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AN EXAMPLE OF A TWO-SIDED WILCOXON SIGNED RANK TEST WHICH IS NOT UNBIASED

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Abstract. Although the theory of rank tests is rather complete in the one-sided case, it was not even known in 1959, whether the Wilcoxon two-sample test and other similiar tests are unbiased against the two-sided alternatives (Lehmann (1959, *Testing Statistical Hypotheses*, p. 240, Wiley, New York)). A partial answer to this question was given by Sugiura in 1965, who found, that the test named above may be biased (Sugiura (1965, *Ann. Inst. Statist. Math.*, **17**, 261–263)). According to Lehmann (1986, *Testing Statistical Hypotheses*, 2nd ed., pp. 322–324, Wiley, New York) it seems to be still open, whether the same is true for the WILCOXON one-sample test, which is also known as WILCOXON signed rank test. This will be shown in the present paper.

Key words and phrases: Rank test, unbiasedness, counter-example.

Let $F(x)$ be a continuous distribution function satisfying $F(-x) = 1 - F(x)$ and define $F_\mu(x) := F(x-\mu)$. We are interested in testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. Therefore let $X = (X_1, \dots, X_n)$ be a sample taken from the distribution $F_\mu(x)$, determine

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

such that $|X_{\sigma(1)}| < |X_{\sigma(2)}| < \cdots < |X_{\sigma(n)}|$ is satisfied, define

$$L_i := \begin{cases} 0 & \text{if } X_{\sigma(i)} < 0 \\ 1 & \text{if } X_{\sigma(i)} > 0 \end{cases}$$

and consider the WILCOXON statistic $T := \sum_{i=1}^n iL_i$. Then the tests

$$\Phi_m(X) := \begin{cases} 0 & \text{if } T \in [m, N-m] \\ 1 & \text{else,} \end{cases}$$

where $N = \frac{n(n+1)}{2}$ and $m \in \{1, \dots, [N/2]\}$, are the nontrivial two-sided WILCOXON signed rank tests for H_0 against H_1 (Lehmann (1975), p. 123). Φ_m is said to be unbiased, if its powerfunction

$$\beta_m(\mu) := \boldsymbol{E}_\mu[\Phi_m(X)]$$

satisfies

$$\beta_m(\mu) \geq \alpha_m \quad \forall \mu \in \boldsymbol{R},$$

where $\alpha_m := \beta_m(0)$ is the level of significance.

Let us now specify the situation: Suppose $F(x)$ has a density

$$f(x) := \begin{cases} 1/2 & \text{if } 1/2 < |x| < 3/2 \\ 0 & \text{else} \end{cases}$$

and let $n := 3$.

We now study the distribution of $L = (L_1, L_2, L_3)$ for $\mu = 0$ and $\mu = 1/2$.

For $\mu = 0$, it is well known (Lehmann (1975), p. 164), that each observation

$$\ell = (\ell_1, \ell_2, \ell_3) \in \{0, 1\}^3$$

appears with the same probability

$$\boldsymbol{P}_{\mu=0}\{L = \ell\} = \frac{1}{2^3} = 1/8.$$

For $\mu = 1/2$ the situation is more complicated. The key to the distribution of L is a look at Fig. 1, which illustrates the density $f_{1/2}$ defined by $f_{1/2}(x) = f(x - 1/2)$.

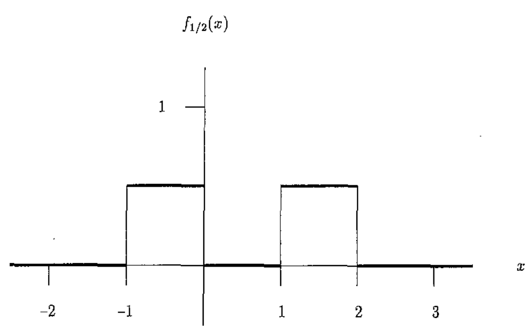

Fig. 1.

We see, that the support of $f_{1/2}$ is divided into the two intervals

$$A_1 := (-1, 0), \qquad A_2 := (1, 2)$$

and that this partition fulfills two important properties: First

$$A_1 \times A_2 \subset \boldsymbol{R}_- \times \boldsymbol{R}_+$$

and second

$$(x_1, x_2) \in A_1 \times A_2 \Longrightarrow |x_1| < |x_2|.$$

Hence $\boldsymbol{P}_{\mu=1/2}\{L_1 \leq L_2 \leq L_3\} = 1$ and it becomes clear, that for example

$$\begin{aligned}
&\boldsymbol{P}_{\mu=1/2}\{L = (0,1,1)\} \\
&\quad = \boldsymbol{P}_{\mu=1/2}(\{X_{\sigma(1)} \in A_1\} \cap \{X_{\sigma(2)} \in A_2\} \cap \{X_{\sigma(3)} \in A_2\}) \\
&\quad = \boldsymbol{P}_{\mu=1/2}(\{X_1 \in A_1\} \cap \{X_2 \in A_2\} \cap \{X_3 \in A_2\}) \cdot 3 \\
&\quad = \frac{3}{2^3}.
\end{aligned}$$

In order to calculate the remaining, non vanishing probabilities $\boldsymbol{P}_{\mu=1/2}\{L = \ell\}$ we proceed in the same manner and obtain the distribution of L as given by Table 1.

Table 1.

$\ell = (\ell_1,$	$\ell_2,$	$\ell_3)$	$t := \sum_{i=1}^{3} i\ell_i$	$\boldsymbol{P}_{\mu=1/2}\{L = \ell\}$
0	0	0	0	1/8
1	0	0	1	0
0	1	0	2	0
0	0	1	3	3/8
1	1	0	3	0
1	0	1	4	0
0	1	1	5	3/8
1	1	1	6	1/8

It follows, that the two-sided WILCOXON signed rank test

$$\Phi_3(X) = \begin{cases} 0 & \text{if } T = 3 \\ 1 & \text{else} \end{cases}$$

is not unbiased, since

$$\begin{aligned}
\beta_3(1/2) &= 1 - \boldsymbol{P}_{\mu=1/2}(T = 3) = 5/8 \\
&< 6/8 = 1 - \boldsymbol{P}_{\mu=0}(T = 3) = \alpha_3.
\end{aligned}$$

It may be desirable to achieve a more convenient level of significance than $\alpha = 3/4$. To this end it is possible to construct similiar examples, which, however, will be more complex.

We conclude this note with some remarks concerning the difference between the one-sample and the two-sample problem.

By reasons of symmetry we have $\beta_m(-\mu) = \beta_m(\mu)$ and thus Φ_3 is biased in both directions, what does not apply to SUGIURA's test. Furthermore his test compares to the case $m = 1$, which allows a comprehensive analytic discussion of the powerfunction. But in our one-sample situation the test

$$\Phi_1(X) = \begin{cases} 0 & \text{for } T \in [1, N-1] \\ 1 & \text{for } T \in \{0, N\} \end{cases}$$

is unbiased at the level of significance $\alpha_1 = \frac{1}{2^{n-1}}$ for all n and for any distribution function $F(x)$, which is continuous and symmetric with respect to the origin.

This can be seen as follows:

$$\begin{aligned}
\beta_1(\mu) &= \boldsymbol{P}_\mu\{T = 0\} + \boldsymbol{P}_\mu\{T = N\} \\
&= \boldsymbol{P}_\mu\left(\bigcap_{i=1}^{n}\{X_i < 0\}\right) + \boldsymbol{P}_\mu\left(\bigcap_{i=1}^{n}\{X_i > 0\}\right) \\
&= F_\mu(0)^n + [1 - F_\mu(0)]^n \\
&\geq 2 \cdot \left[\frac{1 - F_\mu(0)}{2} + \frac{F_\mu(0)}{2}\right]^n \\
&= \frac{1}{2^{n-1}} = \alpha_1.
\end{aligned}$$

Thereby the inequality holds because the function $z \mapsto z^n$ is convex.

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