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LR TEST FOR RANDOM-EFFECTS COVARIANCE STRUCTURE IN A PARALLEL PROFILE MODEL

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Abstract. We consider a parallel profile model which is useful in analyzing parallel growth curves of several groups. The likelihood ratio criterion for a hypothesis concerning the adequacy of a random-effects covariance structure is obtained under the parallel profile model. The likelihood ratio criterion for the hypothesis in the general one-way MANOVA model is also obtained. Asymptotic null distributions of the criteria are derived when the sample size is large. We give a numerical example of these asymptotic results.

Key words and phrases: Parallel profile model, random-effects covariance structure, asymptotic null distribution, likelihood ratio criterion.

1. Introduction

The serial measurements of a response variable x at several occasions or treatments on each individual are called repeated measures or longitudinal data. It is assumed that each of N individuals has been observed at p different occasions. Now consider a k-sample case, and let $\boldsymbol{x}_j^{(g)}$ be the p-vector of measurements on the j-th individual in the g-th group, where $j = 1, \dots, N_g$, $g = 1, \dots, k$, $N = N_1 + \cdots + N_k$. We assume that the $\boldsymbol{x}_j^{(g)}$ are independent,

$$\boldsymbol{x}_j^{(g)} \sim N_p(\boldsymbol{\mu}^{(g)}, \Sigma), \quad \boldsymbol{\mu}^{(g)} = \delta_g \mathbf{1}_p + \boldsymbol{\mu}, \qquad g = 1, \dots, k, \tag{1.1}$$

where $\mathbf{1}_p$ is a p-vector of ones. Here, without loss of generality, we may assume that $\delta_k = 0$. Then

$$X \sim N_{N\times p}(A_1 \boldsymbol{\delta} \mathbf{1}_p' + \mathbf{1}_N \boldsymbol{\mu}', \Sigma \otimes I_N), \tag{1.2}$$

where $X = [\boldsymbol{x}_1^{(1)}, \dots, \boldsymbol{x}_{N_1}^{(1)}, \dots, \boldsymbol{x}_1^{(k)}, \dots, \boldsymbol{x}_{N_k}^{(k)}]'$,

$$A_1 = \begin{pmatrix} \mathbf{1}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{N_{k-1}} \\ \hdashline & 0 & \end{pmatrix}$$

is an $N \times (k-1)$ between-individual design matrix of rank $k-1(\leq N-p-1)$, $\boldsymbol{\delta} = (\delta_1, \dots, \delta_{k-1})'$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)'$ are vectors of unknown parameters, Σ is an unknown positive definite matrix. The model (1.2) is called a parallel profile model. It may be noted (Verbyla and Venables (1988)) that the model (1.2) can be applied to analysis of repeated measurements with parallel profiles.

In this paper, we are interested in a random-effects covariance structure, which is based on random-coefficients models (see, e.g., Rao (1965, 1975)). In our model, the structure can be expressed as

$$\Sigma = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p, \tag{1.3}$$

where $\lambda^2 \geq 0$ and $\sigma^2 > 0$. By making this stronger assumption about Σ, we can get more efficient estimators (see, e.g., Yokoyama and Fujikoshi (1993)) and more powerful tests. However, these are only valid if Σ satisfies (1.3). When Σ has no structures, Geisser and Greenhouse (1958) and Greenhouse and Geisser (1959) investigated the profile MANOVA by a two-way mixed-model ANOVA approach and proposed approximate F tests for equal response means and parallelism of mean profiles. Huynh and Feldt (1970) have found necessary and sufficient conditions on Σ for the Geisser-Greenhouse tests to have exact F distributions and proposed a test statistic for testing the hypothesis that their necessary and sufficient conditions hold. It may be noted (Huynh and Feldt (1970)) that a sufficient condition is that Σ has the uniform covariance structure, i.e.,

$$\Sigma = \sigma_1^2[\rho \mathbf{1}_p \mathbf{1}_p' + (1-\rho) I_p], \tag{1.4}$$

where $\sigma_1^2 > 0$ and $-1/(p-1) < \rho < 1$. Therefore, the test of the Huynh-Feldt hypothesis may be also used for testing the hypothesis that (1.4) holds. Gleser and Olkin (1969) discussed the likelihood ratio (=LR) test for a restricted uniform covariance structure (1.4) with $\rho_0 \leq \rho < 1$ in the multivariate one-sample model which is a simple MANOVA model, where ρ_0 is some number such that $-1/(p-1) < \rho_0 < 1$.

The purpose of this paper is to obtain the LR criterion for a hypothesis concerning the adequacy of the random-effects covariance structure (1.3), which is equivalent to the restricted uniform covariance structure with $0 \leq \rho < 1$, under the parallel profile model (1.2) which is a special case of mixed MANOVA-GMANOVA models or extended growth curve models (see, Chinchilli and Elswick (1985), Verbyla and Venables (1988)). Here, we note that the difference between our test and the Gleser-Olkin test arises from the mean structures of the model considered. In Section 2, we give a canonical reduction. In Section 3, the LR criterion and its asymptotic null distribution are examined. The LR criterion for the random-effects covariance structure (1.3) in the general (non-parallel) one-way MANOVA model is also obtained. In Section 4, the results of Section 3 are exemplified by a data set of repeated measurements. The LR tests for (1.3) are compared with the test of the Huynh-Feldt hypothesis.

2. A canonical reduction

The random-effects covariance structure (1.3) is based on the following model:

$$\boldsymbol{x}_j^{(g)} = (\delta_g + u_j^{(g)})\mathbf{1}_p + \boldsymbol{\mu} + \boldsymbol{e}_j^{(g)}, \tag{2.1}$$

where the $u_j^{(g)}$ and $\boldsymbol{e}_j^{(g)}$ are independent,

$$u_j^{(g)} \sim N(0, \lambda^2), \qquad \boldsymbol{e}_j^{(g)} \sim N_p(\mathbf{0}, \sigma^2 I_p).$$

Here $u_j^{(g)}$'s denote variation between individuals for each group. From (2.1), we have

$$V[\boldsymbol{x}_j^{(g)}] = \Sigma = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p.$$

Therefore, the model of X with random effects can be written as

$$X \sim N_{N\times p}(A_1 \boldsymbol{\delta} \mathbf{1}_p' + \mathbf{1}_N \boldsymbol{\mu}', (\lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p) \otimes I_N). \tag{2.2}$$

In order to examine whether or not the model (2.2) can be assumed, we consider testing the hypothesis

$$H_{01} : \Sigma = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p \quad \text{vs.} \quad H_{11} : \text{not } H_{01} \tag{2.3}$$

under the model (1.2).

We now reduce the model (1.2) to a canonical form. Let $Q = [p^{-1/2}\mathbf{1}_p \;\; Q_2]$ and $H = [N^{-1/2}\mathbf{1}_N \;\; H_2]$ be the orthogonal matrices of orders p and N, respectively. Consider the transformation from X to

$$\begin{bmatrix} \boldsymbol{z}' \\ Y \end{bmatrix} = \begin{bmatrix} N^{-1/2}\mathbf{1}_N' \\ H_2' \end{bmatrix} XQ.$$

Then, letting $Y = [\boldsymbol{y}_1 \;\; Y_2]$,

$$\begin{bmatrix} \boldsymbol{z}' \\ \boldsymbol{y}_1 \;\; Y_2 \end{bmatrix} \sim N_{N\times p}\left(\begin{bmatrix} \boldsymbol{\theta}' \\ \tilde{A}_1 \boldsymbol{\gamma} \;\; 0 \end{bmatrix}, \Psi \otimes I_N \right), \tag{2.4}$$

where $\boldsymbol{\theta}' = N^{-1/2}\mathbf{1}_N' A_1 [\boldsymbol{\gamma} \;\; 0] + N^{1/2}\boldsymbol{\mu}' Q$, $\tilde{A}_1 = H_2' A_1$, $\boldsymbol{\gamma} = p^{1/2}\boldsymbol{\delta}$ and

$$\Psi = Q'\Sigma Q = \begin{pmatrix} \psi_{11} & \boldsymbol{\psi}_{12} \\ \boldsymbol{\psi}_{21} & \Psi_{22} \end{pmatrix}.$$

We can express the hypothesis (2.3) as

$$H_{01} : \Psi = \begin{pmatrix} p\lambda^2 + \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 I_{p-1} \end{pmatrix} \quad \text{vs.} \quad H_{11} : \text{not } H_{01}. \tag{2.5}$$

3. LR tests

We may consider the LR test for the hypothesis (2.5) under (2.4) instead of the one for the hypothesis (2.3) under (1.2). Let

$$S = Y'(I_{N-1} - P_{\tilde{A}_1})Y = \begin{pmatrix} s_{11} & \boldsymbol{s}_{12} \\ \boldsymbol{s}_{21} & S_{22} \end{pmatrix}, \qquad s_{11\cdot 2} = s_{11} - \boldsymbol{s}_{12}S_{22}^{-1}\boldsymbol{s}_{21},$$

where $P_{\tilde{A}_1} = \tilde{A}_1(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'$, and let $L(\boldsymbol{\theta}, \boldsymbol{\gamma}, \Psi)$ be the likelihood function of $[\boldsymbol{z}\ Y']$, i.e., X. Then we have

$$\begin{aligned} g(\boldsymbol{\gamma}, \Psi) &= -2\log L(\hat{\boldsymbol{\theta}}, \boldsymbol{\gamma}, \Psi) \\ &= N\log|\Psi| + \operatorname{tr}\Psi^{-1}[\boldsymbol{y}_1 - \tilde{A}_1\boldsymbol{\gamma}\ Y_2]'[\boldsymbol{y}_1 - \tilde{A}_1\boldsymbol{\gamma}\ Y_2], \end{aligned}$$

where $\hat{\boldsymbol{\theta}} = \boldsymbol{z}$. The minimum of $g(\boldsymbol{\gamma}, \Psi)$ with respect to $\boldsymbol{\gamma}$ and $\Psi > 0$ can be obtained by a well-known technique in a general MANOVA model (see, e.g., Gleser and Olkin (1970)) and is given by

$$\min g(\boldsymbol{\gamma}, \Psi) = N\log\left(\frac{1}{N}s_{11\cdot 2} \times \left|\frac{1}{N}Y_2'Y_2\right|\right) + Np. \tag{3.1}$$

On the other hand, the minimum of $g(\boldsymbol{\gamma}, \sigma^2, \lambda^2)$ under H_{01} is achieved at

$$\begin{aligned} &\hat{\boldsymbol{\gamma}} = (\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'\boldsymbol{y}_1, \qquad \hat{\lambda}^2 = \max\left\{\frac{1}{p}\left[\frac{1}{N}s_{11} - \frac{1}{N(p-1)}\operatorname{tr}Y_2'Y_2\right], 0\right\}, \\ &\hat{\sigma}^2 = \min\left\{\frac{1}{N(p-1)}\operatorname{tr}Y_2'Y_2, \frac{1}{Np}(s_{11} + \operatorname{tr}Y_2'Y_2)\right\} \end{aligned} \tag{3.2}$$

(see Yokoyama and Fujikoshi (1993)). Therefore, from (3.1) the LR criterion can be written as

$$\Lambda = \begin{cases} \Lambda_1, & \text{if } s_1/f_1 \geq s_2/f_2, \\ \Lambda_2, & \text{if } s_1/f_1 < s_2/f_2, \end{cases} \tag{3.3}$$

where $f_1 = N$, $f_2 = N(p-1)$,

$$\Lambda_1 = \frac{\left(\dfrac{s_3}{f_1}\right)^{f_1/2}\left|\dfrac{1}{f_1}S_4\right|^{f_1/2}}{\left(\dfrac{s_1}{f_1}\right)^{f_1/2}\left(\dfrac{s_2}{f_2}\right)^{f_2/2}}, \qquad \Lambda_2 = \frac{\left(\dfrac{s_3}{f_1}\right)^{f_1/2}\left|\dfrac{1}{f_1}S_4\right|^{f_1/2}}{\left(\dfrac{s_1+s_2}{f_1+f_2}\right)^{(f_1+f_2)/2}},$$

and

$$\begin{aligned} &s_1 = s_{11\cdot 2} + \boldsymbol{s}_{12}S_{22}^{-1}\boldsymbol{s}_{21}, \qquad s_2 = \operatorname{tr}(S_{22} + Y_2'P_{\tilde{A}_1}Y_2), \\ &s_3 = s_{11\cdot 2}, \qquad S_4 = S_{22} + Y_2'P_{\tilde{A}_1}Y_2. \end{aligned}$$

We now express the LR criterion (3.3) in terms of the original observations. Let

$$S_t = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})',$$

$$S_w = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})',$$

where $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}^{(g)}$ are the sample mean vectors of observations of all the groups and the g-th group, respectively. Since

$$(\tilde{A}_1'\tilde{A}_1)^{-1} = \mathrm{diag}\left(\frac{1}{N_1}, \dots, \frac{1}{N_{k-1}}\right) + \frac{1}{N_k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}',$$

it is seen that

$$s_1 = \frac{1}{p}\mathbf{1}_p'S_w\mathbf{1}_p, \qquad s_2 = \mathrm{tr}\, S_t - \frac{1}{p}\mathbf{1}_p'S_t\mathbf{1}_p,$$

$$s_3 = \left(\frac{1}{p}\mathbf{1}_p'S_w^{-1}\mathbf{1}_p\right)^{-1}, \qquad |S_4| = \frac{1}{p}\mathbf{1}_p'S_t^{-1}\mathbf{1}_p|S_t|.$$

The statistics Λ_1 and Λ_2 given in (3.3) can be decomposed as

$$\Lambda_1 = \lambda \times \lambda_1 \quad \text{and} \quad \Lambda_2 = \lambda \times \lambda_2, \tag{3.4}$$

where

$$\lambda = \frac{\left(\frac{s_3}{f_1}\right)^{f_1/2}\left|\frac{1}{f_1}S_4\right|^{f_1/2}}{\left(\frac{s_1}{f_1}\right)^{f_1/2}\left(\prod_{i=1}^{p-1}\frac{s_{ii}^{(4)}}{f_1}\right)^{f_1/2}}, \qquad \lambda_1 = \frac{\left(\prod_{i=1}^{p-1}\frac{s_{ii}^{(4)}}{f_1}\right)^{f_1/2}}{\left(\frac{s_2}{f_2}\right)^{f_2/2}},$$

$$\lambda_2 = \frac{\left(\frac{s_1}{f_1}\right)^{f_1/2}\left(\prod_{i=1}^{p-1}\frac{s_{ii}^{(4)}}{f_1}\right)^{f_1/2}}{\left(\frac{s_1+s_2}{f_1+f_2}\right)^{(f_1+f_2)/2}},$$

and $s_{ii}^{(4)}$'s denote the diagonal elements of S_4. It is known (Boik (1988)) that under H_{01}, the statistics λ and λ_i in (3.4) are independent, $i = 1, 2$. The statistic λ is the LR criterion for testing the hypothesis that Ψ is a diagonal matrix, and the statistics λ_1 and λ_2 are the LR criteria for testing equality of the diagonal elements of Ψ_{22} and Ψ, respectively, given that Ψ is diagonal. Therefore, we note that the statistics Λ_1 and Λ_2 are the LR criteria for

$$\tilde{H}_{01} : \Psi = \begin{pmatrix} \tau^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 I_{p-1} \end{pmatrix} \quad \text{and} \quad H_{02} : \Psi = \sigma^2 I_p,$$

respectively, where $\tau^2 > 0$.

THEOREM 3.1. *Let Λ be the LR criterion for testing $H_{01} : \Sigma = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p$ vs. H_{11} : not H_{01} under the parallel profile model (1.2). Then, under H_{01} it holds that*

$$\lim_{N\to\infty} P(-2\log\Lambda \ge c) = \begin{cases} P(\chi_f^2 \ge c), & \text{if } \lambda^2 > 0, \\ \dfrac{1}{2}\{P(\chi_f^2 \ge c) + P(\chi_{f+1}^2 \ge c)\}, & \text{if } \lambda^2 = 0, \end{cases}$$

where $f = (p^2 + p - 4)/2$ and χ_f^2 denotes a χ^2 variate with f degrees of freedom.

PROOF. From the definition of Λ we have

$$\begin{aligned} P(-2\log\Lambda \ge c) = P(&-2\log\Lambda_1 \ge c, s_1/f_1 \ge s_2/f_2) \\ &+ P(-2\log\Lambda_2 \ge c, s_1/f_1 < s_2/f_2). \end{aligned}$$

Under H_{01}, it is well known (see, e.g., Anderson (1984)) that $s_{11\cdot 2}$, $\boldsymbol{s}_{12} S_{22}^{-1} \boldsymbol{s}_{21}$ and S_{22} are independent,

$$\begin{aligned} &s_{11\cdot 2} \sim \tau^2 \chi^2_{N-k-p+1}, \qquad \boldsymbol{s}_{12} S_{22}^{-1} \boldsymbol{s}_{21} \sim \tau^2 \chi^2_{p-1}, \\ &S_{22} \sim W_{p-1}(N-k, \sigma^2 I_{p-1}), \end{aligned}$$

where $\tau = (p\lambda^2 + \sigma^2)^{1/2}$. Let

$$\frac{1}{\sqrt{2N}}\left(\frac{s_{11\cdot 2}}{\tau^2} - N\right) = v, \qquad \frac{1}{\sqrt{2N}}\left(\frac{1}{\sigma^2} S_{22} - N I_{p-1}\right) = U.$$

Then the limiting distribution of v and u_{ii} is $N(0,1)$, and that of u_{ij} is $N(0, 1/2)$, $1 \le i, j \le p-1$, $i \ne j$, and s_1/f_1, s_2/f_2, s_3/f_1 and S_4/f_1 can be expressed as

$$\begin{aligned} &\frac{s_1}{f_1} = \tau^2\left(1 + \sqrt{\frac{2}{f_1}}v + \frac{1}{f_1}t\right), \\ &\frac{s_2}{f_2} = \sigma^2\left(1 + \sqrt{\frac{2}{(p-1)f_2}}\operatorname{tr} U + \frac{1}{f_2}\operatorname{tr} T\right), \\ &\frac{s_3}{f_1} = \tau^2\left(1 + \sqrt{\frac{2}{f_1}}v\right), \qquad \frac{1}{f_1}S_4 = \sigma^2\left(I_{p-1} + \sqrt{\frac{2}{f_1}}U + \frac{1}{f_1}T\right), \end{aligned}$$

where $\tau^2 t = \boldsymbol{s}_{12} S_{22}^{-1} \boldsymbol{s}_{21}$, $\sigma^2 T = Y_2' P_{\tilde{A}_1} Y_2$. Using $\log|I + D| = \operatorname{tr} D - \operatorname{tr} D^2/2 + \cdots$, we can expand $-2\log\Lambda_1$ as

$$-2\log\Lambda_1 = R_1 + O_p(N^{-1/2}),$$

where $R_1 = t + \operatorname{tr} U^2 - (\operatorname{tr} U)^2/(p-1)$. Here we note that the limiting distribution of R_1 is χ_f^2, $f = (p^2 + p - 4)/2$. When $\tau^2 > \sigma^2$, $\lim_{N\to\infty} P(s_1/f_1 \ge s_2/f_2) = 1$ and hence

$$\begin{aligned} \lim_{N\to\infty} P(-2\log\Lambda \ge c) &= \lim_{N\to\infty} P(-2\log\Lambda_1 \ge c) \\ &= P(\chi_f^2 \ge c). \end{aligned}$$

When $\tau^2 = \sigma^2$, we have

$$-2\log\Lambda_2 = R_2 + O_p(N^{-1/2}),$$

where $R_2 = t + v^2 + \operatorname{tr} U^2 - (v + \operatorname{tr} U)^2/p$. Let

$$Z = \sqrt{\frac{p-1}{p}}\left(v - \frac{1}{p-1}\operatorname{tr} U\right).$$

It is easy to verify that Z and R_1 are independent, $R_2 = Z^2 + R_1$. Since the limiting distribution of Z is $N(0,1)$, and $s_1/f_1 \geq s_2/f_2$ is equivalent to $Z \geq 0$, it holds that

$$\begin{aligned}\lim_{N\to\infty} P(-2\log\Lambda \geq c) &= \lim_{N\to\infty}\{P(R_1 \geq c, Z \geq 0) + P(R_2 \geq c, Z < 0)\}\\ &= \frac{1}{2}\{P(\chi_f^2 \geq c) + P(\chi_{f+1}^2 \geq c)\},\end{aligned}$$

which proves the desired result.

Next, we consider the LR test for the hypothesis (2.3) under the general (non-parallel) one-way MANOVA model, i.e.,

$$X \sim N_{N\times p}(A\mu, \Sigma \otimes I_N), \tag{3.5}$$

where

$$A = \begin{pmatrix} \mathbf{1}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{N_k} \end{pmatrix}$$

is an $N \times k$ between-individual design matrix of rank $k(\leq N-p)$, $\mu = [\boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(k)}]'$ is an unknown $k\times p$ parameter matrix. This testing problem is an extension of the multivariate one-sample case due to Gleser and Olkin (1969) to the multivariate k-sample case. In this case, the random-effects covariance structure (1.3) is based on the following model:

$$\boldsymbol{x}_j^{(g)} = \boldsymbol{\mu}^{(g)} + u_j^{(g)}\mathbf{1}_p + \boldsymbol{e}_j^{(g)}, \tag{3.6}$$

where the $u_j^{(g)}$ and $\boldsymbol{e}_j^{(g)}$ are defined in (2.1). We may consider the LR test for the hypothesis (2.5) under a transformed model

$$Y^* \sim N_{N\times p}(A\xi, \Psi \otimes I_N), \tag{3.7}$$

where $Y^* = XQ$, $\xi = \mu Q$ and $\Psi = Q'\Sigma Q$. Let

$$S^* = Y^{*\prime}(I_N - P_A)Y^* = \begin{pmatrix} s_{11}^* & \boldsymbol{s}_{12}^* \\ \boldsymbol{s}_{21}^* & S_{22}^* \end{pmatrix},$$

where $P_A = A(A'A)^{-1}A'$, and let $L^*(\xi, \Psi)$ be the likelihood function of Y^*. Then we have

$$\begin{aligned} g^*(\xi, \Psi) &= -2 \log L^*(\xi, \Psi) \\ &= N \log |\Psi| + \operatorname{tr} \Psi^{-1}[Y^* - A\xi]'[Y^* - A\xi]. \end{aligned}$$

The minimum of $g^*(\xi, \Psi)$ when ξ and Ψ are unrestricted is given by

$$\min g^*(\xi, \Psi) = N \log \left| \frac{1}{N} S^* \right| + Np. \tag{3.8}$$

By the same way as in (3.2), it is easy to see that the MLE's of ξ, λ^2 and σ^2 under H_{01} are given by

$$\begin{aligned} &\tilde{\xi} = (A'A)^{-1}A'Y^*, \qquad \tilde{\lambda}^2 = \max\left\{ \frac{1}{p}\left[\frac{1}{N} s_{11}^* - \frac{1}{N(p-1)} \operatorname{tr} S_{22}^* \right], 0 \right\}, \\ &\tilde{\sigma}^2 = \min\left\{ \frac{1}{N(p-1)} \operatorname{tr} S_{22}^*, \frac{1}{Np} \operatorname{tr} S^* \right\}. \end{aligned} \tag{3.9}$$

Therefore, from (3.8) the LR criterion can be written as

$$\Lambda^* = \begin{cases} \Lambda_1^*, & \text{if } s_{11}^*/f_1 \geq \operatorname{tr} S_{22}^*/f_2, \\ \Lambda_2^*, & \text{if } s_{11}^*/f_1 < \operatorname{tr} S_{22}^*/f_2, \end{cases} \tag{3.10}$$

where f_1 and f_2 are defined in (3.3),

$$\Lambda_1^* = \frac{\left| \frac{1}{f_1} S^* \right|^{f_1/2}}{\left(\frac{1}{f_1} s_{11}^* \right)^{f_1/2} \left(\frac{1}{f_2} \operatorname{tr} S_{22}^* \right)^{f_2/2}}, \qquad \Lambda_2^* = \frac{\left| \frac{1}{f_1} S^* \right|^{f_1/2}}{\left(\frac{1}{f_1 + f_2} \operatorname{tr} S^* \right)^{(f_1+f_2)/2}},$$

and

$$s_{11}^* = \frac{1}{p} \mathbf{1}_p' S_w \mathbf{1}_p, \qquad \operatorname{tr} S_{22}^* = \operatorname{tr} S_w - \frac{1}{p} \mathbf{1}_p' S_w \mathbf{1}_p, \qquad |S^*| = |S_w|.$$

Noting that $|S^*| = s_{11\cdot 2}^* |S_{22}^*|$, let

$$\frac{1}{\sqrt{2N}} \left(\frac{1}{\tau^2} s_{11\cdot 2}^* - N \right) = v^*, \qquad \frac{1}{\sqrt{2N}} \left(\frac{1}{\sigma^2} S_{22}^* - N I_{p-1} \right) = U^*.$$

Then, by an argument similar to the one in the proof of Theorem 3.1, we have the following theorem.

THEOREM 3.2. *Let Λ^* be the LR criterion for testing $H_{01} : \Sigma = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p$ vs. H_{11} : not H_{01} under the general one-way MANOVA model* (3.5). *Then, under H_{01} the limiting distribution of $-2 \log \Lambda^*$ is the same as the one of $-2 \log \Lambda$ in Theorem* 3.1.

From the limiting distributions of the LR criteria in Theorems 3.1 and 3.2, we can suggest a conservative critical value c^* of size α test in an asymptotic sense such that

$$\frac{1}{2}\{P(\chi_f^2 \geq c^*) + P(\chi_{f+1}^2 \geq c^*)\} = \alpha.$$

We note that the MLE's $\hat{\lambda}^2$ and $\hat{\sigma}^2$ in (3.2) are not unbiased, and so are $\tilde{\lambda}^2$ and $\tilde{\sigma}^2$ in (3.9). The usual unbiased estimators of λ^2 and σ^2 under the parallel profile model (2.2) may be defined by

$$\begin{aligned}\hat{\hat{\lambda}}^2 &= \frac{1}{p}\left\{\frac{1}{N-k}s_{11} - \frac{1}{(N-1)(p-1)}\operatorname{tr} Y_2'Y_2\right\},\\ \hat{\hat{\sigma}}^2 &= \frac{1}{(N-1)(p-1)}\operatorname{tr} Y_2'Y_2.\end{aligned}$$

However, there is the possibility that the use of $\hat{\hat{\lambda}}^2$ can lead to a negative estimate of λ^2. Relating to such unbiased estimators, we may propose modified LR criteria by replacing f_1 and f_2 by $N-k$ and $(N-1)(p-1)$ respectively in (3.3) and (3.10). The limiting distributions of the modified LR criteria are the same as ones of the LR criteria in Theorems 3.1 and 3.2.

4. An example

We apply the results of Section 3 to the data (see, e.g., Srivastava and Carter (1983), p. 227) of the price indices of hand soaps packaged in 4 ways, estimated by 12 consumers. For 6 of the consumers, the packages have been labeled with a well-known brand name. For the remaining 6 consumers, no label is used. From this repeated measures data with $p = 4$, $k = 2$ and $N = 12$, we obtain

$$\begin{aligned}&\bar{\boldsymbol{x}}^{(1)} = (.31667, \quad .45833, \quad .47500, \quad .64167)',\\ &\bar{\boldsymbol{x}}^{(2)} = (.60000, \quad .66667, \quad .85000, \quad .96667)',\\ &\bar{\boldsymbol{x}} = (.45833, \quad .56250, \quad .66250, \quad .80417)',\\ &S_t = \begin{pmatrix} .45917 & .32875 & .52375 & .35958\\ & .38563 & .39813 & .41688\\ & & .72563 & .54438\\ & & & .61229\end{pmatrix},\\ &S_w = \begin{pmatrix} .21833 & .15167 & .20500 & .08333\\ & .25542 & .16375 & .21375\\ & & .30375 & .17875\\ & & & .29542\end{pmatrix}.\end{aligned}$$

We now consider testing the "parallelism" hypothesis

$$H_{00} : E[\boldsymbol{x}_j^{(g)}] = \delta_g \mathbf{1}_p + \boldsymbol{\mu} \quad \text{vs.} \quad H_{10} : E[\boldsymbol{x}_j^{(g)}] = \boldsymbol{\mu}^{(g)} \tag{4.1}$$

in the model (1.1) when Σ is unknown positive definite, where $g = 1, \ldots, k$, and $\delta_k = 0$. The LR statistic for the hypothesis (4.1) has been obtained by Chinchilli and Elswick (1985) and Srivastava (1987). The alternative expression of the statistic can be written as

$$\Lambda_0 = \frac{|S_w|}{|S_w + CS_bC'|}, \tag{4.2}$$

where $S_b = S_t - S_w$, $C = I_p - \mathbf{1}_p\mathbf{1}_p' S_w^{-1}(\mathbf{1}_p' S_w^{-1}\mathbf{1}_p)^{-1}$. This gives $\Lambda_0 = .65343$, and

$$-[N - k - (p - k + 1)/2] \log \Lambda_0 = 3.6169 < \chi^2_{(p-1)(k-1)}(.05) = 7.815.$$

The p-value is given by $P(-8.5 \log \Lambda_0 > 3.6169) = P(\chi_3^2 > 3.6169) = .31$. We may consider testing the hypothesis (2.3) under the parallel profile model (1.2). Since $f_1 = N = 12$, $f_2 = N(p-1) = 36$,

$$s_1 = \frac{1}{p}\mathbf{1}_p' S_w \mathbf{1}_p = .76635, \qquad s_2 = \operatorname{tr} S_t - \frac{1}{p}\mathbf{1}_p' S_t \mathbf{1}_p = .35130,$$
$$s_3 = \left(\frac{1}{p}\mathbf{1}_p' S_w^{-1}\mathbf{1}_p\right)^{-1} = .56724, \qquad |S_4| = \frac{1}{p}\mathbf{1}_p' S_t^{-1}\mathbf{1}_p |S_t| = .00043077$$

and $s_1/f_1 > s_2/f_2$, it follows from (3.3) and Theorem 3.1 that

$$\Lambda = \frac{\left(\dfrac{s_3}{f_1}\right)^{f_1/2} \left|\dfrac{1}{f_1} S_4\right|^{f_1/2}}{\left(\dfrac{s_1}{f_1}\right)^{f_1/2} \left(\dfrac{s_2}{f_2}\right)^{f_2/2}} = .0000613,$$

and

$$-2 \log \Lambda = 19.399 > \chi^2_{(p^2+p-4)/2+1}(.05) = 16.919.$$

The p-value is given by

$$P(-2 \log \Lambda > 19.399) = \frac{1}{2}\{P(\chi_8^2 > 19.399) + P(\chi_9^2 > 19.399)\} = .019.$$

Here, we note that this p-value is conservative. We next consider testing the hypothesis (2.3) under the general one-way MANOVA model (3.5). Since

$$s_{11}^* = \frac{1}{p}\mathbf{1}_p' S_w \mathbf{1}_p = .76635, \qquad \operatorname{tr} S_{22}^* = \operatorname{tr} S_w - \frac{1}{p}\mathbf{1}_p' S_w \mathbf{1}_p = .30656,$$
$$|S^*| = |S_w| = .00015966$$

and $s_{11}^*/f_1 > \operatorname{tr} S_{22}^*/f_2$, it follows from (3.10) and Theorem 3.2 that

$$\Lambda^* = \frac{\left|\dfrac{1}{f_1} S^*\right|^{f_1/2}}{\left(\dfrac{1}{f_1} s_{11}^*\right)^{f_1/2} \left(\dfrac{1}{f_2} \operatorname{tr} S_{22}^*\right)^{f_2/2}} = .0000554,$$

and

$$-2\log\Lambda^* = 19.602 > \chi^2_{(p^2+p-4)/2+1}(.05) = 16.919.$$

The p-value is given by

$$P(-2\log\Lambda^* > 19.602) = \frac{1}{2}\{P(\chi^2_8 > 19.602) + P(\chi^2_9 > 19.602)\} = .018.$$

Hence, the random-effects covariance structure (1.3) is rejected at the 5% level (with observed significance levels of a little less than 2%). We note that the difference between the mean structure of (1.2) and that of (3.5) has little effect on the values of the LR criteria Λ and Λ^* in this example.

On the other hand, the alternative expression (Morrison (1990), p. 296) of the test of the Huynh-Feldt hypothesis, which differs from (3.10) only in not taking account of certain restrictions of the range of ρ in (1.4), can be written as

$$W = \frac{(p-1)^{p-1}|S_w|(\mathbf{1}_p' S_w^{-1}\mathbf{1}_p)}{p\left(\operatorname{tr} S_w - \frac{1}{p}\mathbf{1}_p' S_w \mathbf{1}_p\right)^{p-1}} = \frac{|S_{22}^*|}{\left(\frac{1}{p-1}\operatorname{tr} S_{22}^*\right)^{p-1}}. \tag{4.3}$$

This gives $W = .26378$, and

$$-\left[N - k - \frac{1}{6(p-1)}(2p^2 - 3p + 3)\right]\log W = 11.623 > \chi^2_{p(p-1)/2-1}(.05) = 11.071.$$

The p-value is given by $P(-8.72\log W > 11.623) = P(\chi^2_5 > 11.623) = .042$. Hence, the uniform covariance structure (1.4) is also rejected at the 5% level.

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