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ASYMPTOTIC DISTRIBUTION OF MAXIMAL AUTOREGRESSIVE PROCESS WITH WEIGHT TENDING TO 1

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Abstract. The maximal operator plays the similar role as the summation operator in the sense of stability of operation. So, we could discuss *ARMA* processes in the maximal operation by the same way as in the summation operation. However, many papers already treated with moving order statistics. In this paper, we discuss asymptotic behaviors of maximal autoregressive (*MAR*) processes with the weight tending to 1.

Key words and phrases: Extreme value distribution, stability coefficients, maximal autoregressive process, maximum of weighted random variables.

1. Introduction

Although the maximal operator is quite different from the summation in the results of their calculations, they are similar in the sense of stability of operation. That is, when we consider a finite set of real numbers and its subsets which are disjoint coverings of the set, we see that the sum of the subsums of subsets is equal to the total sum of the set, and similarly, the maximum of the submaximums of subsets is equal to the total maximum of the set. Thus, we could discuss maximal *ARMA* processes in the analogical way as summation *ARMA* processes. Several papers discuss the behaviors of order statistics in overlapping samples and moving order statistics, especially moving medians, maximums and ranges (for example, see Inagaki (1980), Mallows (1980), David (1981), David and Rogers (1983)).

In the present paper, we discuss about maximal autoregressive processes. Let $\{Y_0, Y_1, Y_2, \dots, Y_n, \dots\}$ be the process defined recursively as follows: for $t \geq p$,

$$Y_t = \max\{a_1 Y_{t-1}, a_2 Y_{t-2}, \dots, a_p Y_{t-p}, X_t\}, \tag{1.1}$$

where $\{a_1, a_2, \dots, a_p\}$ are positive constants, $\{Y_0, Y_1, \dots, Y_{p-1}\}$ are the initial random variables, and $\{X_n : n = 0, 1, 2, \dots\}$ is the innovation process of positive, independent, and identically distributed random variables with the probability distribution function $F(x)$. We shall call it the maximal autoregressive process of order p (abbreviate to $MAR(p)$). Furthermore, we consider the triangular array of $MAR(p)$ with the transformed innovation process and coefficients depending on the present time n and we denote it by $MAR^*(p)$. That is, letting the transformed innovation process be

$$X^*_{n,t} = \frac{X_t - \mu_n}{\sigma_n}, \qquad 0 \le t \le n,$$

we define $\{Y^*_{n,0}, Y^*_{n,1}, Y^*_{n,2}, \ldots, Y^*_{n,n}\}$ be the process recursively as follows: for $p \le t \le n$,

$$Y^*_{n,t} = \max\{a_{n,1}Y^*_{n,t-1}, a_{n,2}Y^*_{n,t-2}, \ldots, a_{n,p}Y^*_{n,t-p}, X^*_{n,t}\}, \tag{1.2}$$

where $Y^*_{n,t}$, $t = 0, \ldots, p-1$ are any random variables given initially (for example, we may take $Y^*_{n,t} = X^*_{n,t}$, $t = 0, \ldots, p-1$). Our purpose is to investigate the behaviors of maximal autoregressive and transformed processes by this representation, but it is too complicated to deal with, although its derivation is interesting.

In Section 2, we prepare the fundamental lemmas and theorems, some of which are well known results and others are ones extended here. In Section 3, we show that $MAR(p)$ and $MAR^*(p)$ processes are represented as the maximum of weighted i.i.d. random variables:

$$Y_n = \max\{\sigma_{n,t}X_{n-t} : t = 0, 1, \ldots, n\}, \tag{1.3}$$

$$Y^*_{n,n} = \max\{\sigma_{n,t}X^*_{n,n-t} : t = 0, 1, \ldots, n\}, \tag{1.4}$$

similarly as the usual AR process. The nice paper of Daley and Hall (1984) discusses the asymptotic properties of the maximum of weighted i.i.d. random variables in quite general situations.

In the present paper, we discuss only the asymptotic distributions of $MAR^*(1)$ process with the initial value 0 and the weight tending to 1: $a_n = a_{n,1} = (1 + \frac{\alpha}{n})$. However, the maximal $AR(1)$ process may be so important as the usual $AR(1)$ process. Then, the process has the representation as the maximum of geometrically weighted i.i.d. random variables by which we have the limit distribution of the process. Section 4 is devoted to the discussion and includes the very kind and valuable comments due to one of the referees.

2. Fundamental results

In order to obtain the limit distribution of MAR process explicitly, we need and so briefly review classical lemmas for the extreme distribution theory, some of which may be extended in this paper.

Throughout this section, we suppose $X_1, \ldots, X_n$ are independent and identically distributed with a distribution function $F(x)$ and denote the order statistics as follows:

$$X_{n:1} \le X_{n:2} \le \cdots \le X_{n:n}.$$

The following lemma and theorem are due to Smirnov (1952).

LEMMA 2.1. *Let us denote the distribution function of the k-th order statistic* $X_{n:k}$ *by* $F_{n:k}(x)$ *and one of Gamma distribution* $G_A(k,1)$ *by*

$$G_k(x) = \frac{1}{(k-1)!}\int_0^x y^{k-1}e^{-y}dy.$$

Then, we have the following inequalities: for any $x \in \mathcal{R}$,

$$(1-\delta_n)G_k(nF(x)) \le F_{n:k}(x) \le (1+\varepsilon_n)G_k(nF(x)),$$

where

$$\delta_n = 1 - \left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{k}{n+1}\right) \downarrow 0,$$
$$\varepsilon_n = \left(1 + \frac{1}{n-k}\right)\cdots\left(1 + \frac{k}{n-k}\right) - 1 \downarrow 0.$$

Now, we consider the extreme statistic normalized by stabilizing coefficients $\sigma_n > 0$ and μ_n for any fixed k: $Z_{n:k} \equiv \frac{X_{n:k}-\mu_n}{\sigma_n}$, and its distribution function: $G_{n:k}(x) \equiv F_{n:k}(\sigma_n x + \mu_n)$. Smirnov's lemma implies that the limit distribution of $G_{n:k}(x)$ is determined only by the limit function $u(x)$:

$$u_n(\sigma_n x + \mu_n) \equiv nF(\sigma_n x + \mu_n) \to u(x).$$

It is well known that its limit function can be characterized as the power function or the exponential function by using theorems due to Khintchine and Feller (see Gnedenko (1943), Smirnov (1952) and Feller (1971)):

$$\begin{array}{lll} \text{(I)} & u(x) = x^{\rho}, & x > 0,\ \rho > 0, \\ \text{(II)} & u(x) = |x|^{-\rho}, & x < 0,\ \rho > 0, \\ \text{(III)} & u(x) = e^{x}, & -\infty < x < \infty. \end{array}$$

Thus, the well known theorem about extreme value distribution is mentioned for appropriate choice of sequences of stabilizing coefficients, $\{\sigma_n > 0\}$ and $\{\mu_n\}$.

THEOREM 2.1. *There are three types of the limit distribution of* $G_{n:k}(x)$:

$$\begin{array}{lll} \text{(Type I)} & \Psi_{k*}(x) \equiv G_k(x^{\rho}), & x > 0,\ \rho > 0. \\ \text{(Type II)} & \Phi_{k*}(x) \equiv G_k(|x|^{-\rho}), & x < 0,\ \rho > 0. \\ \text{(Type III)} & \Lambda_{k*}(x) \equiv G_k(e^{x}), & -\infty < x < \infty. \end{array}$$

For maximal extreme statistics $X_{n:n-k+1}$, we take $X_1^* = -X_1, \ldots, X_n^* = -X_n$, as i.i.d. samples with the distribution function $F^*(x) = 1 - F(-x)$ and denote the k-th order statistic and its distribution function by $X_{n:k}^*$ and $F_{n:k}^*(x)$, respectively. Then, we see $X_{n:n-k+1} = -X_{n:k}^*$, and thus, $F_{n:n-k+1}(x) = 1 - F_{n:k}^*(-x)$. Hence, we have the following corollary for appropriate choice of sequences of stabilizing coefficients, $\{\sigma_n > 0\}$ and $\{\mu_n\}$.

COROLLARY 2.1. *There are three types of the limit distribution of* $G_{n:n-k+1}(x)$:

$$\begin{array}{lll} \text{(Type I}^*) & \Psi_k{}^*(x) \equiv 1 - \Psi_{k*}(-x), & x < 0, \ \rho > 0. \\ \text{(Type II}^*) & \Phi_k{}^*(x) \equiv 1 - \Phi_{k*}(-x), & x > 0, \ \rho > 0. \\ \text{(Type III}^*) & \Lambda_k{}^*(x) \equiv 1 - \Lambda_{k*}(-x), & -\infty < x < \infty. \end{array}$$

In the sequel of this section, we denote the maximum by Y_n, its distribution function by $F_n(y)$ and $G_1(x)$ by $G(x)$, simply. The last is the exponential distribution with mean 1: $G(x) = 1 - e^{-x}$. Let us denote their survival functions as follows:

$$\overline{F}(x) = 1 - F(x), \qquad \overline{F}_n(y) = 1 - F_n(y), \qquad \overline{G}(y) = 1 - G(y),$$

and so on. Then, three types of the limit distribution in the last corollary are according to the regular variation of the limit function

$$v_n(\sigma_n y + \mu_n) \equiv n\overline{F}(\sigma_n y + \mu_n) \to v(y), \tag{2.1}$$

which is either one of the following functions:

$$\begin{array}{lll} (\text{I}^*) & v(x) = x^{-\rho}, & x > 0, \ \rho > 0, \\ (\text{II}^*) & v(x) = |x|^{\rho}, & x < 0, \ \rho > 0, \\ (\text{III}^*) & v(x) = e^{-x}, & -\infty < x < \infty. \end{array} \tag{2.2}$$

Note that the last convergence is uniform in the sense of the following lemma, which is used in the next section.

LEMMA 2.2. *For any fixed y and any fixed interval $[-M_1, M_2]$ with $0 \le M_1 < 1$, $M_2 > 0$, let*

$$w_n(z) \equiv v_n(\sigma_n y(1+z) + \mu_n) \quad \textit{and} \quad w(z) \equiv v(y(1+z)),$$

for $z \in [-M_1, M_2]$. Then, it holds that

$$w_n(z) \to w(z), \quad \textit{as} \quad n \to \infty,$$

uniformly for $z \in [-M_1, M_2]$.

PROOF. We see from (2.1) that $w_n(z)$ converges to $w(z)$ and these functions are monotone, and furthermore, from (2.2) that the limit function $w(z)$ is continuous. Thus, we can prove this lemma in the same way of Pólya's Theorem about the uniform convergence of a sequence of distributions. □

3. Limit distribution of maximal *AR* process

Let us consider the $MAR(p)$ process: for $t \geq p$,

$$Y_t = \max\{a_1 Y_{t-1}, a_2 Y_{t-2}, \ldots, a_p Y_{t-p}, X_t\},$$

defined in (1.1). Then, we see that $MAR(p)$ process has a representation of the maximum of weighted i.i.d. random variables.

THEOREM 3.1. *$MAR(p)$ process $\{Y_n\}$ is represented as the maximum of weighted i.i.d. random variables, when the initial random variables in (1.3) are denoted by $\{X_0, \ldots, X_{p-1}\}$ in the place of $\{Y_0, \ldots, Y_{p-1}\}$:*

$$Y_n = \max\{\sigma_{n,t} X_{n-t} : t = 0, 1, \ldots, n\}, \tag{3.1}$$

where weight constants $\sigma_{n,t}$ are the maximum of products of constant coefficients $\{a_1, \ldots, a_p\}$:

$$\sigma_{n,t} = \max\left\{ a_1^{j_1} \cdots a_p^{j_p} : \sum_{i=1}^{p} i j_i = t,\ j_i \text{ nonnegative integers} \right\}.$$

PROOF. We see, in turn,

$$\begin{aligned}
Y_p &= \max\{a_1 X_{p-1}, a_2 X_{p-2}, \ldots, a_p X_0, X_p\}, \\
Y_{p+1} &= \max\{a_1^2 X_{p-1}, a_1 a_2 X_{p-2}, \ldots, a_1 a_p X_0; \\
&\qquad a_1 X_p, a_2 X_{p-1}, \ldots, a_p X_1; X_{p+1}\}, \\
Y_{p+2} &= \max\{a_1^3 X_{p-1}, a_1^2 a_2 X_{p-2}, \ldots, a_1^2 a_p X_0; \\
&\qquad a_1^2 X_p, a_1 a_2 X_{p-1}, \ldots, a_1 a_p X_1, a_2^2 X_{p-2}, \ldots, a_2 a_p X_0; \\
&\qquad a_1 X_{p+1}, a_2 X_p, \ldots, a_p X_2; X_{p+2}\}.
\end{aligned}$$

By mathematical induction, we have the result of this theorem. □

In the sequel, we consider the limit distribution of only the $MAR^*(1)$ process with the weight tending to 1 and the initial random variable 0:

$$a_n = a_{n,1} = \left(1 + \frac{\alpha}{n}\right), \qquad Y_{n,0}^* = 0.$$

Then, we have the $MAR^*(1)$ process:

$$\begin{aligned}
Y_{n,n}^* &= \max\{a_n Y_{n,n-1}^*, X_{n,n}^*\} \\
&= \max\{X_{n,n}^*, a_n X_{n,n-1}^*, a_n^2 X_{n,n-2}^*, \ldots, a_n^{n-1} X_{n,1}^*\}
\end{aligned} \tag{3.2}$$

and the distribution function of $Y_{n,n}^*$:

$$G_n^*(y) = P(Y_{n,n}^* \leq y) = \prod_{t=1}^{n} F^*(a_n^{-t+1} y), \tag{3.3}$$

where $F^*(x)$ is the distribution function of $X^*_{n,t}$ and thus, depends on n:

$$F^*(x) = P(X^*_{n,t} \leq x) = F(\sigma_n x + \mu_n). \tag{3.4}$$

We use the following lemma without proof in order to show the limit distribution of $G^*_n(y)$.

LEMMA 3.1. *For real numbers $\{f_i, i = 1, \ldots, n\}$ with $0 \leq f_i \leq 1$, the following inequalities hold*:

$$\prod_{i=1}^{n} f_i \leq \exp\left\{-\sum_{i=1}^{n}(1-f_i)\right\} \leq \prod_{i=1}^{n} f_i + \frac{1}{2}\max_{i=1,\ldots,n}(1-f_i).$$

THEOREM 3.2. *There are three types of the limit distribution $G^*_\alpha(y)$ of $G^*_n(y)$ for sequences of stabilizing coefficients chosen in Corollary* 2.1:

$$\begin{array}{lll}
(\text{Type I}^*_\alpha) & \Psi^*_\alpha(y) = \exp\left\{-\dfrac{e^{\alpha\rho}-1}{\alpha\rho}y^{-\rho}\right\}, & y > 0,\ \rho > 0. \\
(\text{Type II}^*_\alpha) & \Phi^*_\alpha(y) = \exp\left\{-\dfrac{1-e^{-\alpha\rho}}{\alpha\rho}|y|^{\rho}\right\}, & y < 0,\ \rho > 0. \\
(\text{Type III}^*_\alpha) & \Lambda^*_\alpha(y) = \exp\left\{-\displaystyle\int_0^1 \exp(-ye^{-\alpha s})ds\right\}, & -\infty < y < \infty.
\end{array}$$

PROOF. It follows from (2.1), (3.3) and (3.4) that the distribution function of $MAR(1)$ process Y^*_n is rewritten as

$$\begin{aligned}
G^*_n(y) &= \prod_{t=1}^{n}\left[1 - \frac{v_n\left(\sigma_n y\left(1+\dfrac{\alpha}{n}\right)^{-(t-1)} + \mu_n\right)}{n}\right] \\
&= \exp\left\{-\frac{1}{n}\sum_{t=1}^{n} v_n\left(\sigma_n y\left(1+\frac{\alpha}{n}\right)^{-(t-1)} + \mu_n\right)\right\} + o(1),
\end{aligned} \tag{3.5}$$

as $n \to \infty$. That relation is guaranteed by Lemma 3.1 and the uniform convergence to 0 of each term

$$\frac{v_n\left(\sigma_n y\left(1+\dfrac{\alpha}{n}\right)^{-(t-1)} + \mu_n\right)}{n}, \qquad t = 1, \ldots, n,$$

which is proved in the following manner. Since

$$\exp(-|\alpha|) < \left(1+\frac{\alpha}{n}\right)^{-(t-1)} < \exp(|\alpha|), \qquad t = 1, \ldots, n,$$

we have, by Lemma 2.2,

$$v_n\left(\sigma_n y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}+\mu_n\right)-v\left(y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right)\to 0,\qquad \text{as}\quad n\to\infty,$$

uniformly for $t=1,\dots,n$. This leads to the fact:

$$\frac{1}{n}\sum_{t=1}^{n}v_n\left(\sigma_n y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}+\mu_n\right)-\frac{1}{n}\sum_{t=1}^{n}v\left(y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right)\to 0,\quad \text{as}\quad n\to\infty$$

According to (2.2), we have the convergence of the second term:

$$J=\frac{1}{n}\sum_{t=1}^{n}v\left(y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right),$$

as follows.

Type I* case:

$$J=\frac{1}{n}\sum_{t=1}^{n}\left\{y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right\}^{-\rho}=y^{-\rho}\frac{1-\left(1+\dfrac{\alpha}{n}\right)^{\rho n}}{n\left(1-\left(1+\dfrac{\alpha}{n}\right)^{\rho}\right)}$$
$$\to y^{-\rho}\frac{\exp(\rho\alpha)-1}{\rho\alpha}.$$

Type II* case:

$$J=\frac{1}{n}\sum_{t=1}^{n}\left|y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right|^{\rho}=|y|^{\rho}\frac{1-\left(1+\dfrac{\alpha}{n}\right)^{-\rho n}}{n\left(1-\left(1+\dfrac{\alpha}{n}\right)^{-\rho}\right)}$$
$$\to |y|^{\rho}\frac{1-\exp(-\rho\alpha)}{\rho\alpha}.$$

Type III* case:

$$\begin{aligned}J&=\frac{1}{n}\sum_{t=1}^{n}\exp\left\{-y\left(1+\frac{\alpha}{n}\right)^{-(t-1)}\right\}\\&=\frac{1}{n}\sum_{t=1}^{n}\exp\left[-y\exp\left\{-(t-1)\log\left(1+\frac{\alpha}{n}\right)\right\}\right]\\&=\frac{1}{n}\sum_{t=1}^{n}\exp\left[-y\exp\left\{-\alpha\frac{t-1}{n}+o(n^{-1})\right\}\right]\\&\to\int_0^1\exp(-ye^{-\alpha s})ds.\end{aligned}$$

Thus, we have the result of this theorem. □

4. Discussion

It is worth noting in Theorem 3.2 that the limit distribution of $MAR^*(1)$ process belongs to the same domain of attraction as the limit distribution of maximum when its domain is type I* or II*, but the former is quite different from the latter when it belongs to the type III* of domain of attraction.

If we denote Y_n by $Y_n^*(\alpha)$ with parameter α, we could show the limit distribution of the joint one of $(Y_n^*(\alpha), Y_n^*(\beta))$.

Further, we might discuss about problems of statistical inference about the parameters of $MAR(p)$ process, although we treated mainly the $MAR(1)$ process in this paper.

One of the referees pointed that, in the case: $\mu_n = 0$, the original process

$$Y_n = \max\{aY_{n-1}, X_n\}, \qquad Y_0 = 0, \tag{4.1}$$
$$= \max\{X_n, aX_{n-1}, a^2X_{n-2}, \dots, a^{n-1}X_1\}$$

has an approximate distribution:

$$F_n(y) = P(Y_n \le y) \approx G^*_{n(a-1)}(\sigma_n y), \tag{4.2}$$

and a simulation study comparing $F_n(y)$ and the approximation would be of interest, and furthermore, that the limiting distribution of Y_n is obtained:

$$F_n(y) \to \exp\{-(1-a^\rho)^{-1}y^{-\rho}\}, \tag{4.3}$$

if the innovation distribution is max-stable: $F(x) = \exp(-x^{-\rho})$, $0 < a < 1$ (noting that Alpuim (1989) and Alpuim and Athayde (1990) discuss the situations where Y_n has the stationary limit distribution). I appeciate his kindness heartily. By virtue of his valuable comments, I have made this paper clearer. I would like to study the problems suggested above by the referee in the next papers.

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