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SOME PROPERTIES OF CONVEX HULLS GENERATED BY HOMOGENEOUS POISSON POINT PROCESSES IN AN UNBOUNDED CONVEX DOMAIN

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Abstract. Let $C(A)$ be the convex hull generated by a Poisson point process in an unbounded convex set A. A representation of $A\backslash C(A)$ as the union of curvilinear triangles with independent areas is established. In the case when A is a cone the properties of the representation are examined more completely. It is also indicated how to simulate $C(A)$ directly without first simulating the process itself.

Key words and phrases: Vertex of convex hull, Poisson point process.

1. Preliminaries

The paper aims to establish a number of properties of a homogeneous Poisson point process (in the sequel p.p.p.). Although the properties are elementary they do not seem to be known. At least they are not mentioned in Stoyan *et al.* (1987) or Daley and Vere-Jones (1988).

The author's interest in this topic was stimulated by the remarkable paper of Groeneboom (1988) who has made impressive progress in convex hulls theory. His arguments are based on the original idea of using a Poisson approximation to the binomial point process close to the boundary of the support of the underlying distribution. This idea allowed him to reduce the problems he was considering to an examination of the asymptotic properties of a p.p.p. Further arguments were concerned with the application of powerful techniques such as martingales, mixing stationary processes, etc. It turns out that in many cases the above mentioned elementary properties of a p.p.p. provide easier arguments.

Consider a p.p.p. restricted to an unbounded convex set in the plane. The convex hull of such a process is one of its most interesting characteristics, both from the view-point of stochastic geometry and from that of statistics.

Within the framework of the Groeneboom approach we need to know how many vertices of the convex hull occur in any finite part of the plane, how close to the restricting set boundary they lie, etc.

It turns out that the vertices generate a triangularization of the region complementary to the convex hull within the restricting set. Under the triangularization the region is represented as a union of curvilinear triangles with independent areas, and the problem of counting vertices is reduced to a boundary problem for sums of i.i.d.r.v's (see Nagaev and Khamdamov (1991)).

2. General case

Consider a homogeneous p.p.p. in R^2 of intensity one; i.e. the Lebesgue measure $\lambda(\cdot)$ is its intensity measure. Denote by $C(A)$ the convex hull generated by the restriction of the process to the Borel set A. Let Γ_B denote the boundary of the Borel set B. It is convenient to denote by $l(w, w')$ and $|l(w, w')|$ the rectilinear segment with endpoints w and w', and its length, respectively.

Assume that A satisfies the following conditions:

a) A is convex, unbounded, with $\lambda(A) = \infty$;

b) the point $(0, 0)$ lies on its boundary;

c) each straight line $Z_c = ((x, y) : y = c)$ cuts off from A a region having finite area, i.e. Z_c intersects Γ_A at two points.

Let $w_0 = (u_0, v_0)$ be the vertex of $C(A)$ with minimum y-coordinate, having the line Z_{v_0} as a line of support (see Fig. 1). This choice uniquely determines the labels of the other vertices $w_j = (u_j, v_j)$, $j = \ldots, -1, 0, 1, \ldots$, by, say, numbering them in anticlockwise order.

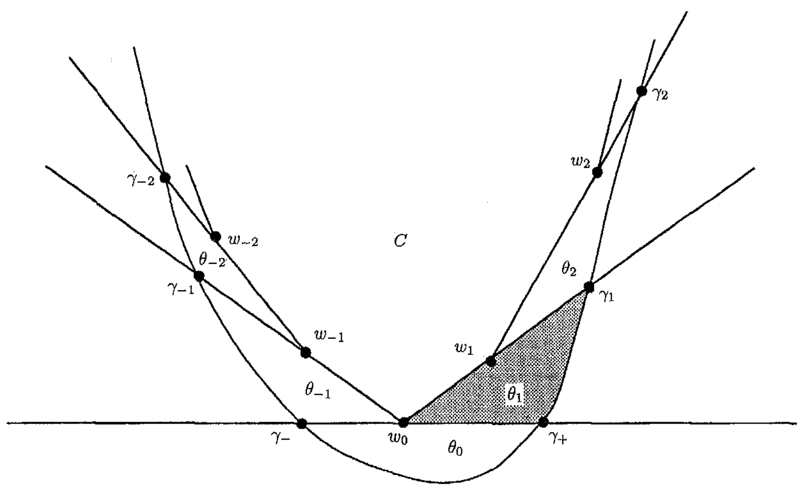

Fig. 1. Notation and schematic representations for the general case.

Let, further, θ_0 be the region cut off from A by Z_{v_0}, where Z_{v_0} cuts Γ_A at γ_- and γ_+. Now we define a sequence of regions θ_j, $j = \pm 1, \pm 2, \ldots$. Denote by $Z_{j-1,j}$ the straight line passing through w_{j-1} and w_j. Suppose that $j = 1$. The lines Z_{v_0} and $Z_{0,1}$ together with Γ_A bound the region θ_1, which is a (curvilinear) triangle with the vertices w_0, γ_+ and the point γ_1 where $Z_{0,1}$ meets Γ_A. Similarly, the region θ_{-1} is bounded by Z_{v_0}, $Z_{-1,0}$ and Γ_A. The corresponding vertices are w_0, γ_- and the point γ_{-1} in which $Z_{-1,0}$ meets Γ_A.

For $j \geq 2$ the region θ_j is determined by the lines $Z_{j-2,j-1}$ and $Z_{j-1,j}$ which are defined analogously to $Z_{0,1}$ and $Z_{0,-1}$ above, and Γ_A, and has the vertices w_{j-1}, γ_{j-1} and γ_j. Similarly, for $j \leq -2$ the region θ_j is determined by $Z_{j,j+1}$, $Z_{j+1,j+2}$ and Γ_A, and has the vertices w_{j+1}, γ_{j+1} and γ_j.

It is easy to see that the regions θ_{j+1}, $j \geq 1$, and also the regions θ_{j-1}, $j \leq -1$, depend only on w_j and on the slopes of the lines $Z_{j-1,j}$ and $Z_{j,j+1}$. As for θ_1 and θ_{-1}, they depend on w_0 and the slopes of Z_{01} and $Z_{-1,0}$ respectively. However, certain functionals of θ_j constitute sequences of i.i.d.r.v's, as we shall show below.

Note that the assumptions a) and c) above imply that the sequence of regions $\{\theta_j\}$ is doubly infinite (since any finite number of the θ_j cut off an area of A that has at most finite area and so leaves a part of A having infinite area and containing, therefore, at least one further point of the process).

Let $\xi_j = \lambda(\theta_j)$ be the area of the random set θ_j and set

$$\eta_j = \begin{cases} |l(\gamma_-, w_0)||l(\gamma_-, \gamma_+)|^{-1} & \text{if } j = 0 \\ |l(w_{j-1}, w_j)|^2 |l(w_{j-1}, \gamma_j)|^{-2} & \text{if } j \geq 1 \\ |l(w_j, w_{j+1})|^2 |l(\gamma_j, w_{j+1})|^{-2} & \text{if } j \leq -1. \end{cases} \tag{2.1}$$

We can now state the main result of the paper.

THEOREM 2.1. *Under the conditions stated, the variables ξ_j, η_k for $j, k = \dots, -1, 0, 1, \dots$ are jointly independent. Furthermore,*

$$P(\xi_j < x) = 1 - e^{-x}, \qquad x > 0,$$

and

$$P(\eta_k < x) = x, \qquad 0 < x < 1.$$

PROOF. Let $j = k = 0$. It is easily seen from properties of the Poisson process that the joint density function of $(u_0, v_0) = w_0$ has the form

$$p_0(u, v) = \begin{cases} \exp(-\lambda(\theta_0(v)))dudv, & \text{if } (u, v) \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where $\theta_0(v)$ is the region cut off from A by the line Z_v. (We allow here the usual duality in notation that occurs in mathematical statistics.) Formula (2.2) implies that given $v_0 = v$ the variable u_0 is uniformly distributed in (γ_-, γ_+), whence it follows that η_0 is uniformly distributed on $(0, 1)$. Since the function $\lambda(\theta_0(v))$ is increasing in v, the variable ξ_0 is exponentially distributed with parameter one. ξ_0 and η_0 are independent.

Assume for definiteness that the axes are chosen so that $u_j < u_{j+1}$ a.s., $j \geq 0$. Let $j = k = 1$. Note that the conditional joint density function of $w_1 = (u_1, v_1)$ given w_0 may be written as follows,

$$p_1(x, y) = \begin{cases} \exp(-\lambda(\theta_1(x, y)))dxdy, & \text{if } x > u_0,\ y > v_0,\ (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Here $\theta_1(x,y)$ is the region bounded by the lines Z_{v_0}, $((u,v) : \frac{v-v_0}{y-v_0} = \frac{u-u_0}{x-u_0})$ and Γ_A.

Choose w_0 as origin of polar coordinates and $l(w_0,\gamma_+)$ as a fixed direction. The area $\theta_1(x,y)$ depends on w_0 and the slope of the line Z_{01}. The polar coordinates ν and ψ of w_1 have the conditional joint distribution density function, given w_0, of the form

$$p_1^*(\nu,\psi) = \begin{cases} \exp(-\lambda(\theta_1^*(\psi)))\nu d\nu d\psi, & \text{if } 0 < \psi < \pi/2 \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where $\theta_1^*(\psi)$ is the representation of $\theta_1(x,y)$ under the new parameterization. From (2.3) it immediately follows that the conditional distribution of $|l(w_0,w_1)|^2$ given w_0 is uniform in $(0,|l(w_0,\gamma_1)|^2)$. Hence, that of η_1 is $(0,1)$-uniform. Since $\gamma(\theta_1^*(\psi))$ increases for every fixed w_0 the area of θ_1 has the exponential distribution with parameter one. Moreover, ξ_0, η_0, ξ_1 and η_1 are jointly independent.

In the case $j = k > 1$ the conditional joint distribution of w_j given w_{j-2}, w_{j-1} has the density function

$$p_j(x,y) = \begin{cases} \exp(-\lambda(\theta_j(x,y)))dxdy, \\ \qquad \text{if } (x,y) \in A \text{ and } x > u_{j-1}, \\ \qquad y > v_{j-2} + \dfrac{v_{j-1}-v_{j-2}}{u_{j-1}-u_{j-2}}(x-u_{j-2}) \\ 0 \quad \text{otherwise,} \end{cases} \tag{2.4}$$

where $\theta_j(x,y)$ is the region bounded by the lines $Z_{j-2,j-1}$, $Z_{j-1,j}$, and Γ_A.

Choosing w_{j-1} as origin of polar coordinates and $l(w_{j-2},w_{j-1})$ as the fixed direction and repeating the above arguments we easily obtain from (2.4) that $\xi_j = \lambda(\theta_j)$ and η_j do not depend on (η_k, y_k), $k = 0,1,\ldots,j-1$ and have the joint distribution

$$P(\xi_j < x; \eta_j < y) = (1-e^{-x})y, \qquad x > 0, \quad 0 < y < 1.$$

Similar arguments can be applied for $j \leq -1$ if we proceed clockwise, and the theorem is proved.

Let A^C and θ_j^C be closures of A and θ_j respectively. Let further C^0 be the interior of $C(A)$. Then $(A\backslash C^0) = \bigcup_j \theta_j^C$ and by Theorem 2.1 $\lambda(A\backslash C^0) = \infty$ a.s.

3. Properties of homogeneous p.p.p. in a cone

Suppose that A is a cone having the point $(0,0)$ as a vertex and rays $l_i = (w : w = te^{(i)})$, $i = 1,2$, as generators. Here the unit vectors $e^{(i)} = (\cos\phi_i, \sin\phi_i)$ are such that $0 < \phi_1 < \phi_2 < \pi$.

In this particular case the properties of $C(A)$ can be investigated more completely. Here the θ_j are ordinary triangles. Define δ_j as the distance from the vertex w_j to the ray l_1 if $j \geq 1$, and to the ray l_2, if $j \leq -1$. If $j = 0$, then δ_0 and δ_0' denote the distances from w_0 to l_1 and l_2 respectively. Note that

$$\delta_0 = |v_0\cos\phi_1 - u_0\sin\phi_1|, \qquad \delta_0' = |v_0\cos\phi_2 - u_0\sin\phi_2|. \tag{3.1}$$

Consider the variables

$$\tau_j = \begin{cases} \delta_j/\delta_{j-1}, & \text{if } j \ge 1, \\ \delta_j/\delta_{j+1}, & \text{if } j \le -2 \end{cases} \tag{3.2}$$

and $\tau_{-1} = \delta_{-1}/\delta_0'$.

It easily seen that

$$\tau_j = 1 - \eta_j^{1/2}. \tag{3.3}$$

In accordance with Theorem 2.1, τ_j, $(j \neq 0)$ are jointly independent and identically distributed. Furthermore

$$E\tau_1 = \frac{1}{3}, \qquad E\tau_1^2 = \frac{1}{6}, \qquad E(-\ln\tau_1) = \frac{3}{2}, \qquad \mathrm{Var}(\ln\tau_1) = \frac{5}{4}. \tag{3.4}$$

The variable δ_n can be represented as

$$\delta_n = \begin{cases} \delta_0 \prod_{j=1}^{n} \tau_j, & \text{if } n \ge 1 \\ \delta_0' \prod_{j=n}^{-1} \tau_j, & \text{if } n \le -1 \end{cases} \tag{3.5}$$

whence in view of (3.4) we obtain

$$E\delta_n = \begin{cases} 3^{-n}E\delta_0, & \text{if } n \ge 1, \\ 3^{-n}E\delta_0', & \text{if } n \le -1, \end{cases} \qquad E\delta_n^2 = \begin{cases} 6^{-n}E\delta_0^2, & \text{if } n \ge 1, \\ 6^{-n}(E\delta_0')^2, & \text{if } n \le -1. \end{cases} \tag{3.6}$$

Note that τ_j, $j \ge 1$, and τ_j, $j \le -1$, do not depend on δ_0 and δ_0' respectively.

From (3.4) and (3.5) it follows that $\ln\delta_n$ is asymptotically normal as $n \to \infty$. Denote by ν_t the number of vertices which fall in the disk S_t having radius t and centre $(0,0)$. On the basis of the above properties it can be proved that

$$E\nu_t \sim (4/3)\ln t = \alpha(t), \qquad \mathrm{Var}(\nu_t) \sim (5/4)\ln t = \beta^2(t) \tag{3.7}$$

and the variable $(\nu_t - \alpha(t))/\beta(t)$ is asymptotically normally distributed $N(0,1)$ (see Groeneboom (1988), Nagaev and Khamdamov (1991)). It is worth noting that the total number of points falling in S_t has the expectation $(\phi_2 - \phi_1)t^2/2$.

From (3.6) it follows that $\delta_n \to 0$ a.s. as $n \to \infty$, that is the rays l_i, $i = 1, 2$, serve as asymptotes for $\Gamma_{C(A)}$. The relations (3.6) and (3.7) show how rapidly $\Gamma_{C(A)}$ approaches Γ_A. Another way to describe the closeness of these lines is to estimate the difference between the lengths of the parts $\Gamma_{C(A)}$ and Γ_A which are cut off by S_t. To do this we need additional rather cumbersome notation. Put

$$d = 1/\sin\alpha + \mathrm{ctg}\,\alpha, \tag{3.8}$$

where $\alpha = \phi_2 - \phi_1$.

Further, put

$$\rho_j = \begin{cases} \operatorname{tg}\left(\dfrac{\pi}{2} - \phi_1\right) + 2\delta_0^{-2} \displaystyle\sum_{k=1}^{j} \xi_k \prod_{l=1}^{k-1} \tau_l^{-2}, & \text{if } j \geq 1 \\ \operatorname{tg}\left(\phi_2 - \dfrac{\pi}{2}\right) + 2(\delta_0')^{-2} \displaystyle\sum_{k=j}^{-1} \xi_k \prod_{l=k+1}^{-1} \tau_l^{-2}, & \text{if } j \leq -1 \end{cases} \tag{3.9}$$

and

$$\begin{cases} \sigma = \displaystyle\sum_{j=1}^{\infty} \left(d + \rho_j - \sqrt{1+\rho_j^2}\right)(1-\tau_j) \prod_{k=1}^{j-1} \tau_k, \\ \sigma' = \displaystyle\sum_{j=-\infty}^{-1} \left(d + \rho_j - \sqrt{1+\rho_j^2}\right)(1-\tau_j) \prod_{k=j+1}^{-1} \tau_k. \end{cases} \tag{3.10}$$

Note that σ and σ' are conditionally independent, given δ_0 and δ_0'. Finally, let $\mathcal{L}_t$ be the length of the part of $\Gamma_{C(A)}$ cut off by S_t.

THEOREM 3.1. *If* $0 < \phi_1 < \pi/2 < \phi_2 < \pi$, *then* $2t - \mathcal{L}_t \to \xi = \delta_0\sigma + \delta_0'\sigma'$ *a.s. where* δ_0, δ_0', σ *and* σ' *are defined in* (3.1) *and* (3.10).

PROOF. Consider the triangles $\tilde{\theta}_j$ constructed in the following way (see Fig. 2).

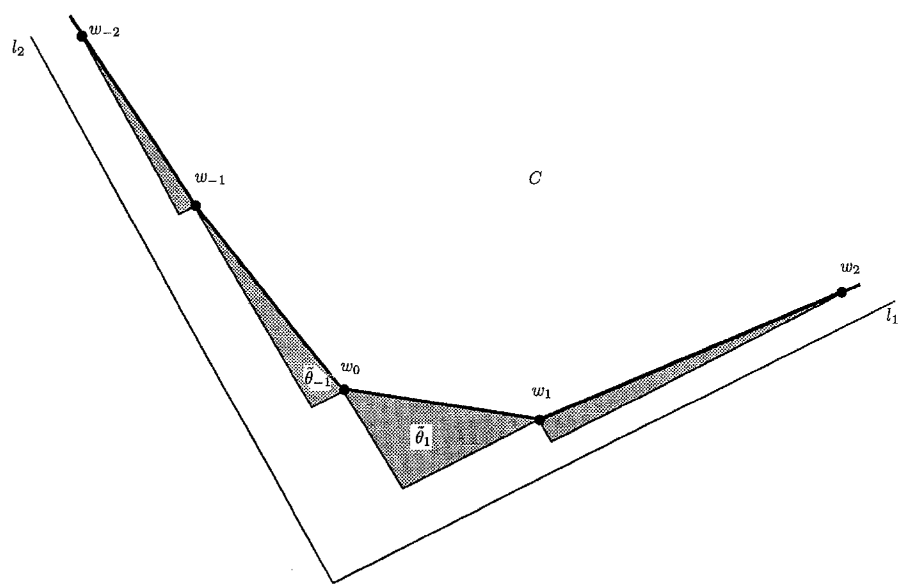

Fig. 2. Notation and schematic representation for the special case of a cone.

For $j \geq 1$, $\tilde{\theta}_j$ is formed by the line (w_{j-1}, w_j) and by the straight lines drawn through w_{j-1} and w_j parallel to l_2 and l_1 respectively. The length of the side of $\tilde{\theta}_j$ lying on $\Gamma_{C(A)}$ will be denoted by c_j, i.e.

$$c_j = \begin{cases} |l(w_{j-1}, w_j)|, & \text{if } j \geq 1 \\ |l(w_j, w_{j+1})|, & \text{if } j \leq -1. \end{cases}$$

Further let a_j and b_j be the lengths of the sides of $\tilde{\theta}_j$ which are parallel to l_1 and l_2 respectively. Put

$$\epsilon_j = a_j + b_j - c_j. \tag{3.11}$$

Consider the straight lines passing through the vertices w_{-m} and w_n parallel to l_1 and l_2 respectively. Let $t'_m e^{(2)}$ and $t_n e^{(1)}$ be the points at which those lines intersect l_2 and l_1 respectively. Denote by l_{mn} the length the part of $\Gamma_{C(A)}$ between w_{-m} and w_n. It is obvious that

$$t_n + t'_m - l_{mn} = \sum_{j=-m}^{n} \epsilon_j - (\delta_n + \delta_{-m})/\sin\alpha. \tag{3.12}$$

Let ρ'_j be tangent of the angle between $l(w_{j-1}, w_j)$ and the perpendicular to l_1, if $j \geq 1$, and that between $l(w_j, w_{j+1})$ and the perpendicular to l_2, if $j \leq -1$. It is easy to see that

$$\xi_j = \lambda(\theta_j) = \begin{cases} \delta_{j-1}^2(\rho'_j - \rho'_{j-1})/2, & \text{if } j \geq 1 \\ \delta_{j+1}^2(\rho'_j - \rho'_{j+1})/2, & \text{if } j \leq -1 \end{cases}$$

while $\xi_{-1} = (\delta'_0)^2(\rho'_{-1} - \rho'_0)/2$, $\rho_0 = \operatorname{tg}(\frac{\pi}{2} - \phi_1)$, $\rho'_0 = \operatorname{tg}(\phi_2 - \frac{\pi}{2})$.

Therefore

$$\rho'_j = \begin{cases} 2\sum_{k=1}^{j-1} \xi_k \delta_{k-1}^{-2} + \operatorname{tg}\left(\frac{\pi}{2} - \phi_1\right), & \text{if } j \geq 1 \\ 2\sum_{k=j}^{-1} \xi_k \delta_{k+1}^{-2} + \operatorname{tg}\left(\phi_2 - \frac{\pi}{2}\right), & \text{if } j \leq -1. \end{cases} \tag{3.13}$$

Combining (3.2), (3.5) and (3.13) we obtain $\rho_j = \rho'_j$. With the help of elementary arguments from (3.2), (3.8), (3.9), (3.11) and (3.13) we derive

$$\epsilon_j = \begin{cases} \delta_{j-1}(1 - \tau_j)\left(d + \rho_j - \sqrt{1 + \rho_j^2}\right), & \text{if } j \geq 1 \\ \delta_{j+1}(1 - \tau_j)\left(d + \rho_j - \sqrt{1 + \rho_j^2}\right), & \text{if } j \leq -1. \end{cases} \tag{3.14}$$

From (3.5) and (3.6) it follows that the series $\sum_{j \geq 1} \delta_j$ and $\sum_{j \leq -1} \delta_j$ converge almost surely. In view of (3.14) we have for some $c > 0$

$$\epsilon_j < \begin{cases} cv_{j-1}, & \text{if } j \geq 1 \\ cv_{j+1}, & \text{if } j \leq -1. \end{cases}$$

Thus the series $\sum_{j\neq 0} \epsilon_j$ converges almost surely.

In view of (3.11) and (3.12) we obtain that as $n \to \infty$, $m \to \infty$

$$t_n + t'_m - l_{mn} \to \zeta \qquad \text{a.s.}$$

and, therefore, the desired result follows. The theorem is proved.

4. Some remarks on simulation of convex hulls

The properties of homogeneous p.p.p. considered above provide a method of directly simulating the convex hull without first simulating the process itself. Here we indicate how to realize such a procedure in the case when A is a cone. The general case can be treated similarly.

Let (ξ_j, η_j), $j = 0, 1, 2, \ldots$ be i.i.d. random vectors with independent components. Assume that ξ_j and η_j have the standard exponential and $(0,1)$-uniform distributions respectively. For the sake of simplicity assume that l_1 coincides with the halfline $((x,y) : x > 0, y = 0)$, i.e. $\phi_1 = 0$, $\alpha = \phi_2$.

In order to simulate $w_0 = (u_0, v_0)$ we use the formulae

$$u_0 = \sqrt{\frac{2\xi_0}{\sin\alpha}}(1 - 2\eta_0 \sin^2(\alpha/2)), \qquad v_0 = \eta_0\sqrt{2\xi_0 \sin\alpha}. \tag{4.1}$$

Here we have taken into account that $\eta_0 \stackrel{d}{=} 1 - \eta_0$. Before constructing the w_j note that in the case considered we have $\gamma_j = (x_j, 0)$, $\delta_j = v_j$ provided $j \geq 0$ (see also (3.1)). The recurrence formula for the second coordinates follows immediately from (3.2) and (3.3), that is

$$v_j = (1 - \sqrt{\eta_j})v_{j-1}. \tag{4.2}$$

As to the first coordinates it is easy to see that

$$x_j = x_{j-1} + 2\lambda(\theta_j)/v_{j-1}$$

where $\lambda(\theta_j)$ is an area of the triangle θ_j. In view of (1) we have

$$x_j = x_{j-1} + 2\xi_j/v_{j-1}. \tag{4.3}$$

Finally,

$$v_j/v_{j-1} = (x_j - u_j)/(x_j - u_{j-1}) = 1 - \sqrt{\eta_j}$$

whence

$$u_j = x_j\sqrt{\eta_j} + u_{j-1}(1 - \sqrt{\eta_j}). \tag{4.4}$$

The formulae (4.1)–(4.4) determine an algorithm for simulating the sequence w_j, $j \geq 0$, or, in other words, the right wing of the boundary. The algorithm for the left wing is similar.

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