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JOINT DISTRIBUTIONS OF NUMBERS OF SUCCESS-RUNS AND FAILURES UNTIL THE FIRST CONSECUTIVE k SUCCESSES*

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Abstract. Joint distributions of the numbers of failures, successes and success-runs of length less than k until the first consecutive k successes are obtained for some random sequences such as a sequence of independent and identically distributed integer valued random variables, a $\{0, 1\}$-valued Markov chain and a binary sequence of order k. There are some ways of counting numbers of runs with a specified length. This paper studies the joint distributions based on three ways of counting numbers of runs, i.e., the number of overlapping runs with a specified length, the number of non-overlapping runs with a specified length and the number of runs with a specified length or more. Marginal distributions of them can be derived immediately, and most of them are surprisingly simple.

Key words and phrases: Probability generating function, geometric distribution, discrete distributions, Markov chain, waiting time, geometric distribution of order k, binary sequence of order k.

1. Introduction

We denote by τ the waiting time (the number of trials) for the first consecutive k successes in independent Bernoulli trials with success probability p. The distribution of τ is called the geometric distribution of order k (cf. Feller (1968) and Philippou *et al.* (1983)). Aki and Hirano (1994) studied the exact marginal distributions of numbers of failures, successes and overlapping number of success-runs of length l until τ. These distributions are the geometric distribution, $G(p^k)$, the shifted geometric distribution of order $k-1$, $G_{k-1}(p, k)$, and the shifted geometric distribution of order $k-l$, $G_{k-l}(p, k-l+1)$, respectively. Here, we denote by $G_k(p, a)$ the shifted geometric distribution of order k so that its support begins with a.

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In this paper, we study the joint distributions of waiting time τ and number of outcomes such as successes, failures and success-runs until τ in more general situations.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables taking values in $\{1, 2, \ldots, m\}$. Let $p_i = P(X_1 = i)$ for $i = 1, 2, \ldots, m$. We regard the value 1 as success and the remaining $m-1$ values as failures. We consider $m-1$ kinds of failures. If $m = 2$ then the sequence can be regarded as the independent Bernoulli trials. We denote by τ the waiting time (the number of trials) for the first consecutive k successes in $X_1, X_2, \ldots$. Let η_j be the number of occurrences of j among $X_1, X_2, \ldots, X_\tau$. For integers $a_1, \ldots, a_m$, let $d(a_1, \ldots, a_m) = P(\tau = a_1, \eta_2 = a_2, \ldots, \eta_m = a_m)$. Then we have

$$\begin{cases} d(a_1, \ldots, a_m) = 0 \quad \text{if } a_1 < k \text{ or } a_j < 0 \text{ for some } j > 1, \\ d(k, 0, \ldots, 0) = p_1^k, \\ d(k, a_2, \ldots, a_m) = 0 \quad \text{if } a_j \neq 0 \text{ for some } j > 1, \\ d(a_1, \ldots, a_m) = \sum_{j=1}^{k} \sum_{i=2}^{m} d(a_1 - j, a_2 - \delta_{2i}, \ldots, a_m - \delta_{mi}) p_1^{j-1} p_i \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } a_1 > k, \end{cases}$$

by considering all possibilities of the first occurrence of one of the failures. Define the joint probability generating function (pgf) of $(\tau, \eta_2, \ldots, \eta_m)$ by

$$\phi_1(t_1, t_2, \ldots, t_m) = \sum_{a_1=0}^{\infty} \cdots \sum_{a_m=0}^{\infty} d(a_1, \ldots, a_m) t_1^{a_1} \ldots t_m^{a_m}.$$

Then we can obtain a recurrence relation of $\phi_1(t_1, \ldots, t_m)$ from the above equations. By solving it, we have

PROPOSITION 1.1. *The joint pgf $\phi_1(t_1, \ldots, t_m)$ is given by*

$$\phi_1(t_1, \ldots, t_m) = \frac{(1 - p_1 t_1)(p_1 t_1)^k}{1 - p_1 t_1 - t_1 (\sum_{i=2}^{m} p_i t_i)(1 - (p_1 t_1)^k)}.$$

Since the result of Proposition 1.1 is a corollary of Proposition 2.1 in Section 2, we omit a proof of Proposition 1.1. We can regard Proposition 1.1 as a general version of Proposition 2.1 of Aki and Hirano (1994). Moreover, some interesting results on marginal distributions and moments can be derived from it. For example, we can obtain the means, variances and covariances of $\tau, \eta_2, \ldots, \eta_m$ by differentiating the joint pgf.

COROLLARY 1.1. *The means, variances and covariances of $\tau, \eta_2, \ldots, \eta_m$ are given by*

$$E(\tau) = \frac{1 - {p_1}^k}{(1 - p_1) p_1^k},$$

$$E(\eta_j) = p_j \cdot E(\tau), \qquad for\ j = 2, \dots, m,$$
$$\mathrm{Var}(\tau) = \frac{1-(2k+1)(1-p_1)p_1^k - p_1^{2k+1}}{(1-p_1)^2 p_1^{2k}},$$
$$\mathrm{Var}(\eta_j) = \{E(\eta_j)\}^2 + E(\eta_j)$$
$$= \frac{p_j(1-p_1^k)\{p_j(1-p_1^k) + (1-p_1)p_1^k\}}{(1-p_1)^2 p_1^{2k}}, \qquad for\ j = 2, \dots, m,$$
$$\mathrm{Cov}(\tau, \eta_j) = \frac{p_j\{1-p_1^k - k(1-p_1)p_1^k\}}{(1-p_1)^2 p_1^{2k}}, \qquad for\ j = 2, \dots, m,$$
$$\mathrm{Cov}(\eta_l, \eta_j) = E(\eta_l)E(\eta_j) = \frac{p_l p_j (1-p_1^k)^2}{(1-p_1)^2 p_1^{2k}}, \qquad for\ l \neq j.$$

We can easily check that the covariances in Corollary 1.1 are non-negative and hence the variables $\tau, \eta_2, \dots, \eta_m$ are positively correlated.

COROLLARY 1.2. *For any fixed positive integer k ($k \geq 1$), the joint distribution of $\eta_2, \dots, \eta_m$ is $m-1$ dimensional geometric distribution (cf. Johnson and Kotz* (1969)).

It may be noted that Corollary 1.2 generalizes Corollary 2.1 of Aki and Hirano (1994).

In Section 2, the joint distribution of the numbers of successes and failures until the waiting time τ is investigated. Section 3 treats more general case that the joint distribution of the numbers of success-runs of length l ($l = 1, 2, \dots, k$) and the numbers of failures until τ. It is well known that there are different ways of counting the numbers of success-runs (cf. Hirano and Aki (1993)). The joint distributions of them are studied on the basis of three ways of counting. In Section 4, the corresponding results are given based on some dependent sequences.

2. Joint distribution of numbers of successes and failures

In this section, we study the joint distribution of $(\tau, \eta_1, \eta_2, \dots, \eta_m)$. Let $\phi_2(r, t_1, \dots, t_m)$ be the joint pgf of $(\tau, \eta_1, \eta_2, \dots, \eta_m)$. For $i = 0, 1, \dots, k-1$ and $j = 2, 3, \dots, m$, let A_i^j be the event that we start with a "1"-run of length i and "j" occurs just after the "1"-run. We denote by C the event that we start with a "1"-run of length k, i.e., $\{\tau = k\}$. Let $\phi_2(r, t_1, \dots, t_m \mid A_i^j)$ and $\phi_2(r, t_1, \dots, t_m \mid C)$ be pgf's of the conditional joint distributions of $(\tau, \eta_1, \eta_2, \dots, \eta_m)$ given that the event A_i^j occurs and given that the event C occurs, respectively. Then we can easily see that

$$\phi_2(r, t_1, \dots, t_m \mid A_i^j) = r^{i+1} t_1^i t_j \phi_2(r, t_1, \dots, t_m),$$

and

$$\phi_2(r, t_1, \dots, t_m \mid C) = r^k t_1^k.$$

Since A_i^j, $i = 0, 1, \dots, k-1$, $j = 2, 3, \dots, m$ and C construct a partition of the sample space, we have

$$\begin{aligned}\phi_2(r, t_1, \dots, t_m) &= \sum_{j=2}^{m} \sum_{i=0}^{k-1} P(A_i^j) \cdot \phi_2(r, t_1, \dots, t_m \mid A_i^j) \\ &\quad + P(C) \cdot \phi_2(r, t_1, \dots, t_m \mid C) \\ &= \sum_{j=2}^{m} \sum_{i=0}^{k-1} p_1^i p_j r^{i+1} t_1^i t_j \cdot \phi_2(r, t_1, \dots, t_m) + p_1^k r^k t_1^k \\ &= r\left(\sum_{j=2}^{m} p_j t_j\right)\left(\sum_{i=0}^{k-1} p_1^i t_1^i r^i\right) \phi_2(r, t_1, \dots, t_m) + (p_1 r t_1)^k.\end{aligned}$$

Then we obtain

PROPOSITION 2.1. *The joint pgf* $\phi_2(r, t_1, \dots, t_m)$ *is given by*

$$\phi_2(r, t_1, \dots, t_m) = \frac{(1 - p_1 t_1 r)(p_1 r t_1)^k}{1 - p_1 t_1 r - r(\sum_{j=2}^{m} p_j t_j)\{1 - (p_1 t_1 r)^k\}}.$$

If we set $t_1 = 1$ in $\phi_2(r, t_1, \dots, t_m)$, we have the joint pgf of $(\tau, \eta_2, \dots, \eta_m)$, $\phi_1(r, t_2, \dots, t_m)$.

If we set $r = 1$, $t_2 = 1, \dots,$ and $t_m = 1$, then we obtain the pgf of the (marginal) distribution of η_1. Let $q_1 = 1 - p_1$. Then we see that

$$\phi_2(1, t_1, 1, \dots, 1) = \frac{(1 - p_1 t_1) p_1^{k-1} t_1^k}{1 - t_1 + q_1 p_1^{k-1} t_1^k}.$$

The last expression is the well-known pgf of the geometric distribution of order $k-1$ multiplied by t_1, i.e., the pgf of $G_{k-1}(p_1, k)$.

3. Joint distribution of numbers of success-runs and failures

Let l be a positive integer no greater than k. In many situations, numbers of success-runs with length l are counted in various ways. Usually, the number of non-overlapping success-runs is counted as in Feller (1968) for technical reasons. Recently, however, some other ways of counting numbers of runs are treated for practical reasons. For example, Ling (1988) studied the exact distribution of the number of overlapping runs of a specified length until the n-th trial in independent Bernoulli trials with success probability p (cf. also Hirano *et al.* (1991) and Godbole (1992)). Schwager (1983) also obtained the probability of the occurrences of overlapping runs in more general situations. Goldstein (1990) studied the Poisson approximation for the distribution of the number of success-runs with a specified length or more until the n-th trial. Hirano and Aki (1993) obtained the exact distribution of the number of success-runs with a specified length or more until the n-th trial in a $\{0,1\}$-valued Markov chain.

Aki and Hirano (1994) showed that the distribution of the number of overlapping success-runs with length l until the first consecutive k successes in independent Bernoulli trials is the shifted geometric distribution of order $k-l$, $G_{k-l}(p, k-l+1)$. Let $X_1, X_2, \dots, \tau$ and $\eta_1, \eta_2, \dots, \eta_m$ be defined as in the previous sections. Let ξ_l, $l = 1, 2, \dots, k$ be the number of overlapping "1"-runs with length l until τ. We denote by $\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m)$ the joint pgf of $(\tau, \xi_1, \dots, \xi_k, \eta_2, \dots, \eta_m)$.

THEOREM 3.1. *The joint pgf* $\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m)$ *is given by*

$$\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m) = \frac{p_1^k r^k t_1^k t_2^{k-1} \cdots t_k}{1 - r(\sum_{j=2}^m p_j s_j) \sum_{i=0}^{k-1} p_1^i r^i t_1^i t_2^{i-1} \cdots t_i}.$$

PROOF. Let $\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m \mid A_i^j)$ and $\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m \mid C)$ be the pgf's of the conditional joint distributions of $(\tau, \xi_1, \dots, \xi_k, \eta_2, \dots, \eta_m)$ given that the event A_i^j occurs and given that the event C occurs, respectively. It is easy to see that

$$\begin{aligned}&\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m \mid A_i^j)\\ &\quad = r^{i+1} t_1^i t_2^{i-1} \cdots t_i s_j \cdot \phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m),\end{aligned}$$

and

$$\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m \mid C) = r^k t_1^k t_2^{k-1} \cdots t_k.$$

Since the events A_i^j, $i = 0, 1, \dots, k-1$, $j = 2, 3, \dots, m$ and C construct a partition of the sample space, we have

$$\begin{aligned}\phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m) &= \left(\sum_{i=0}^{k-1} p_1^i r^{i+1} t_1^i t_2^{i-1} \cdots t_i \left(\sum_{j=2}^m p_j s_j\right)\right)\\ &\quad \times \phi_3(r, t_1, \dots, t_k, s_2, \dots, s_m) + p_1^k r^k t_1^k t_2^{k-1} \cdots t_k.\end{aligned}$$

This completes the proof.

By setting $r = 1$, $t_1 = 1, \dots, t_{l-1} = 1$, $t_{l+1} = 1, \dots, t_k = 1$, $s_2 = 1, \dots, s_m = 1$, we obtain the pgf of ξ_l as follows:

$$\phi_3(1, \dots, 1, t_l, 1, \dots, 1) = \frac{(1 - p_1 t_l) p_1^{k-l} t_l^{k-l+1}}{1 - t_l + q_1 p_1^{k-l} t_l^{k-l+1}}$$

where $q_1 = 1 - p_1$. Thus, we see that the marginal distribution of ξ_l is the shifted geometric distribution of order $k-l$, $G_{k-l}(p_1, k-l+1)$. A direct and intuitive proof of the fact was given in Aki and Hirano (1994).

Let ν_l, $l = 1, 2, \ldots, k$ be the number of "1"-runs with length greater than or equal to l until τ. We denote by $\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$ the joint pgf of $(\tau, \nu_1, \ldots, \nu_k, \eta_2, \ldots, \eta_m)$.

THEOREM 3.2. *The joint pgf* $\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$ *is given by*

$$\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m) = \frac{p_1^k r^k t_1 t_2 \cdots t_k}{1 - r(\sum_{j=2}^m p_j s_j) \sum_{i=0}^{k-1} p_1^i r^i t_1 t_2 \cdots t_i}.$$

PROOF. Let $\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid A_i^j)$ and $\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid C)$ be the pgf's of the conditional joint distributions of $(\tau, \nu_1, \ldots, \nu_k, \eta_2, \ldots, \eta_m)$ given that the event A_i^j occurs and given that the event C occurs, respectively. It is easily seen that

$$\begin{aligned}&\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid A_i^j)\\ &\quad = r^{i+1} t_1 t_2 \cdots t_i s_j \cdot \phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m),\end{aligned}$$

and

$$\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid C) = r^k t_1 t_2 \cdots t_k.$$

Then, we have the desired result by the same way as in the proof of Theorem 3.1. This completes the proof.

COROLLARY 3.1. *For every* $l = 1, 2, \ldots, k$, *the (marginal) distribution of* ν_l *is the shifted geometric distribution with parameter* p_1^{k-l} *so that its support begins with* 1.

PROOF. By setting $r = 1$, $t_1 = 1, \ldots, t_{l-1} = 1$, $t_{l+1} = 1, \ldots, t_k = 1$, $s_2 = 1, \ldots, s_m = 1$ in $\phi_4(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$, we have

$$\phi_4(1, \ldots, 1, t_l, 1, \ldots, 1) = \frac{p_1^{k-l} t_l}{1 - (1 - p_1^{k-l}) t_l}.$$

This completes the proof.

Let μ_l, $l = 1, 2, \ldots, k$ be the number of non-overlapping "1"-runs with length l until τ by Feller's way of counting. We denote by $\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$ the joint pgf of $(\tau, \mu_1, \ldots, \mu_k, \eta_2, \ldots, \eta_m)$.

THEOREM 3.3. *The joint pgf* $\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$ *is given by*

$$\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m) = \frac{p_1^k r^k t_1^k t_2^{[k/2]} \cdots t_k^{[k/k]}}{1 - r(\sum_{j=2}^m p_j s_j) \sum_{i=0}^{k-1} p_1^i r^i t_1^i t_2^{[i/2]} \cdots t_k^{[i/k]}}.$$

PROOF. Let $\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid A_i^j)$ and $\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid C)$ be the pgf's of the conditional joint distributions of $(\tau, \mu_1, \ldots, \mu_k, \eta_2, \ldots, \eta_m)$

given that the event A_i^j occurs and given that the event C occurs, respectively. By using the same argument as the proofs of the previous theorems, it suffices to check that

$$\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid A_i^j)$$
$$= r^{i+1} t_1^i t_2^{[i/2]} \cdots t_k^{[i/k]} s_j \cdot \phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m),$$

and

$$\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m \mid C) = r^k t_1^k t_2^{[k/2]} \cdots t_k^{[k/k]}.$$

This completes the proof.

COROLLARY 3.2. *For every $l = 1, 2, \ldots, k$, the pgf of the marginal distribution of μ_l is given by*

$$\frac{p_1^k t_l^{[k/l]}}{1 - q_1 \sum_{i=0}^{k-1} p_1^i t_l^{[i/l]}}.$$

In particular, if $k = l(m+1)$ holds for a positive integer m, then the distribution of μ_l is the shifted geometric distribution of order m, $G_m(p_1^l, m+1)$.

PROOF. By setting $r = 1$, $t_1 = 1, \ldots, t_{l-1} = 1$, $t_{l+1} = 1, \ldots, t_k = 1$, $s_2 = 1, \ldots, s_m = 1$ in $\phi_5(r, t_1, \ldots, t_k, s_2, \ldots, s_m)$, we have the above formula immediately. When $k = l(m+1)$, it holds that

$$\phi_5(1, \ldots, 1, t_l, 1, \ldots, 1) = \frac{(p_1^l)^m t_l^{m+1}}{1 - \sum_{i=0}^{m-1} (p_1^l)^i (1 - p_1^l) t_l^{i+1}}.$$

This completes the proof.

4. Dependent sequences

In this section, we consider the problems treated in the previous sections based on some dependent sequences. For simplicity, only sequences of $\{0, 1\}$-valued random variables are considered. However, it is not difficult to study them based on sequences of $\{1, \ldots, m\}$-valued random variables.

First, let $X_0, X_1, X_2, \ldots$ be a $\{0, 1\}$-valued Markov chain with $P(X_0 = 0) = p_0$, $P(X_0 = 1) = p_1$, $P(X_{i+1} = 0 \mid X_i = 0) = p_{00}$, $P(X_{i+1} = 1 \mid X_i = 0) = p_{01}$, $P(X_{i+1} = 0 \mid X_i = 1) = p_{10}$, and $P(X_{i+1} = 1 \mid X_i = 1) = p_{11}$. We also denote by X_n the outcome of the n-th trial and we say success and failure for the outcomes "1" and "0", respectively. We denote by τ the number of trials until the first consecutive k successes in $X_1, X_2, \ldots$. Let η be the number of occurrences of "0" among $X_1, X_2, \ldots, X_\tau$. Let ξ_l, ν_l and μ_l, $l = 1, 2, \ldots, k$ be the numbers of overlapping "1"-runs with length l until τ, the number of "1"-runs with length greater than or equal to l until τ, and the number of non-overlapping "1"-runs with length l until τ, respectively. For $j = 0$ and 1, let $\phi_6^j(r, s, t_1, \ldots, t_k)$, $\phi_7^j(r, s, t_1, \ldots, t_k)$ and $\phi_8^j(r, s, t_1, \ldots, t_k)$ be the joint pgf's of the conditional distributions of $(\tau, \eta, \xi_1, \dots, \xi_k)$, $(\tau, \eta, \nu_1, \dots, \nu_k)$ and $(\tau, \eta, \mu_1, \dots, \mu_k)$ given that $X_0 = j$, respectively.

THEOREM 4.1. *The conditional joint pgf's are given as follows;*

$$\phi_6^0(r, s, t_1, \dots, t_k) = \frac{p_{01}p_{11}^{k-1}r^k t_1^k t_2^{k-1} \cdots t_k}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{i-1} \cdots t_i},$$
$$\phi_6^1(r, s, t_1, \dots, t_k) = \frac{p_{11}^{k-1}r^k t_1^k t_2^{k-1} \cdots t_k\{(p_{01}p_{10} - p_{00}p_{11})rs + p_{11}\}}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{i-1} \cdots t_i},$$
$$\phi_7^0(r, s, t_1, \dots, t_k) = \frac{p_{01}p_{11}^{k-1}r^k t_1 t_2 \cdots t_k}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1 t_2 \cdots t_i},$$
$$\phi_7^1(r, s, t_1, \dots, t_k) = \frac{p_{11}^{k-1}r^k t_1 t_2 \cdots t_k\{(p_{01}p_{10} - p_{00}p_{11})rs + p_{11}\}}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1 t_2 \cdots t_i},$$
$$\phi_8^0(r, s, t_1, \dots, t_k) = \frac{p_{01}p_{11}^{k-1}r^k t_1^k t_2^{[k/2]} \cdots t_k^{[k/k]}}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{[i/2]} \cdots t_k^{[i/k]}},$$

and

$$\phi_8^1(r, s, t_1, \dots, t_k) = \frac{p_{11}^{k-1}r^k t_1^k t_2^{[k/2]} \cdots t_k^{[k/k]}\{(p_{01}p_{10} - p_{00}p_{11})rs + p_{11}\}}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{[i/2]} \cdots t_k^{[i/k]}}.$$

PROOF. For $i = 0, 1, \dots, k-1$, let A_i be the event that we start with a "1"-run of length i and "0" occurs just after the "1"-run. Let C be the event that we start with a "1"-run of length k. For j=0 and 1, let $\phi_6^j(r, s, t_1, \dots, t_k \mid A_i)$ and $\phi_6^j(r, s, t_1, \dots, t_k \mid C)$ be the pgf's of the conditional joint distributions of $(\tau, \eta, \xi_1, \dots, \xi_k)$ given that the event $A_i \cap \{X_0 = j\}$ occurs and given that the event $C \cap \{X_0 = j\}$ occurs, respectively. Note that

$$\phi_6^0(r, s, t_1, \dots, t_k \mid A_i) = r^{i+1}st_1^i t_2^{i-1} \cdots t_i \phi_6^0(r, s, t_1, \dots, t_k),$$

and

$$\phi_6^0(r, s, t_1, \dots, t_k \mid C) = r^k t_1^k t_2^{k-1} \cdots t_k.$$

Since A_i, $i = 0, 1, \dots, k-1$ and C construct a partition of the sample space, we have

$$\begin{aligned}
\phi_6^0(r, s, t_1, \dots, t_k) &= \sum_{i=0}^{k-1} P(A_i)\phi_6^0(r, s, t_1, \dots, t_k \mid A_i) \\
&\quad + P(C)\phi_6^0(r, s, t_1, \dots, t_k \mid C) \\
&= p_{00}rs\phi_6^0(r, s, t_1, \dots, t_k) \\
&\quad + \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{i-1} \cdots t_i \phi_6^0(r, s, t_1, \dots, t_k) \\
&\quad + p_{01}p_{11}^{k-1}r^k t_1^k t_2^{k-1} \cdots t_k.
\end{aligned}$$

Thus, we have

$$\phi_6^0(r, s, t_1, \dots, t_k) = \frac{p_{01}p_{11}^{k-1}r^k t_1^k t_2^{k-1}\cdots t_k}{1 - p_{00}rs - \sum_{i=1}^{k-1} p_{01}p_{11}^{i-1}p_{10}r^{i+1}st_1^i t_2^{i-1}\cdots t_i}.$$

Noting that

$$\begin{aligned}\phi_6^1(r, s, t_1, \dots, t_k) &= p_{10}rs\phi_6^0(r, s, t_1, \dots, t_k) \\ &\quad + \sum_{i=1}^{k-1} p_{11}^i p_{10}r^{i+1}st_1^i t_2^{i-1}\cdots t_i\phi_6^0(r, s, t_1, \dots, t_k) \\ &\quad + p_{11}^k r^k t_1^k t_2^{k-1}\cdots t_k,\end{aligned}$$

we obtain the second equality of the theorem. The remaining equalities in the theorem can be proven similarly.

COROLLARY 4.1. *The conditional joint distributions of* $(\xi_1, \dots, \xi_k)$, $(\nu_1, \dots, \nu_k)$ *and* $(\mu_1, \dots, \mu_k)$ *do not depend on the value of* X_0. *Especially, for* $l = 1, \dots, k-1$, *the marginal distributions of* ξ_l *and* ν_l *are* $G_{k-l}(p_{11}, k-l+1)$ *and* $G_1(p_{11}^{k-l}, 1)$, *respectively.*

PROOF. By setting $r = s = 1$ in the formulas of Theorem 4.1, we see that $\phi_6^0(1, 1, t_1, \dots, t_k) = \phi_6^1(1, 1, t_1, \dots, t_k)$, $\phi_7^0(1, 1, t_1, \dots, t_k) = \phi_7^1(1, 1, t_1, \dots, t_k)$ and $\phi_8^0(1, 1, t_1, \dots, t_k) = \phi_8^1(1, 1, t_1, \dots, t_k)$, since $p_{01}p_{10} - p_{00}p_{11} + p_{11} = p_{01}$. The second statement is easy to prove by substituting 1 for all arguments except for t_l. The marginal distribution of ξ_l was obtained by Aki and Hirano ((1994), Theorem 3.3). This completes the proof.

Remark. By comparing the forms of $\phi_6^0(1, 1, t_1, \dots, t_k)$, $\phi_7^0(1, 1, t_1, \dots, t_k)$ and $\phi_8^0(1, 1, t_1, \dots, t_k)$ with $\phi_3(1, t_1, \dots, t_k, 1, \dots, 1)$, $\phi_4(1, t_1, \dots, t_k, 1, \dots, 1)$ and $\phi_5(1, t_1, \dots, t_k, 1, \dots, 1)$, respectively, we see that the joint distributions of $(\xi_1, \dots, \xi_k)$, $(\nu_1, \dots, \nu_k)$ and $(\mu_1, \dots, \mu_k)$ based on the Markov chain are the same as those based on independent Bernoulli trials, if we put $p_{11} = p$.

Next, we consider the problems based on another dependent sequence called a binary sequence of order k. The sequence was defined by Aki (1985) and studied by Hirano and Aki (1987), Aki and Hirano (1988, 1994) and Aki (1992). The sequence is closely related to the cluster sampling scheme (cf. Philippou (1988) and Xekalaki and Panaretos (1989)). When we consider the reliability of a system called consecutive-k-out-of-n:F system, the dependency in the sequence is related to $(k-1)$-step Markov dependence which was studied by Fu (1986). The definition is the following:

DEFINITION. A sequence $\{X_i\}_{i=0}^{\infty}$ of $\{0, 1\}$-valued random variables is said to be a binary sequence of order k if there exist a positive integer k and k real numbers $0 < p_1, p_2, \dots, p_k < 1$ such that

(1) $X_0 = 0$ almost surely, and

(2) $P(X_n = 1 \mid X_0 = x_0, X_1 = x_1, \dots, X_{n-1} = x_{n-1}) = p_j$ is satisfied for any positive integer n, where $j = r - [(r-1)/k] \cdot k$, and r is the smallest positive integer which satisfies $x_{n-r} = 0$.

Let $X_0, X_1, X_2, \dots$ be a binary sequence of order k defined above. Let τ, η, ξ_l, ν_l and μ_l, $l = 1, 2, \dots, k$ be random variables defined as before. Let $\phi_9(r, s, t_1, \dots, t_k)$, $\phi_{10}(r, s, t_1, \dots, t_k)$ and $\phi_{11}(r, s, t_1, \dots, t_k)$ be the joint pgf's of $(\tau, \eta, \xi_1, \dots, \xi_k)$, $(\tau, \eta, \nu_1, \dots, \nu_k)$ and $(\tau, \eta, \mu_1, \dots, \mu_k)$, respectively.

THEOREM 4.2. *The joint pgf's are given as follows;*

$$\phi_9(r, s, t_1, \dots, t_k) = \frac{p_1 p_2 \cdots p_k r^k t_1^k t_2^{k-1} \cdots t_k}{1 - q_1 rs - \sum_{i=1}^{k-1} p_1 p_2 \cdots p_i q_{i+1} r^{i+1} s t_1^i t_2^{i-1} \cdots t_i},$$

$$\phi_{10}(r, s, t_1, \dots, t_k) = \frac{p_1 p_2 \cdots p_k r^k t_1 t_2 \cdots t_k}{1 - q_1 rs - \sum_{i=1}^{k-1} p_1 p_2 \cdots p_i q_{i+1} r^{i+1} s t_1 t_2 \cdots t_i},$$

and

$$\phi_{11}(r, s, t_1, \dots, t_k) = \frac{p_1 p_2 \cdots p_k r^k t_1^k t_2^{[k/2]} \cdots t_k^{[k/k]}}{1 - q_1 rs - \sum_{i=1}^{k-1} p_1 p_2 \cdots p_i q_{i+1} r^{i+1} s t_1^i t_2^{[i/2]} \cdots t_k^{[i/k]}},$$

where $q_i = 1 - p_i$.

PROOF. Note that $P(A_0) = q_1$, $P(A_i) = p_1 p_2 \cdots p_i q_{i+1}$, for $i = 1, \dots, k-1$, and $P(C) = p_1 p_2 \cdots p_k$. Then, by the same argument as in the proof of Theorem 4.1, we can prove this theorem.

COROLLARY 4.2. *For* $l = 1, 2, \dots, k-1$, *the marginal distributions of* ξ_l *and* ν_l *are* $EG_{k-l}(p_{l+1}, \dots, p_k, k-l+1)$ *and* $G_1(p_{l+1} \cdots p_k, 1)$, *respectively.*

PROOF. We can obtain the result by substituting 1 for all arguments except for t_l in $\phi_9(r, s, t_1, \dots, t_k)$ and $\phi_{10}(r, s, t_1, \dots, t_k)$. In particular, the marginal distribution of ξ_l was given by Aki and Hirano (1994), alternatively. This completes the proof.

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