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NECESSARY CONDITIONS FOR CHARACTERIZATION OF LAWS VIA MIXED SUMS*

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Abstract. Suppose X and Y are independent and identically distributed, and independent of U which satisfies $0 \leq U \leq 1$. Recent work has centered on finding the laws $\mathcal{L}(X)$ for which $X \cong U(X+Y)$ where $\cong$ denotes equality in law. We show that this equation corresponds to a certain projective invariance property under random rotations. Implicitly or explicitly, it has been assumed that the characteristic function of X has an expansion property near the origin. We show that solutions may be admitted in the absence of this condition when $-\log U$ has a lattice law. A continuous version of the basic problem replaces sums with a Lévy process. Instead we consider self-similar processes, showing that a solution exists only when U is constant, and then all processes of a given order are admitted.

Key words and phrases: Distribution theory, characterization, semi-stable laws, mixtures, self-similar processes.

1. Introduction

Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) and U be a r.v., independent of the X's, and whose law, $\mathcal{L}(U)$, is supported in $[0,1]$ and is not degenerate at 0 or 1. Pakes (1992*b*) considers the problem of determining those non-trivial laws $\mathcal{L}(X_1)$ which solve the "in law" mixed sum equation

$$X_1 + \cdots + X_m \cong U(X_1 + \cdots + X_{m+n}) \tag{1.1}$$

where $\cong$ denotes equality in law, and $m, n \in \mathbb{N}$ are fixed. A continuous version posits a Lévy process $(Z_\tau : \tau \geq 0)$, $Z_0 = 0$, independent of U and seeks solutions for $\mathcal{L}(Z_1)$ of

$$Z_u \cong U Z_{u+v} \tag{1.2}$$

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for fixed positive u and v. These formulations cover several special cases in the literature; see Pakes (1992a) for a review which should be supplemented by Shanbhag (1972), Chang (1989) and Alzaid and Al-Osh (1991). The latter is significant in being the first contribution following Lévy's complete solution, stable and semi-stable laws, when U is constant. But being unpublished, Shanbhag's work had little impact. Another early and independent contribution was posed as a problem by Cowan (1980). His challenge was to show that if V_1, V_2 and U are independent and uniform on $[0,1]$, then $W = (V_1V_2)^U$ is uniform on the same interval. This equality is an exponentiated version of (1.1) with $m = n = 1$. Note that neither Cowan or the respondents to his problem mention any sort of converse formulation. Pakes (1992a, 1992b) subsumes the later, and apparently independent, univariate results of Alamatsaz (1993). Generalizations of the multiplication operation in (1.1) and (1.2) give characterizations which admit lattice laws. See van Harn and Steutel (1993), and Pakes (1993), for this.

Fix $\delta > 0$ and $0 \neq A \in \mathbb{C}$. Let $\mathcal{C}(\delta, A)$ be the set of functions ϕ which are defined and continuous on $\mathbb{R}$, $|\phi(t)| \leq 1$, and which have the local expansion $\phi(t) = 1 + At^\delta(1 + o(1))$ as $t \to 0+$. Set $\mathcal{C}(\delta) = \bigcup_{A\neq 0} \mathcal{C}(\delta, A)$. If $\phi \in \mathcal{C}(\delta)$ is a characteristic function (CF) of a law $\mathcal{L}(X)$ then $\delta \leq 2$ and we shall write $\mathcal{L}(X) \in \mathcal{C}(\delta)$.

Pakes (1992b) proved that if (1.1) has a solution $\mathcal{L}(X_1)$ whose CF $\phi \in \mathcal{C}(\delta)$ then

$$E(U^\delta) = \frac{m}{m+n} \tag{1.3}$$

($= \frac{u}{u+v}$ in the case of (1.2) and then ϕ is the CF of Z_1). Assume (1.3) holds. The CF version of (1.1) is

$$(\phi(t))^m = E[(\phi(Ut))^{m+n}] \tag{1.4}$$

and Pakes (1992b) shows this has at most one solution in $\mathcal{C}(\delta, A)$. Thus (1.1) admits at most a two-parameter family of laws in $\mathcal{C}(\delta)$, and any one of these laws determines that of U. It is in this sense that (1.1) can be said to characterize laws. Similar remarks apply to (1.2) which induces an obvious analogue (generalization even) of (1.4):

$$(\phi(t))^u = E[(\phi(Ut))^{u+v}]. \tag{1.5}$$

But note that there is an equivalence between (1.4) and (1.1), i.e., a CF solution of (1.4) induces a solution $\mathcal{L}(X_1)$ of (1.1). On the other hand, a CF solution ϕ of (1.5) must be infinitely divisible (infdiv) for (1.2) to have meaning. This occurs for all known cases but, in principle, (1.5) could have a solution for irrational u or v which is not infdiv.

Pakes (1992b) shows when $\delta \leq 2$ that (1.4) always has a solution in $\mathcal{C}(\delta)$, but only when $m = 1$ is it possible to guarantee it is a CF: more generally, $(\phi(t))^m$ is a CF. Such solutions are mixtures of stable laws, including stable laws themselves (and their reflections) and generalized Linnik laws (i.e., gamma scale mixtures of stable laws). Similar remarks apply to (1.5).

We show in Section 3 that membership of $\mathcal{C}(\delta)$ is, in a general sense necessary for a characterization result. We do this by extending a construction in Pakes (1992*b*) to show that mixtures of semi-stable laws (SS-laws) are admissible when $-\log U$ has a lattice law satisfying some mild constraints. This extension suggests that when $\mathcal{L}(-\log U)$ is non-lattice then solutions must lie in $\mathcal{C}(\delta)$, but this question remains open.

Preliminary material on SS-laws is presented in the next section, together with a geometric invariance interpretation of (1.1) which seems to be new. This supplements the motivations described in Pakes (1992*a*).

Historically, these mixture-type characterizations were first formulated in terms of (1.5), before the probabilistic representation in terms of a Lévy process via (1.2). See Pakes (1992*a*) for references for this. But (1.2) is meaningful for other processes. We illustrate this in Section 4 where (Z_τ) is a self-similar process. In this case (1.2) can be completely solved, but only in a trivial way, and this precludes sufficient uniqueness to give meaningful characterizations. This suggests that restriction to Lévy processes is necessary for a rich theory.

2. Some preliminaries

Say that a CF σ corresponds to a SS-law if it does not vanish on the real line and satisfies the functional equation

$$\sigma(ct) = (\sigma(t))^{\gamma}, \tag{2.1}$$

where c and γ are positive constants; see Kagan *et al.* (1973) or Ramachandran and Lau ((1991), p. 59). Non-trivial solutions exist iff $0 < c < 1$ and then $\gamma = c^\delta$ for some $0 < \delta \leq 2$. Denote the class of all SS-laws corresponding to (δ, c) by $\mathrm{SS}(\delta, c)$ and let $\mathrm{SS}(\delta) = \bigcup_{0<c<1} \mathrm{SS}(\delta, c)$. Usually c is allowed to be negative but our formulation imposes no essential restriction. Any SS-law is infdiv, and if $\delta = 2$ then it is a centered normal law. When $\delta < 2$, σ can be specified more closely in terms of its canonical representation, but we do not need this here. See the above references but note that the centering term given therein can be present only when $\delta = 1$. The laws in $\mathrm{SS}(\delta) \cap \mathcal{C}(\delta)$ comprise the strictly stable laws of index δ, given by

$$\sigma(t) = \exp[-(a + ib\,\mathrm{sgn}(t))|t|^{\delta}] \tag{2.2}$$

where a and b are constants satisfying:

(i) If $\delta \neq 1$ then $a > 0$ and $|b| \leq a|\tan(\pi\delta/2)|$; and

(ii) If $\delta = 1$ then $a \geq 0$ and $a + |b| > 0$.

If $U = c$ (a.s.) in (1.1) then (1.3) becomes

$$c^{\delta} = \frac{m}{m+n} \tag{2.3}$$

and (2.1) is just the CF version of (1.1). Similar remarks apply to (1.2). The case $m = n = 1$, $\delta = 2$ and $c = 1/\sqrt{2}$ provides a well known characterization of the normal law; see Kagan *et al.* ((1973), p. 448).

Still assuming $m = n = 1$, (1.1) has an attractive geometric interpretation which extends in an obvious way. Regard $X_1 + X_2$ as the resultant length of the sum of collinear vectors with signed lengths X_1 and X_2, directed along the line having polar angle θ where $c = \cos\theta$. Then $c(X_1 + X_2)$ is the projection of this vector onto the horizontal axis and our problem seeks those laws for which this projection equals X_1 in law. When $0 < \theta < \pi/4$ then $\delta > 2$ and there are no non-trivial solutions. All centered normal laws appear for $\theta = \pi/4$, as do non-normal SS-laws when $\pi/4 < \theta < \pi/2$. Indeed $2 > \delta \geq 1$ when $\pi/4 < \theta \leq \pi/3$, and the stable laws corresponding to $\theta = \pi/3$ are all Cauchy laws. Finally, $1 > \delta > 0$ when $\pi/3 < \theta < \pi/2$, and this region admits all the one-sided stable laws.

Thus the general problems (1.1) and (1.2) ask for laws which are projectively invariant under random rotations.

3. Mixed semi-stable solutions

Pakes (1992*b*) shows that (1.1) can be solved if a related one-sided law can be determined. Suppose (1.3) holds, $0 < \delta \leq 2$ and there is a non-negative sequence $(Y_\nu : \nu \geq 1)$ independent of U, with $E(Y_1) = 1$, and

$$Y_1 + \cdots + Y_m \cong U^\delta (Y_1 + \cdots + Y_{m+n}). \tag{3.1}$$

Then (1.1) has the solution

$$X_1 \cong S Y_1^{1/\delta}$$

where S is independent of Y_1 and has the above strictly stable law; see (2.2). This is the only solution in $\mathcal{C}(\delta, -a-ib)$. The Laplace-Stieltjes transform (LST) version of (3.1) always has a positive and non-increasing solution λ satisfying $\lambda(0) = 1$, $\lambda'(0+) = -1$, and it *is* an LST when $m = 1$ (thus completely solving (1.1)). There is a version of this construction for (1.2) which requires the existence of a subordinator (Y_τ) with $E(Y_1) = 1$ solving $Y_u \cong U^\delta Y_{u+v}$.

The following theorem extends this construction. Let $(S(\tau) : \tau \geq 0)$ denote a SS(δ, c)-process, i.e., a Lévy process where $S(\tau)$ has the semi-stable CF $(\sigma(t))^\tau$; see (2.1). Let $\mathbb{N}_+ = \{0, 1, \ldots\}$.

THEOREM 3.1. *Let $m, n \in \mathbb{N}$ be fixed and $0 < c < 1$ satisfy (2.3) for some $0 < \delta < 2$. Suppose N is an $\mathbb{N}_+$-valued r.v. having a possibly defective law. Set $U = c^N$ in (3.1), suppose (1.3) is satisfied and that (Y_ν) exists as above with Y_1 independent of $S(\tau)$. Then (1.1) has the solution*

$$X_1 \cong S(Y_1).$$

PROOF. Write the CF of $S(1)$ as $\sigma(t) = \exp(-\omega(t))$. Then (2.1) becomes $\omega(ct) = c^\delta \omega(t)$ which, when iterated, implies (a.s.)

$$\omega(Ut) = U^\delta \omega(t) \tag{3.2}$$

for U as specified above.

If λ is the LST of Y_1 then (3.1) is equivalent to

$$(\lambda(\theta))^m = E[(\lambda(U^\delta\theta))^{m+n}] \tag{3.3}$$

which holds for $\theta \geq 0$, and hence for θ in the open right-hand complex plane. Since $S(1)$ has a non-lattice law, $|\exp(-\omega(t))| < 1$ for $t \neq 0$, whence $\Re(\omega(t)) > 0$. Hence, defining $\phi(t) = \lambda(\omega(t))$, the CF of $S(Y_1)$, we have from (3.2) and then (3.3) that

$$E\{[\lambda(\omega(Ut))]^{m+n}\} = E\{[\lambda(U^\delta\omega(t))]^{m+n}\} = E[\lambda(\omega(t))]^m = (\phi(t))^m.$$

This is the CF version of (1.1), and the proof is complete.

Remarks. 1. When $(S(\tau))$ is a stable process then $S(Y_1) \cong SY_1^{1/\delta}$ where S has the stable law defined by (2.2).

2. The identity (3.2) holds for any U when $S(1)$ is stable, but only for U as in the theorem for any other SS(δ, c)-law. This suggests that $\mathcal{L}(X_1) \in \mathcal{C}(\delta)$ whenever $-\log U$ has a non-lattice law. Indeed, this often is obvious just from (1.4) when U has a smooth density.

3. If f is the probability generating function of N then (1.3) and (2.3) imply $f(\frac{m}{m+n}) = \frac{m}{m+n}$. In particular we must have $E(N) > 1$ and $P(N = 0) > 0$. Conversely, it is obvious now that appropriate choice of $\mathcal{L}(N)$ will satisfy (1.3) and (2.3). For example, let $P(N = 0) = \frac{m}{2m+n} = 1 - P(N = 2)$ giving, when $m = n = 1$, $\lambda(\theta) = \frac{1}{3}(\lambda(\theta))^2 + \frac{2}{3}(\lambda(\frac{\theta}{4}))^2$. This has a unique LST solution with $\lambda'(0) = -1$. If $\mathcal{L}(N)$ is defective then $P(U = 0) > 0$ and hence $\mathcal{L}(X_1)$ has an atom at zero. See Pakes ((1992*b*), Lemma 2.2) and Section 4.

4. Theorem 3.1 has a continuous version: If (Y_τ) is the above subordinator, independent of $(S(\tau))$, then $Z_\tau = S(Y_\tau)$ is a Lévy process and $\mathcal{L}(Z_1)$ solves (1.2).

4. The self-similar case

We consider (1.2) but assume now that (Z_τ) is a non-trivial stochastically continuous self-similar process with $Z_0 = 0$. Self-similarity means an equivalence between spatial and temporal scaling. Specifically, there is a positive number H, which is unique, such that

$$(Z_{a\tau}) \cong (a^H Z_\tau) \tag{4.1}$$

and we say (Z_τ) has order H. See Lamperti (1962), and Maejima (1989) for a survey. Self-similar processes are not in general infdiv and the CF version of (1.2) differs in form from (1.5). Imposing additional restrictions will bring our two interpretations of (1.2) closer. Thus if (Z_τ) also has independent increments then $\mathcal{L}(Z_\tau)$ is infdiv (in fact, of class L). If (Z_τ) is also a Lévy process then it is a stable process and it satisfies (1.2) only when U is a constant. Surprisingly, self-similarity alone implies this restriction.

THEOREM 4.1. *Suppose (Z_τ) is stochastically continuous, self-similar and* $Z_0 = 0$.

(i) *Let $H > 0$ and $U = (\frac{u}{u+v})^H \equiv c$ in (1.2). Then any self-similar process of order H solves (1.2).*

(ii) *Suppose (Z_τ) has order H and solves (1.2). Then $U = c$.*

PROOF. (i) Take $a = \frac{u}{u+v}$ and $\tau = u + v$ for one-dimensional distributions in (4.1).

(ii) Choosing $\tau = 1$ in (4.1), and then $a = u$ and $a = u + v$ for the left and right hand sides of (1.2), reduces (1.2) to

$$Z_1 \cong Z_1 U/c. \tag{4.2}$$

To "divide out" Z_1 let ζ and ψ be the CF's of $\log|Z_1|$ and $\log(U/c)$, respectively. Then (4.2) yields $\zeta(t) = \zeta(t)\psi(t)$ and hence $\psi(t) \equiv 1$ in an open interval containing the origin but no zeros of ζ. It follows that $\psi(t) \equiv 1$ and the assertion follows.

Theorem 4.1 asserts that no self-similar process has the above projective invariance property for random rotations but all have it for a correctly chosen fixed direction.

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