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OPTIMAL ORDER FOR TWO SERVERS IN TANDEM*

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Abstract. We consider two servers (server i, $i = 1, 2$) in tandem for which the order of servers can be changed. Server 1 has a general service time distribution and server 2 has either its 'shifted' or 'truncated' distribution. This permits that the service times at the two servers are overlapping. An unlimited queue is allowed in front of the first server. For the systems having zero buffer capacity between the servers, we show that the sojourn time of every customer is stochastically minimized under any arrival process if server 2 is first. For the systems with infinite buffer capacity and a Poisson arrivals , we show that this order of servers minimizes mean customer delay when traffic is light. Several numerical examples are presented to demonstrate that this optimal order is invariant under any arrival process (the interarrival times are i.i.d. r.v.'s) and mild traffic condition.

Key words and phrases: Tandem queue, optimal order, stochastic ordering, light traffic.

1. Introduction

An important design problem for queueing systems is to determine the optimal order for two or more single-server stations. For given external arrival process and given service-time distributions at the stations, the problem is to determine the order of the stations (to be used by all customers) under suitable optimal criterion. Usually, the criterion is to minimize the sojourn time (or departure process) in some sense for systems without loss of customers and to minimize the loss probability for systems with loss of customers. The optimal order of two servers in tandem depends much on how to select the criterion. This will be discussed in detail in Section 3.

Most of the work on the related problems has concerned systems without loss of customers. Friedman (1965) considered a system with a general arrival process and infinite capacity for waiting customers at each station in which all servers have deterministic service times. He found that the departure process

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does not depend on the order of the servers (stations). Suzuki and Kawashima (1974) showed that this result holds for the system with blocking, i.e. with finite capacity at each station (except the first). Weber (1979) considered a system with a general arrival process and infinite capacity at each station in which all servers have exponential service times and found the same result as Friedman's for the system. Yamazaki (1981) and Chao *et al.* (1989) showed a similar result for Weber's model with two stations and blocking. Recently, Ding and Greenberg (1991*a*) studied Weber's model with blocking and showed that considering the last two stations, the departure process is stochastically faster if the slower server is last.

Tembe and Wolff (1974) considered systems with a general arrival process and infinite capacity at each station. They showed that for a system with two single-server stations, one with constant service times and the other with variable ones, the departure process is stochastically smaller if the deterministic server is first. Kawashima (1975) showed that the optimal order remains valid for the systems with zero buffer capacity between the stations. Tembe and Wolff (1974) also showed that for systems with nonoverlapping service time distributions, the departure process is stochastically minimized if the servers are ordered from slowest to fastest. Dattatreya (1978) showed that this holds for the systems with any buffer capacity between the stations. These results suggest that the optimal order for two single-server stations (servers) in tandem is independent of the buffer capacity between the servers.

The main purpose of this paper is twofold: to give some evidence for this independence, and to give a generalization of the results in Tembe and Wolff (1974). To do so, we consider systems with two servers, servers 1 and 2, in tandem in which the capacity in front of the first server is infinite and the buffer capacity between the servers is either zero or infinite. Server 1 has a general service time distribution and server 2 has either its 'shifted' or 'truncated' distribution. We assume throughout the paper that arriving customer's service times are selected independently. In this settings, note that the service times of both servers may be overlapping.

In Section 3, we consider the systems with zero buffer capacity and show that for any arrival process, the sojourn time of every customer is stochastically minimized when server 2 is first. In Sections 4 and 5, we consider the systems with infinite buffer capacity in which the interarrival times of customers are i.i.d. r.v.'s. In Section 4, we examine the behavior of the systems with Poisson arrivals when traffic is light. It is shown that the mean sojourn time of a customer is minimized when server 2 is first. In Section 5, many results of numerical experiments are presented to show that the optimal order of servers is invariant even under mild traffic condition for 'standard' interarrival time distributions such as deterministic, Erlang, uniform and hyperexponential.

2. Model

Consider two servers, servers 1 and 2, in tandem for which the order of servers can be changed. Then we have two possible arrangements (orders) of the servers. Server 1 first (server1 $\rightarrow$ server 2), which we call system 1, and system 2, which is server 2 first (server 2 $\to$ server 1). Whatever order we choose, it is the same for all customers. For system i ($i = 1, 2$), each arriving customer receives the service at the first server, and then the second before leaving the queue. The service discipline is FCFS. An unlimited queue is allowed in front of the first server. On the buffer capacity between the servers, we consider two extreme cases, that is, zero and infinite. In the former 'blocking' may occur: if the second server is busy when a service is completed at the first server, the completed customer remains at the server and blocks further service until the second becomes free.

Let X_n and Y_n be the service times of n-th arrival customer (C_n) at servers 1 and 2, respectively. We assume that $\{X_n\}$ and $\{Y_n\}$ are independent renewal sequences, and that they are independent of the arrival process. Throughout the paper, the distribution function (d.f.) and its tail distribution of a r.v. are distinguished by adding the parentheses and the mark ($-$) for the same letter: $X(x)$ is the d.f. of X and $\bar{X}(x) = 1 - X(x)$.

3. Tandem queues with zero buffer capacity

In this section, we compare systems 1 and 2 in which the buffer capacity between the servers is zero. Greenberg and Wolff (1988) showed that the order 'faster server first' minimizes mean customer delay under light traffic for two server tandem queues with infinite buffer capacity and Poisson arrivals if both distributions of X and Y are a kind of hyperexponential distribution. Ding and Greenberg (1991*b*) showed that this order minimizes the loss probability for two server tandem queues with Poisson arrivals and zero buffer capacity at both servers. The results suggest that for some two server tandem queues, the order of servers is desirable. We begin to examine this order for our model. We assume below that both systems 1 and 2 start empty.

Let

$$\begin{aligned} D_n^i &= \text{the sum of the waiting (in front of the first server) and} \\ &\quad \text{blocking times of } C_n \text{ (i.e. the delay time of } C_n \text{ in system } i), \quad i = 1, 2, \\ W_n^i &= \text{the sojourn time of } C_n \text{ in system } i, \ (= D_n^i + X_n + Y_n), \quad i = 1, 2. \end{aligned}$$

Then the following expressions for D_n^i were obtained in Sakasegawa and Yamazaki (1977).

$$\begin{aligned} D_n^1 &= 0 \vee (X_n \vee Y_{n-1} - A_n + X_{n-1} - X_n) \\ &\quad \vee (X_n \vee Y_{n-1} - A_n + X_{n-1} \vee Y_{n-2} - A_{n-1} + X_{n-2} - X_n) \vee \cdots \\ &\quad \vee (X_n \vee Y_{n-1} - A_n + X_{n-1} \vee Y_{n-2} - A_{n-1} + \cdots \\ &\qquad + X_2 \vee Y_1 - A_2 + X_1 - X_n) \\ &= \max \left[0, \sum_{j=i+1}^{n} (X_j \vee Y_{j-1} - A_j) + X_i - X_n \ \ (i = 1, 2, \ldots, n-1) \right], \end{aligned} \tag{3.1}$$

$$D_n^2 = \max \left[0, \sum_{j=i+1}^{n} (Y_j \vee X_{j-1} - A_j) + Y_i - Y_n \ \ (i = 1, 2, \ldots, n-1) \right],$$

where A_n is the interarrival time between C_{n-1} and C_n and $a \vee b = \max(a, b)$.

THEOREM 3.1. (a) *If* $X \leq_{st} Y$ *(stochastic order), then*

$D_n^1 + X_n \leq_c D_n^2 + Y_n$ *(convex order; see for example, Wolff* (1989)).

(b) *If* $X \leq_{rh} Y$ *(reverse hazard rate order, i.e.* $X(x)/Y(x)$ *is non-increasing in* x*), then*

$$D_n^1 + X_n \leq_{st} D_n^2 + Y_n.$$

Before proving this theorem, we introduce the following lemma.

LEMMA 3.1. *Let the r.v.'s X and Y be mutually independent and let c be any non-negative constant.*

(a) *If* $X \leq_{st} Y$, *then* $X \vee (Y - c) \leq_c Y \vee (X - c)$.

(b) *If* $X \leq_{rh} Y$, *then* $X \vee (Y - c) \leq_{st} Y \vee (X - c)$.

The proof of the lemma is given in Appendix 1.

PROOF OF THEOREM 3.1. Here we prove only (a) because (b) can be similarly proved. From (3.1) we have

$$D_n^1 + X_n \overset{d}{\sim} \max\left[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \qquad (i = 1, \ldots, n-1)\right], \tag{3.2}$$

where the sign $\overset{d}{\sim}$ denotes the equality of distributions. For a fixed k $(1 \leq k \leq n-1)$ we will look at the right hand side of (3.2) as a function of X_k and Y_k, while other values of X_j, Y_j and A_j are fixed. Let this function be $f(X_k, Y_k)$. Then, one finds that for suitable constants c_1, c_2 and c_3,

$$f(X_k, Y_k) = c_1 \vee (X_k + c_2) \vee (X_k \vee Y_k + c_3).$$

Let

$$g(X_k, Y_k) = c_1 \vee (Y_k + c_2) \vee (Y_k \vee X_k + c_3).$$

If $c_2 \leq c_3$, then $f(X_k, Y_k) = g(X_k, Y_k)$. If $c_2 > c_3$, then $f(X_k, Y_k) = c_1 \vee (X_k + c_2) \vee (Y_k + c_3)$ and (a) of Lemma 3.1 implies $f(X_k, Y_k) \leq_c g(X_k, Y_k)$. Unconditioning we find that

$$\begin{aligned}
&\max\left[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \qquad (i = 1, \ldots, n-1)\right] \\
&\leq_c \max\Bigg[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \qquad (i = 1, \ldots, k-2), \\
&\qquad \sum_{j=1}^{k-1}(X_j \vee Y_j - A_k) + Y_k, \\
&\qquad \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \qquad (i = k, \ldots, n-1)\Bigg].
\end{aligned}$$

Continuing this interchange for all values $k = 1, 2, \ldots, n-1$ leads to

$$\begin{aligned} D_n^1 + X_n \leq_c \max &\left[Y_1, \sum_{j=1}^{i}(Y_j \vee X_j - A_{j+1}) + Y_{i+1} \quad (i = 1, \ldots, n-2), \right. \\ &\left. \qquad \sum_{j=1}^{n-1}(Y_j \vee X_j - A_{j+1}) + X_n \right] \\ \leq_{st} \max &\left[Y_1, \sum_{j=1}^{i}(Y_j \vee X_j - A_{j+1}) + Y_{i+1} \quad (i = 1, \ldots, n-1) \right] \\ &\stackrel{d}{\sim} D_n^2 + Y_n. \end{aligned}$$

Remark 3.1. (a) Consider systems 1 and 2 in which the buffer capacity in front of the first server is finite. For each system, when an arriving customer finds full of the buffer capacity before the first server, the customer arrived, immediately leaves the system without being served. Suppose that customers arrive the systems according to a Poisson process and the buffer capacity between the servers is infinite. If $X \leq_{st} Y$, then

$$[\text{the loss probability in system 1}] \leq [\text{that in system 2}]$$

(see, e.g., Miyazawa (1990)). This supports that the rule 'faster server first' is useful on 'loss probability' in the systems 1 and 2. Is the rule useful for the case where the buffer capacity between the servers is zero? Ding and Greenberg (1991*b*) gave a positive answer to this question for systems with Poisson arrivals in which the buffer capacity in front of the first server is also zero.

Theorem 3.1 gives a physical interpretation of their result and it suggests that the result can be extended to systems with more general arrival process and the finite buffer capacity before the first server, at least, on the loss probability as follows.

This theorem implies that for the time interval from an arrival instant until the commencement of the customer at the last server (or a departure time from the first server), system 1, that is, the order 'faster server first' is desirable at least for this time interval. This suggests that for the model having finite capacity in front of the first server, the order minimizes the loss probability because it depends on the departure times from the first server.

(b) The rule is no longer useful on the sojourn times in systems without loss of customers as shown in part (b) of the following theorem. Combining Greenberg and Wolff (1988) and the (b) implies that the optimal order on the sojourn times in such systems can not be uniquely determined only by usual partial orderings between X and Y such as $X \leq_{st} Y$.

THEOREM 3.2. (Summary of Tembe and Wolff (1974), Kawashima (1975) and Dattatreya (1978)) (a) *For our model or the model with infinite buffer capacity, suppose that* $Y = c$, *a constant, and let* X *be arbitrary. Then,*

$$W_n^1 \geq_{st} W_n^2. \tag{3.3}$$

(b) *For our model or the model with any buffer capacity, suppose that* $P(X \leq Y) = 1$. *Then,*

$$W_n^1 \geq_{st} W_n^2. \tag{3.4}$$

From now on we focus on the sojourn time of a customer in system as the optimal criterion. We proceed to generalize Theorem 3.2 to our model. Let $\{X_n^*\}$ be a renewal sequence which is independent of $\{X_n\}$ and let $X^* \overset{d}{\sim} X$, where X^* is a generic r.v. of the sequence. Then we have

THEOREM 3.3. *For any constant* $c(\geq 0)$, *let* Y_n *be* $X_n^* + c$ *or* $X_n^* \vee c$. *Then,*

$$W_n^1 \geq_{st} W_n^2 \quad \textit{for every} \quad n. \tag{3.5}$$

To prove this theorem we need the following lemma whose proof is given in Appendix 2.

LEMMA 3.2. (a) *For any constant* $c_1(\geq 0)$,

$$\max[X, X^* + c - c_1] \geq_{st} \max[X^*, X^* + c - c_1, X - c_1].$$

(b) *For any constants* c_1, $c_2(\geq 0)$ *and* $c_3(\geq c)$,

$$\begin{aligned}(X^* + c_1) &\vee (X + c \vee X^*) \vee (X \vee c_3 - c_2 + c \vee X^*) \\ &\geq_{st} (X^* + c_1) \vee (c \vee X + X^*) \vee (X \vee c_3 - c_2 + c \vee X^*).\end{aligned}$$

PROOF OF THEOREM 3.3.
(i) Case of $Y_n = X_n^* + c$
From (3.2) we find that for this case,

$$\begin{aligned}
W_n^1 &= D_n^1 + X_n + Y_n \\
&\overset{d}{\sim} X_0^* + c \\
&\quad + \max\left[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \ (i = 1, \ldots, n-1)\right], \\
W_n^2 &= D_n^2 + Y_n + X_n \\
&\overset{d}{\sim} X_0 + c \\
&\quad + \max\left[X_1^*, \sum_{j=1}^{i}(Y_j \vee X_j - A_{j+1}) + X_{i+1}^* \ (i = 1, \ldots, n-1)\right].
\end{aligned} \tag{3.6}$$

To prove the theorem, therefore, it is sufficient to show that

$$(3.7)\ \max\left[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \quad (i=1,\dots,n-1)\right] \geq_{st} \max\left[X_1^*, \sum_{j=1}^{i}(Y_j \vee X_j - A_{j+1}) + X_{i+1}^* \quad (i=1,\dots,n-1)\right].$$

For a fixed $k (k = 1, \dots, n-1)$, we will look at this left hand side as a function of X_k and X_k^*, denoted by $f(X_k, X_k^*)$, while other values are fixed. Using suitable constants c_1, c_2 and c_3, $f(X_k, X_k^*)$ can be written as

$$f(X_k, X_k^*) = \max[c_1, X_k + c_2, X_k \vee (X_k^* + c) + c_3].$$

Let $g(X_k, X_k^*) = \max[c_1, X_k^* + c_2, (X_k^* + c) \vee X_k + c_3]$. If $c_2 \leq c_3$, then $f(X_k, X_k^*) = g(X_k, X_k^*)$. If $c_2 > c_3$, then $f(X_k, X_k^*) = \max[c_1, X_k + c_2, X_k^* + c + c_3]$ and (a) of Lemma 3.2 implies that $f(X_k, X_k^*) \geq_{st} g(X_k, X_k^*)$. Unconditioning yields

$$\begin{aligned}
&\max\left[X_1, \sum_{j=1}^{i}\{X_j \vee (X_j^* + c) - A_{j+1}\} + X_{i+1} \quad (i=1,\dots,n-1)\right] \\
&\geq_{st} \max\Bigg[X_1, \sum_{j=1}^{i}\{X_j \vee (X_j^* + c) - A_{j+1}\} + X_{i+1} \quad (i=1,\dots,k-2), \\
&\qquad \sum_{j=1}^{k-1}\{X_j \vee (X_j^* + c) - A_{j+1}\} + X_k^*, \\
&\qquad \sum_{j=1}^{i}\{X_j \vee (X_j^* + c) - A_{j+1}\} + X_{i+1} \quad (i=k,\dots,n-1)\Bigg].
\end{aligned}$$

Continuing this procedure for all values $k = 1, \dots, n-1$ leads to

$$\begin{aligned}
&\max\left[X_1, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1} \quad (i=1,\dots,n-1)\right] \\
&\geq_{st} \max\Bigg[X_1^*, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1}^* \quad (i=1,\dots,n-2), \\
&\qquad \sum_{j=1}^{n-1}(X_j \vee Y_j - A_{j+1}) + X_n\Bigg] \\
&\overset{d}{\sim} \max\left[X_1^*, \sum_{j=1}^{i}(X_j \vee Y_j - A_{j+1}) + X_{i+1}^* \quad (i=1,\dots,n-1)\right].
\end{aligned}$$

This right hand side is identical with that of (3.7).

(ii) Case of $Y_n = X_n^* \vee c$

For this case we have

$$
\begin{aligned}
W_n^1 &\overset{d}{\sim} \max[c \vee X_0^* + X_1, \sum_{j=1}^{i}(X_j \vee X_j^* \vee c - A_{j+1}) \\
&\qquad + X_{i+1} + c \vee X_0^* \quad (i = 1, \dots, n-1)], \\
W_n^2 &\overset{d}{\sim} \max[X_0 + c \vee X_1^*, \sum_{j=1}^{i}(c \vee X_j^* \vee X_j - A_{j+1}) \\
&\qquad + c \vee X_{i+1}^* + X_0 \quad (i = 1, \dots, n-1)] \\
&\overset{d}{\sim} \max[X_0^* + c \vee X_1, \sum_{j=1}^{i}(c \vee X_j \vee X_j^* - A_{j+1}) \\
&\qquad + c \vee X_{i+1} + X_0^* \quad (i = 1, \dots, n-1)].
\end{aligned}
\tag{3.8}
$$

For a fixed $k(k = 1, \dots, n)$, we look at the right hand side of the first expression in (3.8) as a function of X_0^* and X_k, denoted by $f(X_0^*, X_k)$, while other values are fixed. Then, using suitable constants c_1, c_2, $c_3(\geq c)$ and c_4,

$$f(X_0^*, X_k) = (X_0^* + c_1) \vee (X_k + c_2 + c \vee X_0^*) \vee (X_k \vee c_3 + c_4 + c \vee X_0^*).$$

Let

$$g(X_0^*, X_k) = (X_0^* + c_1) \vee (c \vee X_k + c_2 + X_0^*) \vee (X_k \vee c_3 + c_4 + c \vee X_0^*).$$

If $c_2 \leq c_4$, then $f(X_0^*, X_k) = g(X_0^*, X_k)$. If $c_2 > c_4$, (b) of Lemma 3.2 gives

$$f(X_0^*, X_k) \geq_{st} g(X_0^*, X_k).$$

Unconditioning,

$$
\begin{aligned}
&\max\Bigg[c \vee X_0^* + X_1, \\
&\qquad \sum_{j=1}^{i}(X_j \vee X_j^* \vee c - A_{j+1}) + X_{i+1} + c \vee X_0^* \quad (i = 1, \dots, n-1)\Bigg] \\
&\geq_{st} \max\Bigg[c \vee X_0^* + X_1, \sum_{j=1}^{i}(X_j \vee X_j^* \vee c - A_{j+1}) \\
&\qquad\qquad + X_{i+1} + c \vee X_0^* \quad (i = 1, \dots, k-2), \\
&\qquad\quad \sum_{j=1}^{k-1}(X_j \vee X_j^* \vee c - A_{j+1}) + c \vee X_k + X_0^*, \\
&\qquad\quad \sum_{j=1}^{i}(X_j \vee X_j^* \vee c - A_{j+1}) + X_{i+1} + c \vee X_0^* \quad (i = k, \dots, n-1)\Bigg].
\end{aligned}
$$

Continuing this procedure for all values $k = 1, \ldots, n$ leads to the desired conclusion.

Remark 3.2. Systems 1 and 2 have been compared under FCFS discipline, but it is clear that all results hold under any other discipline which is permitted to choose a customer at a departure instant from the first server.

Remark 3.3. In Theorem 3.3 we set $Y_n = X_n^* + c$. In manufacturing systems the c can be viewed as a set-up time for every job.

4. Tandem queues with infinite buffer capacity and Poisson arrivals in light traffic

In this section we compare systems 1 and 2 in which the buffer capacity between the servers is infinite and customers arrive according to a Poisson process with rate λ when traffic is light. By light traffic, we mean that the service times, X_n and Y_n, are scaled by a parameter $\alpha > 0$ to obtain αX_n and αY_n, where it is assumed that $\alpha \to 0$, and hence $E(\alpha X) \to 0$ and $E(\alpha Y) \to 0$ so that ρ_X $(= \lambda E(\alpha X)) \to 0$ and ρ_Y $(= \lambda E(\alpha Y)) \to 0$. In the following, we use the notations $\rho_X \to 0$ and $\rho_Y \to 0$ to mean that $\alpha \to 0$.

Let

$$\begin{aligned} s_j^i &= \text{the stationary mean sojourn time of a customer} \\ &\quad \text{at the } j\text{-th server in system } i \ (j = 1, 2 \text{ and } i = 1, 2), \\ w^i &= \text{the stationary mean sojourn time of a customer in system } i \\ &\quad (= s_1^i + s_2^i,\ i = 1, 2). \end{aligned}$$

Then, our light traffic approximation leads to

THEOREM 4.1. *For any constant $c (\geq 0)$, let Y be $X^* + c$ or $X^* \vee c$ assuming that $E[X \vee c] > E[X]$. Then,*

$$w^1 - w^2 = k\alpha^2 + O(\alpha^3) \qquad \textit{as} \quad \alpha \to 0,$$

where $k > 0$ is some constant.

PROOF OF THEOREM 4.1.

(i) Case of $Y = X^* + c$

We begin to find w^1 in light traffic. It is well known that s_1^1 can be written as

$$s_1^1 = \frac{1}{2}\lambda E[X^2] + E[X] + O(\rho_X^2)$$

$$\text{as} \quad \rho_X \to 0 \qquad \text{(see for example, Wolff (1989)).}$$

Wolff (1982) showed that

$$s_2^1 = \rho_X E[(Y - X)^+] + \rho_Y E[(Y_e - X)^+] + E[Y] + O(\rho^2) \tag{4.2}$$

$$\text{as} \quad \rho_X \to 0 \quad \text{and} \quad \rho_Y \to 0,$$

where X, Y and Y_e are mutually independent, $\rho = (\rho_X + \rho_Y)/2$, and $(a)^+ = \max(0, a)$. The distribution of r.v. Y_e is given as

$$Y_e(x) = \frac{1}{E[Y]} \int_0^x \bar{Y}(y)dy.$$

The first two terms of right hand side in (4.2) become

$$\begin{aligned}
E[(Y-X)^+] &= E[Y] - \int_0^\infty \bar{X}(x)\bar{Y}(x)dx \\
&= E[X+c] - \int_0^c \bar{X}(x)dx - \int_c^\infty \bar{X}(x)\bar{X}(x-c)dx, \\
E[(Y_e-X)^+] &= E[Y_e] - \int_0^\infty \bar{X}(x)\bar{Y}_e(x)dx \\
&= \frac{1}{2E[Y]}E[(X+c)^2] - \frac{1}{E[Y]}\int_0^c (c-x)\bar{X}(x)dx \\
&\quad - \frac{E[X]}{E[Y]}\int_0^c \bar{X}(x)dx - \frac{1}{E[Y]}\int_c^\infty \int_x^\infty \bar{X}(x)\bar{X}(y-c)dydx.
\end{aligned}$$

Thus we can obtain that, for the case where X and Y are scaled by a parameter α,

$$\begin{aligned}
(4.3)\quad w^1 &= \lambda\alpha^2\frac{1}{2}E[X^2] + \alpha E[X] \\
&\quad + \lambda\alpha^2 E[X]\left\{E[X+c] - \int_0^c \bar{X}(x)dx - \int_c^\infty \bar{X}(x)\bar{X}(x-c)dx\right\} \\
&\quad + \lambda\alpha^2\left\{\frac{1}{2}E[(X+c)^2] - \int_0^c (c-x)\bar{X}(x)dx - E[X]\int_0^c \bar{X}(x)dx\right. \\
&\quad \left. - \int_c^\infty \int_x^\infty \bar{X}(x)\bar{X}(y-c)dydx\right\} + \alpha E[Y] + O(\alpha^3).
\end{aligned}$$

Similarly, for system 2 we have

$$\begin{aligned}
(4.4)\quad w^2 &= \frac{1}{2}\lambda\alpha^2 E[(X+c)^2] + \alpha E[Y] \\
&\quad + \lambda\alpha^2(E[X]+c)\left\{E[X] - \int_0^c \bar{X}(x)dx - \int_c^\infty \bar{X}(x)\bar{X}(x-c)dx\right\} \\
&\quad + \lambda\alpha^2\left\{\frac{1}{2}E[X^2] - \int_0^c x\bar{X}(x)dx - c\int_c^\infty \bar{X}(x)dx\right. \\
&\quad \left. - \int_c^\infty \int_x^\infty \bar{X}(x-c)\bar{X}(y)dydx\right\} + \alpha E[X] + O(\alpha^3).
\end{aligned}$$

Subtracting (4.4) from (4.3) yields

$$w^1 - w^2 = \lambda g(c)\alpha^2 + O(\alpha^3), \tag{4.5}$$

where

$$g(c) = 2\int_0^c x\bar{X}(x)dx - E[X]\int_0^c \bar{X}(x)dx - E[X]F(c) + 2\int_0^\infty \bar{X}(x)F(x+c)dx$$
$$+ c\int_0^\infty \bar{X}(x+c)\bar{X}(x)dx + c\int_c^\infty \bar{X}(x)dx,$$

and

$$F(x) = \int_x^\infty \bar{X}(y)dy.$$

Then $g(0) = 0$, and

$$\begin{aligned} g'(c) &= 2c\bar{X}(c) - E[X]\bar{X}(c) + E[X]\bar{X}(c) - \int_0^\infty \bar{X}(x+c)\bar{X}(x)dx \\ &\quad - c\int_0^\infty \bar{X}(x)dX(x+c) + \int_c^\infty \bar{X}(x)dx - c\bar{X}(c) \\ &= \int_0^\infty \bar{X}(x+c)X(x)dx + c\int_0^\infty X(x)dX(x+c) \\ &> 0 \quad \text{for} \quad c \geq 0. \end{aligned}$$

Hence, $g(c) > 0$ for $c > 0$. Setting $k = \lambda g(c)$ the proof completes.

(ii) Case of $Y = X^* \vee c$

In the same way of (i), we can find that

$$w^1 - w^2 = \lambda g(c)\alpha^2 + O(\alpha^3), \tag{4.6}$$

where

$$\begin{aligned} g(c) &= (E[X \vee c] - E[X])\int_c^\infty \bar{X}(x)\bar{X}(x)dx \\ &\quad - E[X]\int_0^c \bar{X}(x)dx + \int_0^c x\bar{X}(x)dx + \int_0^c F(x)dx. \end{aligned}$$

Let

$$h(c) = -E[X]\int_0^c \bar{X}(x)dx + \int_0^c x\bar{X}(x)dx + \int_0^c F(x)dx.$$

We have

$$\begin{aligned} h'(c) &= -E[X]\bar{X}(c) + c\bar{X}(c) + F(c) \\ &= \bar{X}(c)\left(c - \int_0^c \bar{X}(x)dx\right) + X(c)\int_c^\infty \bar{X}(x)dx \\ &> 0 \quad \text{for} \quad c > 0. \end{aligned}$$

Because of $h(0) = 0$, this implies $h(c) > 0$ for $c > 0$ and hence $k = \lambda g(c) > 0$ for $c > 0$.

In the theorem, we assumed that $E[X \vee c] > E[X]$. For $E[X \vee c] = E[X]$, that is, $\bar{X}(c) = 1$, it is clear that $W_n^1 \overset{d}{\sim} W_n^2$ for all n because of $X^* \vee c = X^*$ (with probability 1) and $X^* \overset{d}{\sim} X$. The theorem suggests that the optimal order for the model with zero buffer capacity is invariant for the case where the buffer capacity between the servers is infinite. In the following section, we will numerically examine this under mild traffic as well as light traffic for other interarrival time distributions.

5. Numerical examples

In this section, we consider two server tandem queues in which the buffer capacity between the servers is infinite and the interarrival times, A_n, are i.i.d. r.v.'s. The purpose is to show that the optimal order of servers obtained in the preceding section is invariant even under mild traffic for standard distributions of the A_n and X_n such as deterministic, Erlang, uniform and hyperexponential (balanced means).

It is impossible to calculate λw^i (the mean queue length in system i) exactly even for the case where both distributions A (a generic interarrival time) and X are of phase-type such as Erlang and hyperexponential, because the service times at server 2 are included in constant c. Alternatively, simulation experiments are used to obtain estimates for the stationary mean sojourn time of a customer in system i. Some results for $Y = X^* + c$ and $Y = X^* \vee c$ are shown in Tables 1 and 2, respectively. The D, E_k, $U(a,b)$ and $H(p)$ used to describe the d.f.'s for both A and X indicate, respectively, deterministic distribution, k-Erlang distribution, uniform distribution on (a,b) and hyperexponential distribution whose form is given as

$$p(1 - e^{-2p\mu x}) + (1-p)(1 - e^{-2(1-p)\mu x}),$$

where $1/\mu$ is the mean of the distribution. Models parameters are shown in each table. Individual figures for w^i in the tables are obtained from 100 estimates, each of which is an average of 10^3 customers after skipping the first 10^2 customers when the simulation starts with system empty. Each 95% confidence interval is computed using the central limit theorem.

The results suggest that the optimal order for two servers obtained the preceding section can be extended to other systems with infinite buffer capacity and at least a standard interarrival time distribution under mild traffic as well as light traffic.

Concluding Remarks. This paper has mainly examined an optimal order for servers 1 and 2 in tandem where server 1 has a general service time distribution and server 2 has either its shifted or truncated distribution. For such a case, we have shown that for both systems with zero and infinite buffer capacities, the sojourn time is minimized in some sense at least for standard distributions of the interarrival times and the service times at server 1 if server 2 is first. Our independence assumption permits that the service times of both servers are overlapping (depending on the service time distribution at server 1), and hence this result can

Table 1. Mean sojourn times in system i ($i = 1, 2$) for $Y = X^* + c$.

(1) $c = 0.1$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.2, 0.4)$	0.301	0.401	0.702 ± 0.000	0.702 ± 0.000
	1.00	E_5	0.302	0.403	0.706 ± 0.001	0.705 ± 0.001
	1.00	M	0.303	0.402	0.763 ± 0.004	0.753 ± 0.004
	1.00	$H(0.1)$	0.293	0.398	1.24 ± 0.03	1.21 ± 0.03
E_5	1.01	$U(0.2, 0.4)$	0.301	0.401	0.709 ± 0.001	0.708 ± 0.001
	1.00	E_5	0.302	0.402	0.725 ± 0.001	0.720 ± 0.001
	1.01	M	0.303	0.400	0.811 ± 0.004	0.796 ± 0.004
	1.00	$H(0.1)$	0.294	0.399	1.31 ± 0.03	1.26 ± 0.03
M	1.01	$U(0.2, 0.4)$	0.301	0.401	0.839 ± 0.003	0.837 ± 0.003
	1.01	E_5	0.302	0.402	0.878 ± 0.004	0.869 ± 0.004
	1.01	M	0.302	0.402	1.04 ± 0.01	1.01 ± 0.01
	1.01	$H(0.1)$	0.294	0.395	1.56 ± 0.05	1.49 ± 0.04
$H(0.1)$	0.996	$U(0.2, 0.4)$	0.301	0.401	1.03 ± 0.01	1.03 ± 0.01
	0.986	E_5	0.303	0.401	1.11 ± 0.01	1.09 ± 0.01
	0.992	M	0.301	0.402	1.39 ± 0.02	1.35 ± 0.02
	0.980	$H(0.1)$	0.295	0.394	2.23 ± 0.07	2.09 ± 0.06

Remark. $a \pm b$ shows that $[a - b, a + b]$ is a 95% confidence interval.

(2) $c = 0.3$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.2, 0.4)$	0.301	0.601	0.902 ± 0.000	0.902 ± 0.000
	1.00	E_5	0.302	0.603	0.911 ± 0.001	0.906 ± 0.001
	1.00	M	0.303	0.602	1.02 ± 0.01	0.977 ± 0.004
	1.00	$H(0.1)$	0.293	0.598	1.72 ± 0.05	1.56 ± 0.04
E_5	1.00	$U(0.2, 0.4)$	0.301	0.601	0.946 ± 0.001	0.943 ± 0.001
	1.00	E_5	0.302	0.602	0.976 ± 0.002	0.958 ± 0.002
	1.00	M	0.303	0.600	1.12 ± 0.01	1.06 ± 0.01
	1.00	$H(0.1)$	0.294	0.599	1.84 ± 0.05	1.64 ± 0.04
M	1.01	$U(0.2, 0.4)$	0.301	0.601	1.35 ± 0.01	1.35 ± 0.01
	1.01	E_5	0.302	0.602	1.38 ± 0.01	1.36 ± 0.01
	1.01	M	0.302	0.602	1.59 ± 0.02	1.52 ± 0.02
	1.01	$H(0.1)$	0.294	0.595	2.34 ± 0.07	2.09 ± 0.06
$H(0.1)$	0.996	$U(0.2, 0.4)$	0.301	0.601	2.42 ± 0.06	2.41 ± 0.06
	0.986	E_5	0.303	0.601	2.50 ± 0.07	2.48 ± 0.07
	0.992	M	0.301	0.602	2.78 ± 0.07	2.69 ± 0.07
	0.980	$H(0.1)$	0.295	0.594	3.95 ± 0.15	3.53 ± 0.13

Table 1. (continued).

(3) $c = 0.1$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.4, 0.6)$	0.501	0.601	1.10 ± 0.00	1.10 ± 0.00
	1.00	E_5	0.503	0.605	1.15 ± 0.00	1.14 ± 0.00
	1.00	M	0.505	0.603	1.58 ± 0.02	1.53 ± 0.01
	1.00	$H(0.1)$	0.488	0.597	3.73 ± 0.15	3.54 ± 0.15
E_5	1.01	$U(0.4, 0.6)$	0.501	0.601	1.15 ± 0.00	1.14 ± 0.00
	1.01	E_5	0.504	0.604	1.25 ± 0.00	1.23 ± 0.00
	1.01	M	0.504	0.601	1.71 ± 0.02	1.66 ± 0.02
	1.01	$H(0.1)$	0.490	0.598	3.92 ± 0.14	3.69 ± 0.13
M	1.01	$U(0.4, 0.6)$	0.501	0.601	1.55 ± 0.01	1.55 ± 0.01
	1.01	E_5	0.504	0.603	1.73 ± 0.02	1.71 ± 0.02
	1.01	M	0.504	0.603	2.36 ± 0.04	2.30 ± 0.03
	1.01	$H(0.1)$	0.490	0.591	4.56 ± 0.21	4.29 ± 0.19
$H(0.1)$	0.996	$U(0.4, 0.6)$	0.501	0.601	2.62 ± 0.06	2.62 ± 0.06
	0.986	E_5	0.505	0.602	3.00 ± 0.08	2.96 ± 0.08
	0.992	M	0.502	0.603	4.01 ± 0.11	3.91 ± 0.10
	0.980	$H(0.1)$	0.492	0.591	6.94 ± 0.32	6.49 ± 0.29

Table 2. Mean sojourn times in system i ($i = 1, 2$) for $Y = X^* \vee c$.

(1) $c = 0.1$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.0, 0.6)$	0.303	0.311	0.614 ± 0.001	0.614 ± 0.001
	1.00	E_5	0.302	0.303	0.605 ± 0.001	0.605 ± 0.001
	1.00	M	0.303	0.316	0.663 ± 0.004	0.662 ± 0.004
	1.00	$H(0.1)$	0.293	0.321	1.09 ± 0.03	1.08 ± 0.03
E_5	1.01	$U(0.0, 0.6)$	0.302	0.312	0.632 ± 0.002	0.631 ± 0.002
	1.01	E_5	0.302	0.303	0.617 ± 0.001	0.617 ± 0.001
	1.01	M	0.303	0.315	0.700 ± 0.004	0.697 ± 0.004
	1.01	$H(0.1)$	0.294	0.321	1.16 ± 0.03	1.14 ± 0.03
M	1.01	$U(0.0, 0.6)$	0.304	0.311	0.760 ± 0.003	0.757 ± 0.003
	1.01	E_5	0.302	0.302	0.722 ± 0.003	0.722 ± 0.003
	1.01	M	0.302	0.316	0.876 ± 0.007	0.869 ± 0.007
	1.01	$H(0.1)$	0.294	0.317	1.36 ± 0.04	1.33 ± 0.04
$H(0.1)$	0.996	$U(0.0, 0.6)$	0.303	0.311	0.914 ± 0.006	0.907 ± 0.006
	0.986	E_5	0.303	0.302	0.852 ± 0.004	0.850 ± 0.004
	0.992	M	0.301	0.316	1.12 ± 0.01	1.11 ± 0.01
	0.980	$H(0.1)$	0.295	0.317	1.87 ± 0.06	1.82 ± 0.05

Table 2. (continued).

(2) $c = 0.3$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.2, 0.4)$	0.301	0.325	0.627 ± 0.000	0.627 ± 0.000
	1.00	E_5	0.302	0.354	0.656 ± 0.001	0.656 ± 0.001
	1.00	M	0.303	0.411	0.763 ± 0.004	0.750 ± 0.003
	1.00	$H(0.1)$	0.293	0.446	1.30 ± 0.03	1.23 ± 0.03
E_5	1.01	$U(0.2, 0.4)$	0.301	0.326	0.630 ± 0.000	0.629 ± 0.000
	1.01	E_5	0.302	0.354	0.669 ± 0.001	0.665 ± 0.001
	1.01	M	0.303	0.410	0.808 ± 0.004	0.785 ± 0.004
	1.01	$H(0.1)$	0.294	0.446	1.38 ± 0.03	1.29 ± 0.03
M	1.01	$U(0.2, 0.4)$	0.301	0.326	0.713 ± 0.002	0.711 ± 0.002
	1.01	E_5	0.302	0.353	0.788 ± 0.003	0.779 ± 0.003
	1.01	M	0.302	0.411	1.02 ± 0.01	0.985 ± 0.007
	1.01	$H(0.1)$	0.294	0.442	1.66 ± 0.05	1.54 ± 0.04
$H(0.1)$	0.996	$U(0.2, 0.4)$	0.301	0.326	0.813 ± 0.004	0.809 ± 0.004
	0.986	E_5	0.303	0.353	0.946 ± 0.006	0.928 ± 0.005
	0.992	M	0.301	0.411	1.37 ± 0.02	1.30 ± 0.02
	0.980	$H(0.1)$	0.295	0.442	2.44 ± 0.08	2.21 ± 0.07

(3) $c = 0.1$

d.f. of A	λ	d.f. of X	ρ_X	ρ_Y	w^1	w^2
D	1.00	$U(0.0, 1.2)$	0.607	0.609	1.36 ± 0.00	1.36 ± 0.00
	1.00	E_5	0.603	0.606	1.29 ± 0.00	1.29 ± 0.00
	1.00	M	0.606	0.611	2.03 ± 0.03	2.02 ± 0.02
	1.00	$H(0.1)$	0.587	0.609	5.49 ± 0.27	5.45 ± 0.29
E_5	1.007	$U(0.0, 1.2)$	0.604	0.611	1.54 ± 0.01	1.54 ± 0.01
	1.007	E_5	0.605	0.605	1.43 ± 0.01	1.43 ± 0.01
	1.009	M	0.605	0.608	2.18 ± 0.03	2.18 ± 0.03
	1.009	$H(0.1)$	0.589	0.609	5.69 ± 0.23	5.58 ± 0.23
M	1.01	$U(0.0, 1.2)$	0.608	0.610	2.23 ± 0.03	2.23 ± 0.03
	1.01	E_5	0.605	0.604	2.02 ± 0.02	2.02 ± 0.02
	1.01	M	0.605	0.611	3.02 ± 0.05	2.99 ± 0.05
	1.01	$H(0.1)$	0.588	0.601	6.48 ± 0.35	6.37 ± 0.37
$H(0.1)$	0.996	$U(0.0, 1.2)$	0.605	0.609	4.00 ± 0.12	4.00 ± 0.11
	0.986	E_5	0.606	0.603	3.70 ± 0.12	3.68 ± 0.11
	0.992	M	0.602	0.611	5.29 ± 0.16	5.25 ± 0.15
	0.980	$H(0.1)$	0.591	0.601	9.57 ± 0.49	9.37 ± 0.48

be viewed as a generalization of results by Tembe and Wolff (1974), Kawashima (1975) and Dattatreya (1978). From a practical viewpoint, we have other important problems, e.g., the problem of whether or not the optimal order for a two server system with zero buffer capacity can be extended to the system with infinite or any finite buffer capacity. The reason is that selecting the optimal order for two server system with zero buffer capacity is easier than that for the system with infinite buffer capacity as shown in this paper, and perhaps that for the system with finite buffer capacity. More work needs to be done in this direction.

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Appendix

1. Proof of Lemma 3.1

(a): Let

$$Z_1 = X \vee (Y - c),$$
$$Z_2 = Y \vee (X - c).$$

Then we find that

$$\text{(A.1)}\qquad \begin{aligned} Z_1(x) &= X(x)Y(x+c), \\ Z_2(x) &= Y(x)X(x+c). \end{aligned}$$

From the definition of convex ordering, to prove (a) it is sufficient to show that for any constant $a(\geq 0)$,

$$\int_a^\infty [\bar{Z}_2(x) - \bar{Z}_1(x)]dx \geq 0.$$

Since $X \leq_{st} Y$, there exists a function $\epsilon(x)$ satisfying

$$\bar{Y}(x) = \bar{X}(x) + \epsilon(x) \qquad \text{for all} \quad x \geq 0.$$

Using $\epsilon(x)$ yields

$$\begin{aligned} \int_a^\infty [\bar{Z}_2(x) - \bar{Z}_1(x)]dx &= \int_a^\infty \epsilon(x)dx - \int_a^\infty \epsilon(x+c)dx \\ &\quad - \int_a^\infty \bar{X}(x+c)\epsilon(x) + \int_a^\infty \bar{X}(x)\epsilon(x+c)dx \\ &= \int_a^{a+c} \epsilon(x)dx - \int_a^\infty \bar{X}(x+c)\epsilon(x)dx \\ &\quad + \int_{a+c}^\infty \bar{X}(x-c)\epsilon(x)dx \\ &= \int_a^{a+c} \epsilon(x)X(x+c)dx \\ &\quad + \int_{a+c}^\infty [\bar{X}(x-c) - \bar{X}(x+c)]\epsilon(x)dx \\ &\geq 0. \end{aligned}$$

(b): This immediately follows from (A.1) and the definition of reversed hazard rate order.

2. Proof of Lemma 3.2

(a): Let

$$Z_1 = \max(X, X^* + c - c_1),$$
$$Z_2 = \max(X^*, X^* + c - c_1, X - c_1).$$

If $c \geq c_1$, then

$$Z_2 = \max(X^* + c - c_1, X - c_1) \leq Z_1.$$

If $c < c_1$, then

$$Z_2 = \max(X^*, X - c_1) \overset{d}{\sim} \max(X, X^* - c_1) \leq Z_1.$$

Thus (a) follows.

(b): Let

$$Z_1 = (X^* + c_1) \vee (X + c \vee X^*) \vee (X \vee c_3 - c_2 + c \vee X^*),$$
$$Z_2 = (X^* + c_1) \vee (c \vee X + X^*) \vee (X \vee c_3 - c_2 + c \vee X^*).$$

Letting $Z = c \vee X^*$ and $c_4 = c_3 - c_2$, we have

$$\begin{aligned} Z_1 &= (X^* + c_1) \vee (X + Z) \vee (c_4 + Z), \\ Z_2 &= (X^* + c_1) \vee (c + X^*) \vee \{X + (c - c_2) \vee X^*\} \vee (c_4 + Z). \end{aligned} \tag{A.2}$$

Consider the following three cases.

Case 1. Let $c_4 \geq c$. From (A.2), Z_1 and Z_2 become

$$Z_1 = (X^* + c_1) \vee (X + Z) \vee (c_4 + Z),$$
$$Z_2 = (X^* + c_1) \vee \{X + (c - c_2) \vee X^*\} \vee (c_4 + Z).$$

Because of $Z \geq (c - c_2) \vee X^*$, we have

$$Z_1 \geq Z_2. \tag{A.3}$$

Case 2. Let $0 < c_4 < c$. Using $c_3 \geq c$ in (A.2) gives

$$Z_2 \leq (X^* + c_1) \vee (c + X^*) \vee (X + c_4 \vee X^*) \vee (c_4 + Z) \overset{d}{=} Z_2^*.$$

If $c_1 \geq c$, then

$$Z_1 \geq (X^* + c_1) \vee (X + c_4 \vee X^*) \vee (c_4 + Z) = Z_2^*.$$

If $0 < c_1 < c$, then

$$\begin{aligned} Z_2^* &= (X^* + c) \vee (X + c_4) \vee (X + X^*) \vee (c_4 + c) \\ &\overset{d}{\sim} (X + c) \vee (X^* + c_4) \vee (X^* + X) \vee (c_4 + c) \\ &\leq (X^* + c_1) \vee (X + c) \vee (X^* + c_4) \vee (X^* + X) \vee (c_4 + c) = Z_1. \end{aligned}$$

If $c_1 \leq 0$, then

$$\begin{aligned} Z_1 &= (X + Z) \vee (c_4 + Z) \\ &\overset{d}{\sim} (X^* + c \vee X) \vee (c_4 + c \vee X) = Z_2^*. \end{aligned}$$

Thus we conclude that for this case,

$$Z_1 \geq_{st} Z_2. \tag{A.4}$$

Case 3. Let $c_4 \leq 0$. For this case, in a similar manner to Case 2 we can find that $Z_1 \geq_{st} Z_2$, which completes the proof.

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