l = 1, 2, 3, 4$, the terms $x_{1,i}$ and $x_{2,i}$ can be any real numbers. Similarly, by rescaling the masses $\pi'_i(\Theta_1 \cup \Theta_2)$ and $\pi'_i(\Theta_3 \cup \Theta_4)$, we can see that $x_{3,i}$ and $x_{4,i}$ can be any positive real numbers and therefore $(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)})$ can take any value from the set $\Gamma' = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_3 > 0, \alpha_4 > 0, |\alpha_1| + |\alpha_2| + \alpha_3 + \alpha_4 = 1\}$. The remainder of the proof is analogous to the last part of the proof of Theorem 2.2. □

Remark 3.1. If H_A is a one-sided hypothesis, then Theorems 3.2 and 3.3 can easily be adapted to this case. It suffices to put in (3.4) $\alpha_1 = \alpha_3 = \beta_2 = 0$, $\Theta_3 = \Theta_0$ or $\alpha_2 = \alpha_4 = \beta_1 = 0$, $\Theta_2 = \Theta_0$ when the alternative is right-sided or left-sided, respectively.

Remark 3.2. Set $\theta_2 = -\theta_1 = b > 0$, $\theta_0 = 0$, $\theta'_2 = -\theta'_1 = a > b$ in (3.2) and consider the testing problem of $H_0 : |\theta| \leq b$ versus $H_A : |\theta| > b$ for the family of PDFs defined by (2.13). This problem is invariant under the two-element group of transformations $\mathcal{G}$ from Remark 2.3. Let $\alpha(\theta)$ be given by (2.2) and let $g_\theta(x)$ be given by (2.14). Consider testing of null hypothesis $H_0 : 0 \le \theta \le b$ versus $H_A : \theta > b$, i.e. the original problem reduced by invariance. By Remark 3.1, we find that the essentially complete class Φ'_{inv} for the $\mathcal{G}$-invariant tests consists of tests of the form (2.5) with

$$
\begin{aligned}
(3.10)\qquad d(x) = {} & \frac{\alpha}{2}[f_b(-x) + f_b(x)] \\
& + \frac{\alpha'}{2}\int_0^a \frac{f_\theta(-x) + f_\theta(x) - f_b(-x) - f_b(x)}{\theta - b} F(d\theta) \\
& + \int_a^{+\infty} \frac{f_\theta(-x) + f_\theta(x)}{f_0(x)} G(d\theta) \\
& + \beta[R(x_1)\cdots R(x_n) + R(-x_1)\cdots R(-x_n)],
\end{aligned}
$$

where $\alpha' \ge 0$, $|\alpha| + \alpha' = 1$, $\beta \ge 0$, $F \in \mathcal{P}([0,a])$ and $G \in \mathcal{F}([a,+\infty))$.

4. One-sided null and alternative hypotheses

We consider a problem of testing hypothesis

$$
(4.1)\qquad H_0 : \theta \in \Theta_0 \subseteq (-\infty, \theta_0] \quad \text{versus} \quad H_A : \theta \in \Theta_A \subseteq (\theta_0, +\infty),
$$

where θ_0 is a limit point of Borel sets Θ_0 and Θ_A.

Define sets

$$
\begin{aligned}
&\Theta_1 = (-\infty, \theta'_1) \cap \Theta_0, \qquad \Theta_2 = [\theta'_1, \theta_0] \cap \Theta_0, \\
&\Theta_3 = (\theta_0, \theta'_2] \cap \Theta_A \quad \text{and} \quad \Theta_4 = (\theta'_2, -\infty) \cap \Theta_A,
\end{aligned}
$$

where θ'_1 and θ'_2 are real numbers such that $\theta'_1 < \theta_0 < \theta'_2$.

ASSUMPTION 4.1. This assumption is identical to Assumption 2.1.

From now on we use the convention that whenever we integrate $R_\theta(x)$ over the set $\bar{\Theta}_1$ or $\bar{\Theta}_4$ we put

$$
R_{\theta_1}(x) = \lim_{\theta \to \theta'_1 -} R_\theta(x) \quad \text{and} \quad R_{\theta_2}(x) = \lim_{\theta \to \theta'_2 +} R_\theta(x).
$$

ASSUMPTION 4.2. $R_\theta(x)$ has first partial derivative with respect to θ at θ_0 for μ-almost all $x \in \mathcal{X}$.

ASSUMPTION 4.3. There exist positive functions $R_+(x)$ and $R_-(x)$ such that

$$
\lim_{\theta \to +\infty} R_\theta(x) = R_+(x) \quad \text{and} \quad \lim_{\theta \to -\infty} R_\theta(x) = R_-(x)
$$

for μ-almost all $x \in \mathcal{X}$. We assume that $R_+(x)$ is essentially different from $R_-(x)$.

For μ-almost all $x \in \mathcal{X}$, we define the following function of θ on $\bar{\Theta}_2 \cup \bar{\Theta}_3$:

$$D_\theta(x) = \begin{cases} \dfrac{R_\theta(x) - R_{\theta_0}(x)}{\theta - \theta_0}, & \text{if } \theta \neq \theta_0, \\ \left.\dfrac{\partial R_\theta(x)}{\partial \theta}\right|_{\theta=\theta_0}, & \text{if } \theta = \theta_0. \end{cases} \tag{4.2}$$

Let $\mathcal{D}(B)$ denote the family of all probability measures having total masses not greater than 1 on a Borel set $B \subseteq \boldsymbol{R}$.

Let $\Gamma = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_2 \geq 0, \alpha_3 \geq 0, |\alpha_1| + \alpha_2 + \alpha_3 = 1\}$. For $F \in \mathcal{P}(\bar{\Theta}_2 \cup \bar{\Theta}_3)$, $H \in \mathcal{D}(\bar{\Theta}_1)$, $G \in \mathcal{F}(\bar{\Theta}_4)$, $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$ and $\beta \geq 0$, define

$$\begin{aligned} d(x) &= \alpha_1 R_{\theta_0}(x) + \alpha_2 \int_{\bar{\Theta}_2 \cup \bar{\Theta}_3} D_\theta(x) F(d\theta) \\ &\quad - \alpha_3 \left(\int_{\bar{\Theta}_1} R_\theta(x) H(d\theta) + [1 - H(\bar{\Theta}_1)] R_-(x) \right) \\ &\quad + \int_{\bar{\Theta}_4} R_\theta(x) G(d\theta) + \beta R_+(x). \end{aligned} \tag{4.3}$$

Under Assumptions 4.1–4.3 we define Φ'' as the class of all tests of the form (2.5) with $d(x)$ given by (4.3).

THEOREM 4.1. *Under Assumptions* 4.1–4.3, Φ'' *is an essentially complete class of tests for testing hypotheses* (4.1).

PROOF. The proof uses similar techniques to those of Theorems 2.1 and 3.2. Consider Bayes tests $\{\phi_i\}$ of hypotheses (4.1) defined by (3.5). Assume that, the sequence $\{\phi_i\}$ is weak* convergent to a test ϕ. We shall show that $\phi \in \Phi''$. We can assume that $\pi_i(\Theta_1 \cup \Theta_2) > 0$ for almost all i.

Set $\pi_i'(d\theta) = \alpha(\theta)^{-1} \pi_i(d\theta)$, and note that

$$\begin{aligned} &\int_{\Theta_A} f_\theta(x) \pi_i(d\theta) - \int_{\Theta_0} f_\theta(x) \pi_i(d\theta) \\ &\quad = \int_{\Theta_4} R_\theta(x) \pi_i'(d\theta) + \int_{\Theta_3} [R_\theta(x) - R_{\theta_0}(x)] \pi_i'(d\theta) \\ &\quad + \pi_i'(\Theta_3) R_{\theta_0}(x) - \int_{\Theta_2} [R_\theta(x) - R_{\theta_0}(x)] \pi_i'(d\theta) \\ &\quad - \pi_i'(\Theta_2) R_{\theta_0}(x) - \int_{\Theta_1} R_\theta(x) \pi_i'(d\theta). \end{aligned} \tag{4.4}$$

Define probability measures $F_i \in \mathcal{P}(\Theta_2 \cup \Theta_3)$ by

$$F_i(d\theta) = \begin{cases} \dfrac{I_{\Theta_2 \cup \Theta_3}(\theta) |\theta - \theta_0| \pi_i'(d\theta)}{\int_{\Theta_2 \cup \Theta_3}(\theta) |\theta - \theta_0| \pi_i'(d\theta)}, \\ \qquad \text{if } \displaystyle\int_{\Theta_2 \cup \Theta_3} |\theta - \theta_0| \pi_i'(d\theta) > 0, \\ \pi(d\theta), \qquad \text{otherwise,} \end{cases} \tag{4.5}$$

where π is an arbitrary element of $\mathcal{P}(\Theta_2 \cup \Theta_3)$. Further, for each i define probability measure $H_i \in \mathcal{P}(\Theta_1)$ by

$$H_i(d\theta) = \begin{cases} \dfrac{I_{\Theta_1}(\theta)\pi_i'(d\theta)}{\pi_i'(\Theta_1)}, & \text{if } \pi_i'(\Theta_1) > 0, \\ \nu(d\theta), & \text{otherwise}, \end{cases} \tag{4.6}$$

where ν is an arbitrary element of $\mathcal{P}(\Theta_1)$.

Using (4.2), (4.5) and (4.6) we can rewrite the right-hand side of (4.4) as

$$\begin{aligned} &[\pi_i'(\Theta_3) - \pi_i'(\Theta_2)]R_{\theta_0}(x) + \int_{\Theta_2 \cup \Theta_3} |\theta - \theta_0| \pi_i'(d\theta) \int_{\Theta_2 \cup \Theta_3} D_\theta(x) F_i(d\theta) \\ &\quad - \pi_i'(\Theta_1) \int_{\Theta_1} R_\theta(x) H_i(d\theta) + \int_{\Theta_4} R_\theta(x) \pi_i'(d\theta). \end{aligned} \tag{4.7}$$

Set

$$S_i = |\pi_i'(\Theta_3) - \pi_i'(\Theta_2)| + \int_{\Theta_2 \cup \Theta_3} |\theta - \theta_0| \pi_i'(d\theta) + \pi_i'(\Theta_1).$$

Since $\pi_i(\Theta_1 \cup \Theta_2) > 0$ for almost all i, we have that $S_i > 0$ for almost all i.

Define $G_i \in \mathcal{F}(\Theta_4)$ by

$$G_i(d\theta) = I_{\Theta_4}(\theta) S_i^{-1} \pi_i'(d\theta).$$

Dividing the expression (4.7) by S_i, for each i such that S_i is positive, we obtain

$$\begin{aligned} &\alpha_1^{(i)} R_{\theta_0}(x) + \alpha_2^{(i)} \int_{\bar{\Theta}_2 \cup \bar{\Theta}_3} D_\theta(x) F_i(d\theta) \\ &\quad - \alpha_3^{(i)} \int_{\bar{\Theta}_1} R_\theta(x) H_i(d\theta) + \int_{\bar{\Theta}_4} R_\theta(x) G_i(d\theta), \end{aligned} \tag{4.8}$$

for some $(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}) \in \Gamma$. The remainder of the proof is analogous to the last part of the proof of Theorem 2.1. □

THEOREM 4.2. *Under Assumptions* 4.1–4.3, *for every* $F \in \mathcal{P}(\bar{\Theta}_2 \cup \bar{\Theta}_3)$, $H \in \mathcal{D}(\bar{\Theta}_1)$, $G \in \mathcal{F}(\bar{\Theta}_4)$ *and* $\beta > 0$ *the corresponding test of the form* (2.5) *with* $d(x)$ *given by* (4.3) *is a weak* limit of a certain sequence of Bayes tests with respect to prior probabilities concentrated on finite sets.*

PROOF. Similarly as in the proofs of Theorems 2.2 and 3.2, we may assume that π_i' can be any finite measure. By Lemma 3.1 we can also assume that $\pi_i'(\{\theta_0\}) = 0$.

Let $\{F_i'\} \subseteq \mathcal{P}(\Theta_2 \cup \Theta_3)$ and $\{H_i\} \subseteq \mathcal{P}(\Theta_1)$ be sequences of probability measures that converge weakly to $F \in \mathcal{P}(\bar{\Theta}_2 \cup \bar{\Theta}_3)$ and $H \in \mathcal{P}(c(\Theta_1))$, respectively. Let ξ_i, ζ_i, p_i, q_i be real numbers such that $\xi_i < \theta_0 < \zeta_i$, $p_i \geq 0$, $q_i \geq 0$, $p_i + q_i \leq 1$ and $p_i + q_i \to 0$. Define probability measures $F_i = (1 - p_i - q_i)F_i' + p_i \delta_{\xi_i} + q_i \delta_{\zeta_i}$. Let $\pi_i \in \mathcal{F}(\Theta_2 \cup \Theta_3)$ be a measure such that (4.5) is satisfied for all i. Extend π_i'

on $\Theta_1 \cup \Theta_2 \cup \Theta_3$ in such a way that (4.6) holds for all i. The first coefficient in (4.7) can be expressed in the following way:

$$\pi_i'(\Theta_3) - \pi_i'(\Theta_2) = \int_{\Theta_2 \cup \Theta_3} |\theta - \theta_0| \pi_i'(d\theta) \left(\int_{\Theta_3} \frac{F_i(d\theta)}{|\theta - \theta_0|} - \int_{\Theta_2} \frac{F_i(d\theta)}{|\theta - \theta_0|} \right).$$

Put

$$x_i = \int_{\Theta_3} \frac{F_i(d\theta)}{|\theta - \theta_0|} - \int_{\Theta_2} \frac{F_i(d\theta)}{|\theta - \theta_0|} \quad \text{and} \quad y_i = \frac{\pi_i'(\Theta_1)}{\int_{\Theta_2 \cup \Theta_3} |\theta - \theta_0| \pi_i'(d\theta)}.$$

Then we can rewrite the coefficients in (4.8) as

$$\alpha_1^{(i)} = \frac{x_i}{|x_i| + 1 + y_i}, \quad \alpha_2^{(i)} = \frac{1}{|x_i| + 1 + y_i}, \quad \alpha_3^{(i)} = \frac{y_i}{|x_i| + 1 + y_i}. \tag{4.9}$$

By a suitable choice of ξ_i and ζ_i, the term x_i can take an arbitrary real value. By a proper setting of the proportion between the masses $\pi_i'(\Theta_1)$ and $\pi_i'(\Theta_2 \cup \Theta_3)$, the term y_i can attain any positive real value. The remaining part of the proof is analogous to the last part of the proof of Theorem 2.4. □

5. Regularly varying functions and scale or location families

In this section we point out relationships between the notion of regular variation of functions, which has a number of applications in probability theory (see e.g. Bingham *et al.* (1987)), and testing for scale or location parameters. If we deal with scale parameters, then the asymptotic assumptions of previous sections are equivalent to regular variation of probability density functions. In the case of location parameters these assumptions are equivalent to additive regular variation defined below.

Recall that a positive function $g : [0, +\infty) \to \boldsymbol{R}$ is regularly varying (r.v.) at infinity if

$$\lim_{t \to +\infty} \frac{g(tx)}{g(t)} = S(x) < +\infty \tag{5.1}$$

for every $x > 0$. In the natural way the above definition extends to even functions. A function g is said to be r.v. at the origin if $g(x^{-1})$ is r.v. at infinity. If (5.1) holds, then $S(x)$ is of the form x^p, where $-\infty < p < +\infty$, see Feller (1966).

We say that a positive function $f : \boldsymbol{R} \to \boldsymbol{R}$ is additively regularly varying (a.r.v.) at infinity if

$$\lim_{t \to +\infty} \frac{f(t + x)}{f(t)} = R(x) < +\infty \tag{5.2}$$

for every x.

Note that if g is r.v. at infinity, then $f(x) \equiv g(\exp(x))$ is a.r.v. at infinity and conversely if f is a.r.v. at infinity, then $g(x) \equiv f(\ln(x))$ is r.v. at infinity.

Therefore, if (5.2) is satisfied, then there exists a positive number a such that $R(x) = a^x$ for every $x \in \boldsymbol{R}$.

One-parameter exponential families, which do not satisfy the condition (5.1) belong to the class of functions of rapid variation, see Bingham *et al.* (1987).

Example 5.1. Consider the problem of testing the scale parameter for n i.i.d. random variables in the case of Cauchy distribution with PDF

$$f(x) = \pi^{-n} \frac{1}{1+x_1^2} \cdots \frac{1}{1+x_n^2},$$

where $x = (x_1, \ldots, x_n)$, for $H_0 : \theta \in (0, a]$ versus $H_A : \theta \in (a, +\infty)$. Note that for $n = 1$ the Cauchy distribution is r.v. at infinity and at the origin and Assumptions 3.1–3.3 are satisfied.

Similar consideration can be applied to the generalised Cauchy distribution.

LEMMA 5.1. *If a function* $h : [0, +\infty) \to \boldsymbol{R}$ *is concave and non-decreasing and a function* $\epsilon : [0, +\infty) \to \boldsymbol{R}$ *is such that*

$$\lim_{x \to +\infty} \epsilon(x) = 0,$$

then the function

$$f(x) = \exp(\epsilon(|x|) - h(|x|)) \tag{5.3}$$

is a.r.v. at infinity.

PROOF. Put $s = t - x$ and note that

$$\frac{f(t-x)}{f(t)} = \frac{f(s)}{f(s+x)}. \tag{5.4}$$

If (5.2) holds for every $x > 0$, then by letting s diverge to infinity, we see that the right-hand side of (5.4) converges to a^{-x} for some $a > 0$ and all $x > 0$ and so does the left-hand side. Therefore it suffices to verify (5.2) for every positive x. To this end, we show the existence of a number c such that $\lim_{t \to +\infty}[h(t + x) - h(t)] = cx$ for all $x > 0$. It is known that if h is a concave function, then $h'_+(t) \geq [h(t+x) - h(t)]/x \geq h'_-(t+x)$, where h'_+ and h'_- denote the left and the right derivatives of h, and both functions h'_+ and h'_- are non-increasing, see e.g. Ash (1972). Since h is non-decreasing, the functions h'_+ and h'_- are non-negative and therefore there exist non-negative numbers c_+ and c_- such that $\lim_{t \to +\infty} h'_+(t) = c_+$ and $\lim_{t \to +\infty} h'_-(t) = c_-$. Since h is differentiable except for at most countably many points we have that $c_+ = c_- = c$. □

If f is a function satisfying the assumptions of Lemma 5.1, then $\lim_{t \to -\infty} f(t+x)/f(t) = \lim_{t \to +\infty} f(t-x)/f(t)$. Thus f is a.r.v. at (negative) infinity.

As it turns out in the next section, in the case of location families generated by PDFs of the form (5.3) we can prove admissibility of many tests and conditions for admissibility are very simple.

Example 5.2. Denote by $f_{a,b}$, g, h_m and d the densities of the generalised Cauchy, double exponential, generalised logistic and hyperbolic secant distributions, respectively, i.e.

$$\begin{aligned} f_{a,b}(x) &= \frac{a\Gamma(b)}{2\Gamma(a^{-1})\Gamma(b-a^{-1})}\frac{1}{(1+|x|^a)^b}, \qquad g(x) = \frac{1}{2}\exp(-|x|), \\ h_m(x) &= \frac{(2m-1)!}{[(m-1)!]^2}\{\exp(-x)[1+\exp(-x)]^{-2}\}^m, \\ d(x) &= \frac{2}{\pi}\frac{1}{\exp(x)+\exp(-x)}, \end{aligned}$$

where $a, b > 0$, $ab > 1$, $m = 1, \dots$. It is easy to show that $f_{a,b}$, g, h_m and d are of the form (5.3).

In testing problems involving n i.i.d. random variables and location families generated by $f_{a,b}$, g and d, if for each of these families we take $\alpha(\theta)$ of the form (2.2), then the function $R_+(x) \equiv 1$ for $f_{a,b}$ and $R_+(x) = \exp(\sum_{i=1}^n x_i)$ for the densities g and d, where $x = (x_1, \dots, x_n)$. In all the cases above $R_-(x) = R_+(-x)$, for all $x \in R^n$. For the density h_m we have $R_+(x) = m\exp(\sum_{i=1}^n x_i)$.

6. Conditions for admissibility of tests

In this section, we present certain sufficient conditions for admissibility of tests for location families generated by a.r.v. PDFs. To prove admissibility of tests which are weak* limits of Bayes tests, we use the following lemma, based on Blyth (1951):

LEMMA 6.1. *Consider testing the hypothesis $H_0 : \theta \in \Theta_0$ versus $H_A : \theta \in \Theta_A$ for a family of probability density functions $\{f_\theta : \theta \in \Theta\}$, and assume that the following conditions are satisfied:*

(i) *There exists a sequence of priors $\{J_i\} \subseteq \mathcal{F}(\Theta)$ such that*

$$d_i(x) = \int_{\Theta_A} f_\theta(x)J_i(d\theta) - \int_{\Theta_0} f_\theta(x)J_i(d\theta) \to d(x) \tag{6.1}$$

for μ-almost all $x \in \mathcal{X}$.

(ii) *If $\{\phi_i\}$ is a sequence of Bayes tests with respect to J_i, then*

$$\liminf_{i\to\infty} \int_{\mathcal{X}} [\phi(x) - \phi_i(x)]d_i(x)\mu(dx) = 0.$$

(iii) $\mu(\{x : d(x) = 0\}) = 0$.

Then the test of the form (2.5) is admissible.

PROOF. The proof is analogous to the proof of Lemma 3.2 in Brown and Marden (1989). □

We assume (abusing the notation) that the measure μ on the σ-field of Borel sets of $\boldsymbol{R}^n$ is the n-fold product of the same measure. In the next theorem, we give simple conditions for admissibility of tests for a location parameter in the case of invariant testing problems considered in Remark 2.3.

THEOREM 6.1. *Let f be a PDF of the form* (5.3) *satisfying the assumptions of Lemma* 5.1 *and such that the function $\epsilon(|s|) - h(|s|)$ is twice differentiable and*

$$\sup_{s\in\boldsymbol{R}} |\epsilon'(|s|)| - h'(|s|)| < +\infty \qquad \textit{and} \qquad \sup_{s\in\boldsymbol{R}} |\epsilon''(|s|)| - h''(|s|)| < +\infty. \tag{6.2}$$

If $d(x)$ is of the form (2.15) *and $\mu(\{x \in \boldsymbol{R}^n : d(x) = 0\}) = 0$, then the corresponding invariant test defined by* (2.5) *is admissible in the class of all invariant tests for testing $H_0 : \theta = 0$ versus $H_A : \theta \neq 0$ for the location family of distributions generated by* (5.3).

Before proving Theorem 6.1, we establish two technical lemmas.

LEMMA 6.2. *If a sequence of functions $\{h_i\} \subseteq L^1(\mathcal{X}, \mathcal{F}, \mu)$ is convergent in L^1 to a function h and a sequence $\{\psi_i\} \subseteq L^\infty(\mathcal{X}, \mathcal{F}, \mu)$ is weak* convergent to a function ψ, then*

$$\int_{\mathcal{X}} h_i(x)\psi_i(x)\mu(dx) \to \int_{\mathcal{X}} h(x)\psi(x)\mu(dx).$$

PROOF. Note that

$$\int_{\mathcal{X}} h_i(x)\psi_i(x)\mu(dx) = \int_{\mathcal{X}} [h_i(x) - h(x)]\psi_i(x)\mu(dx) + \int_{\mathcal{X}} h(x)\psi_i(x)\mu(dx)$$

and

$$\left| \int_{\mathcal{X}} [h_i(x) - h(x)]\psi_i(x)\mu(dx) \right| \leq M \int_{\mathcal{X}} |h_i(x) - h(x)|\mu(dx),$$

where M is an upper bound for the L^∞ norms of ψ_i for $i = 1, 2, \ldots$. □

LEMMA 6.3. *If $f : \boldsymbol{R} \to \boldsymbol{R}$ is a positive continuous function and $G \in \mathcal{F}(\boldsymbol{R})$ is such that*

$$\int_{-\infty}^{+\infty} f(x)G(dx) < +\infty,$$

then there exists a sequence $\{G_i\}$ of finite measures, which are concentrated on finite sets of points, such that

$$G_i \to G \quad \textit{weakly and} \quad \int_{-\infty}^{+\infty} f(x)G_i(dx) \to \int_{-\infty}^{+\infty} f(x)G(dx). \tag{6.3}$$

Proof. Let $\{I_k\}$ be a sequence of bounded intervals such that $G(I_k) > 0$, $G(\partial I_k) = 0$, $I_k \subseteq I_{k+1}$ and $\bigcup I_k = \boldsymbol{R}$. Denote by G_k the restriction of the measure G to the interval I_k. The sequence of measures $\{G_k\}$ converges weakly to the measure G and

$$\int_{-\infty}^{+\infty} f(x)G_k(dx) = \int_{-\infty}^{+\infty} I_{I_k}(x)f(x)G(dx) \to \int_{-\infty}^{+\infty} f(x)G(dx).$$

On the other hand, we know that for every measure G_k there exists a sequence of probability measures $\{G_{k,l}\}$, which are concentrated on finite number of points, such that $\lim_{l\to\infty} G_{k,l} = G_k$ weakly and since the function $I_{I_k}(x)f(x)$ is continuous G-a.e. and bounded we have

$$\lim_{l\to\infty} \int_{-\infty}^{+\infty} I_{I_k}(x)f(x)G_{k,l}(dx) = \int_{-\infty}^{+\infty} I_{I_k}(x)f(x)G_k(dx),$$

see Ash (1972). Since the set of probability measures with the topology of weak convergence is a metric space, we can extract a subsequence of probability measures from the double subsequence $\{G_{k,l}\}$, which after relabeling we denote $\{G_i\}$, such that it is weakly convergent to the measure G and (6.3) is satisfied. □

PROOF OF THEOREM 6.1. The proof is based on Lemma 6.1. Take $\alpha(\theta)$ defined by (2.2) and consider the original testing problem reduced by invariance, i.e. the problem of testing the hypothesis $H_0 : \theta = 0$ versus $H_A : \theta > 0$ for the family of PDFs g_θ given by (2.14).

From the proof of Theorem 2.2 and Remarks 2.1 and 2.2 it follows that there exists a sequence of prior measures $\{J_i\} \subseteq \mathcal{F}([0, +\infty))$ concentrated on finite sets such that condition (6.1) holds and $d_i(x)$ can be expressed by

$$\alpha_1^{(i)} f_0(x) + \alpha_3^{(i)} \int_0^a D_\theta(x)F_i(d\theta) + \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta),$$

where $\alpha_3^{(i)} \geq 0$, $|\alpha_1^{(i)}| + \alpha_3^{(i)} = 1$, $(\alpha_1^{(i)}, \alpha_3^{(i)}) \to (\alpha_1, \alpha_3)$, $\mathcal{P}((0, a]) \ni F_i \to F$ weakly, $\mathcal{F}((a, +\infty)) \ni G_i \to \bar{G} \in \mathcal{F}([a, +\infty) \cup \{+\infty\})$ weakly, the measure $\bar{G}$ is such that its restriction to the set $[a, +\infty)$ is the measure G and $\bar{G}(\{+\infty\}) = 2\beta$ and $D_\theta(x)$ is defined by (2.16).

Now we verify that the condition (ii) of Lemma 6.1 holds. We break this into two parts. First, we prove that

$$\int_{\mathcal{X}} [\phi(x) - \phi_i(x)] \left\{ \alpha_1^{(i)} f_0(x) + \alpha_3^{(i)} \int_0^a D_\theta(x)F_i(d\theta) \right\} \mu(dx) \to 0, \tag{6.4}$$

and in the next step we prove that

$$\int_{\mathcal{X}} [\phi(x) - \phi_i(x)] \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta)\mu(dx) \to 0. \tag{6.5}$$

If we show that

$$\left|\int_0^a D_\theta(x)F_i(d\theta)\right| \le K f_0(x)$$

for some positive constant K then, by Lebesgue's dominated convergence theorem, condition (6.4) is satisfied.

It is known (see e.g. Feller (1966)) that the convergence in (5.2) is uniform on every bounded interval I and therefore

$$\sup_{\theta\in I}\frac{f(t+\theta)}{f(t)} \le M_0 \tag{6.6}$$

for a certain $M_0 > 0$.

By (6.2) and (6.6) we obtain

$$\begin{aligned}\sup_{\theta\in I}\left|\frac{\partial f(t+\theta)}{\partial\theta}\right| &= \sup_{\theta\in I}\left|\frac{\partial \ln f(t+\theta)}{\partial\theta}\right| f(t+\theta)\\ &= \sup_{\theta\in I}\left|\frac{\partial}{\partial\theta}[\epsilon(|t+\theta|)-h(|t+\theta|)]\right| f(t+\theta) \le M_1 f(t)\end{aligned}\tag{6.7}$$

for a certain positive constant M_1 and it follows that

$$\begin{aligned}&\sup_{\theta\in I}\left|\frac{\partial^2 f(t+\theta)}{\partial\theta^2}\right|\\ &\quad= \sup_{\theta\in I}\left|\frac{\partial^2 \ln f(t+\theta)}{\partial\theta^2}f(t+\theta) + \frac{\partial \ln f(t+\theta)}{\partial\theta}\frac{\partial f(t+\theta)}{\partial\theta}\right|\\ &\quad= \sup_{\theta\in I}\left|\frac{\partial^2}{\partial\theta^2}[\epsilon(|t+\theta|)-h(|t+\theta|)]\right.\\ &\qquad\left.+\frac{\partial}{\partial\theta}[\epsilon(|t+\theta|)-h(|t+\theta|)]\frac{\partial f(t+\theta)}{\partial\theta}\right|\\ &\quad\le M_2 f(t)\end{aligned}\tag{6.8}$$

for a certain positive constant M_2. Bearing in mind (6.6)–(6.8), it is easy to see that

$$\begin{aligned}\left|\int_0^a D_\theta(x)F_i(d\theta)\right| &\le \sup_{\theta\in[0,a]}\frac{1}{4}\left|\frac{\partial^2[f_\theta(x)+f_{-\theta}(x)]}{\partial\theta^2}\right| \le \frac{1}{2}\sup_{\theta\in[-a,a]}\left|\frac{\partial^2 f_{\theta(x)}}{\partial\theta^2}\right|\\ &= \frac{1}{2}\sup_{\theta\in[-a,a]}\left|\sum_{i_1+\cdots+i_n=2}\frac{2f^{(i_1)}(x_1-\theta)\cdots f^{(i_n)}(x_n-\theta)}{i_1!\cdots i_n!}\right|\\ &\le K_1 f(x_1)\cdots f(x_n),\end{aligned}$$

where $f^{(0)}(t-\theta) = f(t-\theta)$ and $f^{(i)}(t-\theta)$ denote partial derivatives of order i of the function f with respect to θ for $i = 1, 2$ and K_1 is a positive constant. The first of the inequalities above results from Taylor's expansion of the second order.

Now we show that the condition (6.5) is satisfied. Let r be the maximal integer such that $0 \le r \le n$ and $\int_a^{+\infty} \exp(rh(\theta))\bar{G}(d\theta) < +\infty$. Denote by $x_{[1]}, \ldots, x_{[n]}$ the decreasing rearrangement of the sequence $|x_1|, \ldots, |x_n|$ and set $x_{[0]} = 0$.

By (5.3) we obtain

$$\begin{aligned}
(6.9)\quad & \frac{g_\theta(x)}{f_\theta(0)f_0(x)} \\
&= \frac{\exp\left(\sum_{k=1}^{n}[\epsilon(|x_k-\theta|)-h(|x_k-\theta|)]\right)+\exp\left(\sum_{k=1}^{n}[\epsilon(|x_k+\theta|)-h(|x_k+\theta|)]\right)}{2\exp(n\epsilon(|\theta|)-nh(|\theta|))\exp\left(\sum_{k=1}^{n}[\epsilon(|x_k|)-h(|x_k|)]\right)} \\
&\ge c\frac{\exp\left(-\sum_{k=1}^{n}h(|x_k-\theta|)\right)+\exp\left(-\sum_{k=1}^{n}h(|x_k+\theta|)\right)}{\exp\left(-nh(|\theta|)-\sum_{k=1}^{n}h(|x_k|)\right)} \\
&\ge 2c\frac{\exp\left(-\sum_{k=1}^{n}h\left(\frac{1}{2}[|x_k-\theta|+|x_k+\theta|]\right)\right)}{\exp\left(-nh(|\theta|)-\sum_{k=1}^{n}h(|x_k|)\right)} \\
&= 2c\exp\left(\sum_{k=1}^{n}h(|x_k|)+nh(|\theta|)-\sum_{k=1}^{n}\sup\{h(|x_k|),h(|\theta|)\}\right)
\end{aligned}$$

where c is some positive constant. The first of the two inequalities in (6.9) follows from the boundedness of the function $\epsilon(x)$ and the second results from the convexity of the exponential function and the concavity of the function h on the set of non-negative real numbers.

If $|\theta| \le x_{[n-r]}$ for a certain integer $r \in \{0, \ldots, n\}$, then, since the function h is non-decreasing on the set of non-negative real numbers, we have

$$\begin{aligned}
(6.10)\quad & \exp\left(\sum_{k=1}^{n}h(|x_k|)+nh(|\theta|)-\sum_{k=1}^{n}\sup\{h(|x_k|),h(|\theta|)\}\right) \\
&= \exp\left(\sum_{k=1}^{n-r-1}h(|x_{[k]}|)+nh(|\theta|)-\sum_{k=1}^{n-r-1}\sup\{h(|x_{[k]}|),h(|\theta|)\}\right) \\
&\ge \exp((r+1)h(|\theta|)).
\end{aligned}$$

By (6.5), (6.9) and (6.10) we obtain

$$\begin{aligned}
(6.11)\quad d_i(x) &\ge -|\alpha_1^{(i)}f_0(x)+\alpha_3^{(i)}\int_0^a D_\theta(x)F_i(d\theta)|+\int_a^{+\infty}\frac{g_\theta(x)}{f_\theta(0)}G_i(d\theta) \\
&\ge f_0(x)\left(\int_a^{+\infty}\frac{g_\theta(x)}{f_0(x)f_\theta(0)}G_i(d\theta)-K'\right) \\
&\ge f_0(x)\left(2c\int_a^{x_{[n-r]}}\exp([r+1]h(|\theta|)G_i(d\theta)-K'\right),
\end{aligned}$$

for some constant $K' > 0$. For a positive number δ, define the set $C_\delta = \{x \in \boldsymbol{R}^n : x_{[n-r]} \leq \delta\}$. Now we can rewrite (6.5) as

$$\int_{C_\delta} [\phi(x) - \phi_i(x)] \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta)\mu(dx) + \int_{C_\delta^c} [\phi(x) - \phi_i(x)] \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta)\mu(dx). \tag{6.12}$$

The first integral in (6.12) can be rewritten as

$$\int_{\mathcal{X}} [\phi(x) - \phi_i(x)] I_{C_\delta}(x) \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta)\mu(dx). \tag{6.13}$$

By Lemma 6.3, we can assume that the sequence $\{G_i\}$ converges weakly to the measure G and

$$\int_a^{+\infty} \exp(rh(\theta))G_i(d\theta) \to \int_a^{+\infty} \exp(rh(\theta))G(d\theta). \tag{6.14}$$

If we show that

$$I_{C_\delta}(x) \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta) \to I_{C_\delta}(x) \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G(d\theta) \quad \text{in} \quad L^1, \tag{6.15}$$

then, by Lemma 6.2, the integral (6.13) converges to 0 as $i \to \infty$. To this end, note that for μ-almost all x the integrand in (6.15) is a continuous and bounded function of θ and therefore the left-hand term of (6.15) converges μ-almost everywhere to the right-hand one.

Define measures

$$G_i'(d\theta) = \exp(rh(\theta))G_i(d\theta) \qquad \text{and} \qquad G'(d\theta) = \exp(rh(\theta))G(d\theta).$$

Obviously G_i' converges vaguely to G' and by (6.14) it follows that it also converges weakly, see Bauer (1981).

If we show that

$$\int_{\mathcal{X}} I_{C_\delta}(x) \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G_i(d\theta)\mu(dx) \to \int_{\mathcal{X}} I_{C_\delta}(x) \int_a^{+\infty} \frac{g_\theta(x)}{f_\theta(0)} G(d\theta)\mu(dx), \tag{6.16}$$

then, since all integrands in the above integrals (with respect to measure μ) are non-negative, we have proven that (6.15) holds, see e.g. Parthasarathy (1978). By Fubini's theorem, rewrite the left-hand part of (6.16) as

$$\begin{aligned} &\int_a^{+\infty} \int_{\mathcal{X}} I_{C_\delta}(x) \frac{g_\theta(x)}{f_\theta(0)} \mu(dx) G_i(d\theta) \\ &\quad = \int_a^{+\infty} \int_{\mathcal{X}} I_{C_\delta}(x) \frac{g_\theta(x)}{f_\theta(0)} \exp(-rh(\theta))\mu(dx)G_i'(d\theta). \end{aligned} \tag{6.17}$$

Define the set $O_\delta = \{x : |x_1| \leq \cdots \leq |x_{n-r}| \leq \delta\}$ and note that

$$\begin{aligned}
&\int_{\mathcal{X}} I_{C_\delta}(x) \frac{g_\theta(x)}{f_\theta(0)} \exp(-rh(\theta))\mu(dx) \\
&= (n-r)! \int_{O_\delta} \frac{g_\theta(x)}{f_\theta(0)} \exp(-rh(\theta))\mu(dx) \\
&\leq (n-r)! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \frac{g_\theta(x)}{f_\theta(0)} \right. \\
&\qquad \left. \cdot \exp(-rh(\theta))\mu(dx_1) \cdots \mu(dx_{n-r}) \right) \mu(dx_{n-r+1}) \cdots \mu(dx_n) \\
&\leq (n-r)! \left(\sup_\theta \int_{-\delta}^{\delta} \frac{f(t-\theta)}{f(\theta)} dt \right)^{n-r} \sup_\theta \exp(r\epsilon(\theta)) < +\infty.
\end{aligned}$$

Hence the integrand with respect to G_i' in (6.17) is bounded and since the mapping $\theta \to g_\theta \in L^1$ is continuous, this integrand is also a continuous function of θ. Therefore (6.16) holds.

To complete the proof we have to show that the second integral in (6.12) converges to zero as $i \to \infty$. If δ is sufficiently large, then by the inequality (6.11) the function $d_i(x)$ is positive for all $x \notin C_\delta$. Therefore $\phi_i(x) = \phi(x)$ and the considered integral vanishes. □

THEOREM 6.2. *Let f be a PDF of the form* (5.3) *satisfying the assumptions of Lemma* 5.1 *such that the function $\epsilon(|s|) - h(|s|)$ is differentiable except possibly at zero where left and right derivatives exist and*

$$\sup_{s \in \boldsymbol{R}} |\epsilon'_-(|s|) - h'_-(|s|)| < +\infty \qquad \text{and} \qquad \sup_{s \in \boldsymbol{R}} |\epsilon'_+(|s|) - h'_+(|s|)| < +\infty.$$

If $d(x)$ is of the form (3.10), *and $\mu(\{x : d(x) = 0\}) = 0$, then the corresponding invariant test defined by* (2.5) *is admissible in the class of all invariant tests for testing $H_0 : |\theta| \leq b$ versus $H_A : |\theta| > b$ for the location family of distributions generated by* (5.3).

PROOF. The proof is similar to the proof of Theorem 6.1 so that we only concentrate on some technical differences.

We have to prove that the condition (ii) of Lemma 6.1 holds with

$$d_i(x) = \alpha_i g_b(x) + \alpha_i' \int_0^a \frac{g_\theta(x) - g_b(x)}{\theta - b} F_i(d\theta) + \int_a^{+\infty} \frac{g_\theta(x)}{f_0(x)} G_i(d\theta),$$

where $\alpha_i' \geq 0$, $|\alpha_i| + \alpha_i' = 1$, $F_i \in \mathcal{P}([0,a])$, $G_i \in \mathcal{F}([a,+\infty))$ and $g_\theta(x)$ is defined by (2.14), see Remark 3.2. First consider

$$\int_{\mathcal{X}} [\phi(x) - \phi_i(x)]\{\alpha_i g_b(x) + \alpha_i' \int_0^a \frac{g_\theta(x) - g_b(x)}{\theta - b} F_i(d\theta)\}\mu(dx). \tag{6.18}$$

If we show that

$$\left|\int_0^a \frac{g_\theta(x) - g_b(x)}{\theta - b} F_i(d\theta))\mu(dx)\right| \le K f_0(x)$$

for a certain positive constant K then, by Lebesgue's dominated convergence theorem, the integral (6.18) converges to zero as $i \to \infty$. Note that

$$\begin{aligned}
&\left|\int_0^a \frac{g_\theta(x) - g_b(x)}{\theta - b} F_i(d\theta)\right| \\
&\quad \le \sup_{\theta\in[-a,a]} \left|\frac{\partial f_\theta(x)}{\partial\theta-}\right| + \sup_{\theta\in[-a,a]} \left|\frac{\partial f_\theta(x)}{\partial\theta+}\right| \\
&\quad \le \sup_{\theta\in[-a,a]} \sum_{i_1+\cdots+i_n=1} |f_-^{(i_1)}(x_1-\theta)\cdots f_-^{(i_n)}(x_n-\theta)| \\
&\qquad + \sup_{\theta\in[-a,a]} \sum_{i_1+\cdots+i_n=1} |f_+^{(i_1)}(x_1-\theta)\cdots f_+^{(i_n)}(x_n-\theta)| \\
&\quad \le \left(\sup_{\theta\in\boldsymbol{R}} |\epsilon'_-(|s|) - h'_-(|s|)| + \sup_{\theta\in\boldsymbol{R}} |\epsilon'_+(|s|) - h'_+(|s|)|\right) f(x_1)\cdots f(x_n)
\end{aligned}$$

where the symbols $\frac{\partial}{\partial\theta-} \equiv f_-^{(1)}$ and $\frac{\partial}{\partial\theta+} \equiv f_+^{(1)}$ denote respectively the firs left and right partial derivatives with respect to θ and $f_-^{(0)} \equiv f_+^{(0)} \equiv f$. The remaining part of the proof, i.e. showing that

$$\int_{\mathcal{X}} [\phi(x) - \phi_i(x)] \int_a^{+\infty} \frac{g_\theta(x)}{f_0(x)} G_i(d\theta)\mu(dx) \to 0,$$

is identical with the analogous part of the proof of Theorem 6.1. □

THEOREM 6.3. *Let f be a PDF of the form* (5.3) *satisfying the assumptions of Lemma* 5.1 *such that $\epsilon(|s|) - h(|s|)$ is differentiable except possibly at θ_0 where left and right derivatives exist and*

$$\sup_{s\in\boldsymbol{R}} |\epsilon'_-(|s|) - h'_-(|s|)| < +\infty \quad \textit{and} \quad \sup_{s\in\boldsymbol{R}} |\epsilon'_+(|s|) - h'_+(|s|)| < +\infty.$$

If $d(x)$ of the form (4.3) *is such that $H(\bar\Theta_1) = 1$, $\beta = 0$, $\int_{\bar\Theta_1} \exp(nh(\theta))H(d\theta) < +\infty$, $\int_{\bar\Theta_4} \exp(nh(\theta))G(d\theta) < +\infty$ and $\mu(\{x : d(x) = 0\}) = 0$, then the corresponding test of the form* (2.5) *is admissible for testing hypotheses* (4.1) *for the location family of distributions generated by* (5.3).

PROOF. The condition (ii) of Lemma 6.1 can be checked by splitting $d_i(x)$ of the form (4.8) into functions

$$\alpha_1^{(i)} R_{\theta_0}(x) + \alpha_2^{(i)} \int_{\bar\Theta_2\cup\bar\Theta_3} D_\theta(x) F_i(d\theta)$$

and

$$- \alpha_3^{(i)} \int_{\bar{\Theta}_1} R_\theta(x) H_i(d\theta) + \int_{\bar{\Theta}_4} R_\theta(x) G_i(d\theta)$$

and proceeding in a way similar to the proofs of Theorems 6.1 and 6.2. □

Remark 6.1. Theorem 6.1 can be applied to the generalised Cauchy, generalised logistic and hyperbolic secant distributions. The double exponential distribution does not satisfy the condition (6.2). However from the proof of the theorem mentioned above, it follows that if there is no local term in $d(x)$ defined by (2.15), i.e. $\alpha_3 = 0$, and the set $\{x : d(x) = 0\}$ has Lebesgue measure zero, then the corresponding test of the form (2.5) is admissible. Theorems 6.2 and 6.3 apply to the double exponential, generalised logistic, hyperbolic secant and generalised Cauchy distributions.

7. Locally best tests

Under Assumption 2.1, consider a test ϕ of the form (2.5) of hypothesis $H_0 : \theta = \theta_0$ versus $H_A : \theta > \theta_0$ with $d(x)$ given by

$$d(x) = \beta R_+(x) - f_{\theta_0}(x), \tag{7.1}$$

where $\beta > 0$, such that $\mu(x : d(x) = 0) = 0$. The expression (7.1) is for μ-almost all $x \in \mathcal{X}$ the limit of the sequence $d_i(x) = \beta_i f_{\theta_i}(x) - R_{\theta_0}(x)$, where $\{\beta_i\}$ is a sequence of positive numbers converging to β and a sequence $\{\theta_i\} \subseteq (\theta_0, +\infty)$ diverges to infinity.

Assume that for all such sequences $\{\theta_i\}$, the sequences $\{\beta_i\}$ are such that the corresponding measures $J_i = \delta_{\theta_0} + \beta i \delta_{\theta_i}$ satisfy the condition (ii) of Lemma 6.1. By Lemma 6.1, the limit test ϕ determined by $d(x)$ is admissible for testing $H_0 : \theta = \theta_0$ versus $H_A : \theta \in \{\theta_i\}$ for every sequence $\{\theta_i\} \subseteq (\theta_0, +\infty)$ diverging to infinity. The test ϕ has the following optimum property: for any test ψ such that

$$r_{\theta_0}(\psi) \le r_{\theta_0}(\phi), \tag{7.2}$$

there exists a number $c \ge \theta_0$ such that

$$r_\theta(\phi) \le r_\theta(\psi) \tag{7.3}$$

for every $\theta \ge c$, where r is the risk function (1.1). To prove that ϕ has the stated property, assume that (7.2) holds and the subsequent condition (7.3) does not. Then there exists a sequence $\{\theta_i\}$ diverging to infinity such that $r_{\theta_i}(\phi) > r_{\theta_i}(\psi)$. But this contradicts the admissibility of the test ϕ for the alternative space consisting of the sequence $\{\theta_i\}$.

In this light, the test ϕ can be called locally best at positive infinity. In a similar way, for a left-sided alternative, we can define a locally best test at negative infinity. These tests are good at detecting big departures from a null hypothesis.

Another extreme case of tests are locally best tests defined by Neyman and Pearson (1936, 1938). In the expression $d(x)$ defined by (2.4), we can find terms involving first and second partial derivatives with respect to a parameter, that are characteristic for locally best unbiased tests. Similarly the expressions $d(x)$ given by (3.4) and (4.3) contain terms present in locally best tests for one-sided hypotheses, see e.g. Ferguson (1967). These tests are weak* limits of Bayes tests relative to priors which concentrate masses arbitrarily close to the boundary of a null parameter space. Locally best tests, locally best at infinity tests and tests which contain combinations of local and asymptotic terms have usually quite simple forms, which makes them easy to use.

8. Examples

In this section we present some examples of tests from the complete classes obtained in the previous sections.

Example 8.1. Let f be an additively regularly varying PDF such that

$$R_+(x) \equiv \lim_{t\to+\infty} \frac{f(x-t)}{f(t)} \neq \lim_{t\to-\infty} \frac{f(x-t)}{f(t)} \equiv R_-(x)$$

for almost all $x \in \boldsymbol{R}$. Both functions $R_+(x)$ and $R_-(x)$ are exponential so that $R_+(x) = a^x$ and $R_-(x) = b^x$, for some positive numbers a and b. We assume that $ab \neq 1$.

The densities of the double exponential, generalized logistic and hyperbolic secant distributions are examples of probability density functions which have the above properties.

Consider testing the hypothesis $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ for a location parameter of the density f. For the case of n i.i.d. random variables take the function d of the form (4.3) given by

$$d(x) = \beta \exp\left(\ln a \sum_{i=1}^{n} x_i\right) - \exp\left(-\ln b \sum_{i=1}^{n} x_i\right), \tag{8.1}$$

where $x = (x_1, \ldots, x_n)$. The test based on the function d given by (8.1) is identical with the test based on the sample mean, i.e.

$$\phi(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i > k, \\ 0, & \text{if } \sum_{i=1}^{n} x_i < k, \end{cases}$$

where $k = \ln \beta^{-1} / \ln ab$.

Since tests based on the sample mean were obtained as limits of Bayes tests relative to priors with masses arbitrarily far from the point θ_0, we can expect, that for the one-sided testing problems under consideration, these tests are sensitive to big departures from the central point θ_0. Figure 1(a) presents the power functions of the tests based on the sample mean for samples of size 5, for testing the

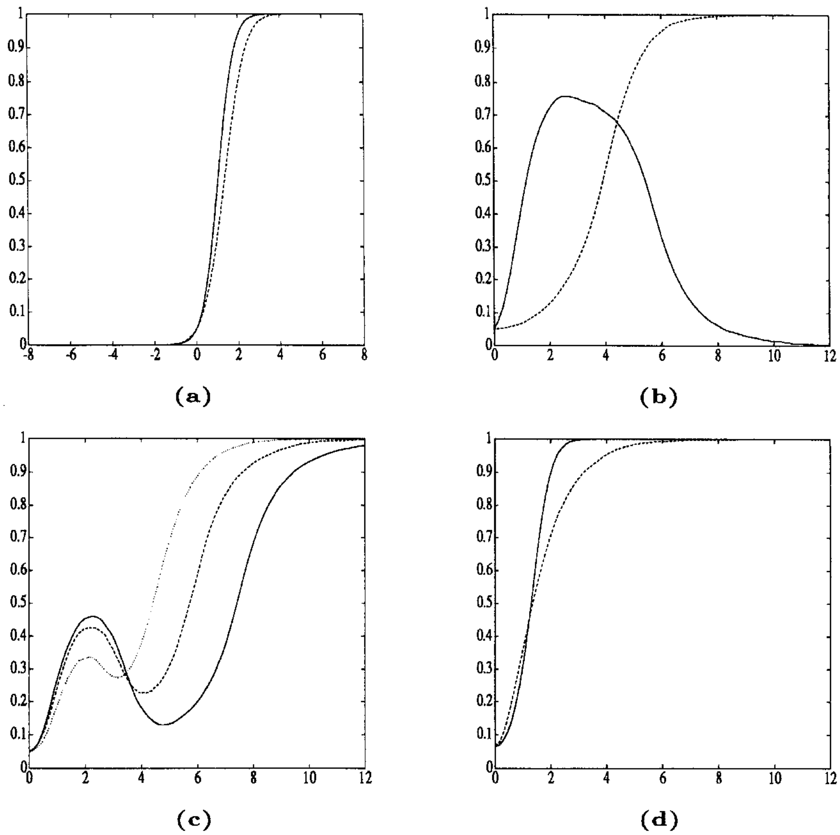

Fig. 1. Power functions of tests.

hypothesis $H_0 : \theta \leq 0$ versus $H_A : \theta > 0$ for the double exponential distribution ($k = 5.1$, solid line) and for the logistic distribution ($k = 6.68$, dashed line). The size of both tests is $\alpha = 0.05$. From the plots, we can see that both tests are very sensitive even to moderate departures from 0 and their power functions increase quite sharply in the neighbourhood of 0.

Example 8.2. Consider the invariant testing problem of the simple null hypothesis arising from Remark 2.3 for the Cauchy distributions. The function $R(x)$ is in this case the constant equal 1. Consider a test of the form (2.5) with $d(x)$ given by (2.15) such that only local and asymptotic terms are involved, i.e.

$$d(x) = \alpha_1 f_0(x) + \alpha_3 \frac{1}{2} \frac{\partial^2}{\partial\theta^2} f_\theta(x)\bigg|_{\theta=0} + \beta, \tag{8.2}$$

where $\alpha_3 \geq 0$, $|\alpha_1| + \alpha_3 = 1$ and $\beta \geq 0$. If $\beta = 0$ in the expression (8.2), then the invariant test based on $d(x)$ is the locally best unbiased invariant test, see Ferguson (1967). If $\alpha_3 = 0$, then the test based on $d(x)$ is locally best at infinity for the testing problem reduced by invariance, i.e. for the one-sided alternative space and the family of distributions with densities of the form (2.14). If we divide (8.2) by $f_0(x)$ we can see that the test based on (8.2) is identical with the test based on

$$d'(x) = \alpha_1 + \alpha_3 \sum_{i=1}^{n} \frac{x_i^2 - 1}{(1+x_i^2)^2} + \beta \prod_{i=1}^{n} (1+x_i^2). \tag{8.3}$$

By Theorem 6.1, all invariant tests determined by $d'(x)$ are admissible in the class of all invariant tests for the testing problem under consideration.

The function $d'(x)$ given by (8.3) attains its minimum when $x_i = 0$ for all i and $d'(0,\dots,0) = \alpha_1 - n\alpha_3 + \beta$. If $\alpha_1 - n\alpha_3 + \beta \geq 0$, then $d'(x) > 0$ for almost all x. Hence, to obtain non-trivial tests we have to impose the condition $\beta < -\alpha_1 + n\alpha_3 = -\alpha_1 + n(1 - |\alpha_1|)$, where $|\alpha_1| \leq 1$. Numerical computations have shown that in order to obtain low sizes of tests based on $d'(x)$ we should choose negative coefficients α_1. For instance if the sample size is 5, and α_1 is non-negative, then the lowest size of such a test is attained when $\alpha_1 = \beta = 0$ and it is approximately equal 0.25.

If $\alpha_1 < 0$ and $\beta = 0$, then the test determined by (8.3) is locally best and unbiased. It is easy to see that the power of this test tends to zero as the parameter θ diverges to infinity. A method of improving the test is to combine it with locally best at infinity tests by choosing positive coefficients β in (8.3). Such combined tests appear to have, as might be expected, quite big powers in a certain neighbourhood of 0 and their power functions approach 1 if the parameter θ tends to infinity. Figure 1(b) presents the power functions of the locally best test (solid line) with $\alpha_1 = -0.6465$ and $\beta = 0$ and the locally best at infinity test (dashed line) with $\alpha_1 = -1$ and $\beta = 5.4 \times 10^{-7}$. Figure 1(c) presents the power functions of the combined tests with $\alpha_1 = -0.79$ and $\beta = 10^{-9}$ (solid line), $\alpha_1 = -0.8183$ and $\beta = 10^{-8}$ (dashed line) and $\alpha_1 = -0.8564$ and $\beta = 10^{-7}$ (dotted line), respectively. In all the cases above, the sample size is 5 and constants α_1 and β are chosen in such a way that the size of all tests is 0.05. Since the above tests are invariant with respect to the symmetry about the origin, Figs. 1(b) and 1(c) show plots for non-negative parameter values only.

Example 8.3. Consider testing the hypothesis $H_0 : \theta = 0$ versus $H_A : \theta \neq 0$ for the double exponential distribution. Take a locally best unbiased invariant test determined by

$$\begin{aligned} d(x) = & -\gamma 2^{-n} \exp\left(-\sum_{i=1}^{n} |x_i|\right) \\ & + (1-\gamma) 2^{-n+1} \frac{\partial^2}{\partial \theta^2} \exp\left(-\sum_{i=1}^{n} |x_i - \theta|\right)\Bigg|_{\theta=0}, \end{aligned}$$

where $0 < \gamma < 1$, see Remark 2.3. After simple calculations we obtain that

$$\begin{aligned} d(x) = & -\gamma 2^{-n} \exp\left(-\sum_{i=1}^{n} |x_i|\right) \\ & + (1-\gamma) 2^{-n+1} \left(\sum_{i=1}^{n} \operatorname{sgn}(x_i)\right)^2 \exp\left(-\sum_{i=1}^{n} |x_i|\right) \end{aligned}$$

and therefore the test based on $d(x)$ appears to be a two-sided sign test. It is easy to see that sizes of such tests can only be of the form $r/2^{n-1}$, where $r = 1, \ldots, 2^{n-1}$.

The test obtained from (2.15) by taking $\alpha_1 = -1$, $\alpha_3 = 0$, $\beta > 0$ and G being the zero measure is identical with the test

$$(8.4)\qquad \phi(x) = \begin{cases} 1, & \text{if } \exp\left(\sum_{i=1}^n |x_i| - \sum_{i=1}^n x_i\right) + \exp\left(\sum_{i=1}^n |x_i| + \sum_{i=1}^n x_i\right) > k, \\ 0, & \text{if } \exp\left(\sum_{i=1}^n |x_i| - \sum_{i=1}^n x_i\right) + \exp\left(\sum_{i=1}^n |x_i| + \sum_{i=1}^n x_i\right) < k, \end{cases}$$

for a certain real number $k > 0$. By Remark 6.1, this test is admissible in the class of invariant tests and it is also locally best at infinity for the testing problem reduced by invariance. Figure 1(d) shows plots of the power functions of the two-sided sign test given by

$$\phi(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^5 \operatorname{sgn} x_i > 4, \\ 0, & \text{if } \sum_{i=1}^5 \operatorname{sgn} x_i < 4, \end{cases}$$

(dashed line) and the test defined by (8.4) with $k = 10^6$ (solid line). Both tests are of size $\alpha = 2^{-4} = 0.0625$. From the plots we can see that although the sign test is slightly better in the neighbourhood of 0, the power of the second test increases faster from a certain point.

It is easy to show that for testing one-sided hypotheses for the double exponential location family, the sign test (which is locally best) belongs to the class Φ''.

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