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ON EQUIVALENCE OF WEAK CONVERGENCE AND MOMENT CONVERGENCE OF LIFE DISTRIBUTIONS

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Abstract. We investigate the equivalence of weak convergence and moment convergence of life distributions in a family which is larger than the HNBUE family. Also, we point out that within the HNBUE family the exponential distribution can be characterized by one value of its Laplace-Stieltjes transform.

Key words and phrases: Weak convergence, moment convergence, HNBUE family, convex order, characterization of distribution.

1. Introduction

Let F be a life distribution, namely F has the support contained in the interval $[0, \infty)$, and let its mean $\mu_F \in (0, \infty)$. Then F is said to be harmonic new better than used in expectation (HNBUE) if

$$\int_t^\infty \bar{F}(x)dx \le \mu_F \exp(-t/\mu_F) \qquad \text{for all} \quad t \ge 0,$$

where $\bar{F} = 1 - F$ (see, e.g., Klefsjö (1982)). For convenience, the degenerate distribution at the point zero, denoted by F_0, is also defined to be HNBUE. It is known that the HNBUE family is closed under weak convergence and that the weak convergence within this family is equivalent to the convergence of each moment sequence of positive order to the corresponding moment of limiting distribution (see Basu and Bhattacharjee (1984) and note that in order to guarantee the closedness of weak convergence it is necessary to define F_0 to be HNBUE). In this paper we first introduce a family $\mathcal{M}$ which is larger than the HNBUE family (Section 2), and then extend the above-mentioned results from the HNBUE family to the family $\mathcal{M}$ (Section 3). Finally, we point out that within the HNBUE family the exponential distribution can be characterized by one value of its Laplace-Stieltjes transform; a general result for the convex ordering family is also given (Section 4).

2. The family $\mathcal{M}$

In this section we denote by F the general life distribution and by G the exponential distribution. Further, let X and Y be two random variables (r.v.s) having distributions F and G, respectively. For convenience, we also call F_0 an exponential distribution with mean zero, and denote $1/\mu = +\infty$ if $\mu = 0$. The HNBUE family can be written as

$$\mathcal{H} = \bigcup_{\mu \geq 0} \left\{ F : \mu_F = \mu_G = \mu, \int_t^\infty \bar{F}(x)dx \leq \int_t^\infty \bar{G}(x)dx \text{ for all } t \geq 0 \right\}.$$

Here we define three more families of life distributions:

$$\begin{aligned} \mathcal{M}_1 &= \bigcup_{\mu \geq 0} \{F : \mu_F = \mu_G = \mu, EX^r \leq EY^r \text{ for all } r \geq 1\}; \\ \mathcal{M}_2 &= \bigcup_{\mu \geq 0} \{F : \mu_F = \mu_G = \mu, Ee^{sX} \leq Ee^{sY} \text{ for all } s < 1/\mu\}; \\ \mathcal{M} &= \mathcal{M}_1 \cap \mathcal{M}_2. \end{aligned}$$

(See Stoyan ((1983), p. 34) for related life families.) It is easy to see that $\mathcal{M}_1 \not\subset \mathcal{M}_2$ and $\mathcal{M}_2 \not\subset \mathcal{M}_1$. We now prove that $\mathcal{M}$ is larger than $\mathcal{H}$.

LEMMA 2.1. *$\mathcal{H} \subset \mathcal{M}$ and $\mathcal{H} \neq \mathcal{M}$.*

PROOF. The fact $\mathcal{H} \subset \mathcal{M}$ follows immediately from the characterization theorem (Lemma 2.2 below) for the HNBUE distributions. To see $\mathcal{H} \neq \mathcal{M}$, define the distribution K by

$$K(x) = \begin{cases} 0 & \text{if } x < 0.3 \\ 0.3 & \text{if } 0.3 \leq x < 3 \\ 1 & \text{if } 3 \leq x. \end{cases}$$

Then $K \in \mathcal{M}$ but $K \notin \mathcal{H}$ (see also Klefsjö ((1983), p. 617)). The proof is complete.

LEMMA 2.2. *Let F be a life distribution and G an exponential distribution with mean $\mu_G = \mu_F \in [0, \infty)$. Further, let X and Y be two r.v.s having distributions F and G, respectively. Then $F \in \mathcal{H}$ if and only if the inequality $Eh(X) \leq Eh(Y)$ holds for all convex function h defined on $[0, \infty)$.*

PROOF. See Stoyan ((1983), Theorem 1.3.1).

3. Weak convergence within the family $\mathcal{M}$

The main result is stated as follows.

THEOREM 3.1. *Let $F_n \in \mathcal{M}$ and be obeyed by a r.v. X_n, where $n = 1, 2, \ldots$.*
(i) *If $F_n \to F$ weakly for some life distribution F, then $F \in \mathcal{M}$ and*

$$\lim_{n\to\infty} EX_n^r = EX^r < \infty \quad \textit{for all} \quad r > 0, \tag{3.1}$$

where X is a r.v. obeying distribution F.
(ii) *Conversely, if the equality in* (3.1) *holds for each integer $r > 0$ and for some nonnegative r.v. X with finite mean, then $F_n \to F$ weakly, where F is the distribution of X and $F \in \mathcal{M}$.*

To prove Theorem 3.1 we need the following lemma.

LEMMA 3.1. *Let $F \in \mathcal{M}_1 \cup \mathcal{M}_2$ and be obeyed by a r.v. X. Then F is uniquely determined by the sequence of its moments $\{EX^n\}_{n=1}^{\infty}$.*

PROOF. It is clear that the conclusion holds for the degenerate distribution F_0. Suppose now that $F \neq F_0$. Then in case $F \in \mathcal{M}_1$, we have

$$EX^n \le EY^n = n!\mu_G^n \quad \text{for} \quad n = 1, 2, \ldots.$$

Note that the Carleman criterion holds for F, namely $\sum_{n=1}^{\infty}(EX^n)^{-1/(2n)} = \infty$, so the distribution F is uniquely determined by the sequence of its moments $\{EX^n\}_{n=1}^{\infty}$. In case $F \in \mathcal{M}_2$, we have

$$Ee^{sX} \le Ee^{sY} = (1 - \mu_G s)^{-1} \quad \text{for all} \quad s < 1/\mu_G,$$

and hence F is characterized by the sequence of its moments (see, e.g., Chow and Teicher ((1988), pp. 283–285)). The proof is complete.

PROOF OF THEOREM 3.1. Denote by G_n the exponential distribution with mean $\mu_{G_n} = \mu_{F_n} \equiv \mu_n$, and by Y_n a r.v. obeying G_n, where $n = 1, 2, \ldots$.
(i) Suppose $F_n \to F$ weakly, then we want to prove that (3.1) holds and that $F \in \mathcal{M}$. By the assumption $F_n \in \mathcal{M} \subset \mathcal{M}_2$, we have

$$(1 + \mu_n)^{-1} \ge Ee^{-X_n} \to Ee^{-X} \in (0, \infty) \quad \text{as} \quad n \to \infty,$$

so that there exists a constant $B \in (0, \infty)$ such that

$$\mu_n \in [0, B) \quad \text{for all} \quad n = 1, 2, \ldots. \tag{3.2}$$

For each fixed $r > 0$, take a constant $s > \max\{r, 1\}$. Then since $F_n \in \mathcal{M} \subset \mathcal{M}_1$ it follows that

$$EX_n^s \le EY_n^s = \Gamma(s+1)\mu_n^s \le \Gamma(s+1)B^s \quad \text{for all} \quad n \ge 1,$$

in which the last inequality follows from (3.2). Hence the sequence $\{EX_n^s\}_{n=1}^{\infty}$ is bounded. By the Moment Convergence Theorem (see, e.g., Chow and Teicher ((1988), p. 259)) we conclude that $\lim_{n\to\infty} EX_n^r = EX^r < \infty$. This is the desired result (3.1). Next, we want to prove that $F \in \mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$. Letting $r = 1$, the equality (3.1) reduces to $\lim_{n\to\infty} \mu_n = \mu_F$, and hence for each $r \geq 1$,

$$EX^r = \lim_{n\to\infty} EX_n^r \leq \lim_{n\to\infty} EY_n^r = \lim_{n\to\infty} \Gamma(r+1)\mu_n^r = \Gamma(r+1)\mu_F^r.$$

Therefore $F \in \mathcal{M}_1$. To prove the part $F \in \mathcal{M}_2$, let us define the r.v. $Z_n = \exp(X_n)$ for $n = 1, 2, \ldots$. Then for each fixed $s < 1/\mu_F$, take a positive constant $t \in (s, 1/\mu_F)$, and we have

$$EZ_n^t = Ee^{tX_n} \leq Ee^{tY_n} = \frac{1}{1-\mu_n t} \qquad \text{for all} \quad n = 1, 2, \ldots,$$

the last term tending to $1/(1-\mu_F t)$ as n goes to infinity. Hence the sequence $\{EZ_n^t\}_{n=1}^{\infty}$ is bounded. Applying the Moment Convergence Theorem again, we obtain that for each $s < 1/\mu_F$,

$$Ee^{sX} = \lim_{n\to\infty} Ee^{sX_n} \leq \lim_{n\to\infty} Ee^{sY_n} = \lim_{n\to\infty} (1-\mu_n s)^{-1} = (1-\mu_F s)^{-1}.$$

Therefore $F \in \mathcal{M}_2$. The proof of $F \in \mathcal{M}$ is complete.

(ii) Suppose that the equality in (3.1) holds for each integer $r > 0$ and for some nonnegative r.v. X with finite mean. Denote by F the distribution of X. Then we want to prove that $F_n \to F$ weakly and that $F \in \mathcal{M}$. Note first that the sequence $\{F_n\}_{n=1}^{\infty}$ is tight due to the assumption $\mu_F < \infty$. Let $\{F_{n'}\}$ be any convergent subsequence of $\{F_n\}$, say $F_{n'} \to F_*$ weakly for some life distribution F_*, and let X_* be a r.v. obeying F_*. Then by the part (i) we have $F_* \in \mathcal{M}$ and $\lim_{n'\to\infty} EX_{n'}^r = EX_*^r$ for each integer $r > 0$, so that the distributions F and F_* have the same finite moments of all orders. On the other hand, it can be proved that F is characterized by the sequence of its moments due to the conditions on F_n. From Lemma 3.1 we conclude that $F_* = F$ and hence each convergent subsequence of $\{F_n\}$ has the same limiting distribution F. Therefore $F_n \to F$ weakly by Fréchet-Shohat Theorem (see, e.g., Chow and Teicher ((1988), p. 265)). Finally, the desired result $F \in \mathcal{M}$ follows immediately from the part (i). The proof is complete.

Note that to prove the equality (3.1) above we have applied the Moment Convergence Theorem. By the same way, we can simplify the proof of Basu and Bhattacharjee's (1984) result.

4. A characterization result for the life distributions within the convex ordering family

We first define the convex ordering for life distributions: the life distributions F and G are said to satisfy $F \leq_c G$ if and only if

$$\int_t^{\infty} \bar{F}(x)dx \leq \int_t^{\infty} \bar{G}(x)dx \qquad \text{for all} \quad t \geq 0$$

(see also Stoyan ((1983), p. 8)). Denote the convex ordering family $\mathcal{F}_G = \{F : F \leq_c G\}$. Then we have the following characterization theorem in terms of one value of the Laplace-Stieltjes transform.

THEOREM 4.1. *Let F and G be two life distributions with the same mean $\mu \in (0, \infty)$, and be obeyed by r.v.s X and Y, respectively. Assume further $F \in \mathcal{F}_G$. Then the equality*

$$Ee^{-sX} = Ee^{-sY} \tag{4.1}$$

holds for some $s > 0$ if and only if $F = G$.

PROOF. The sufficiency part is trivial. It remains to prove the necessity part. Suppose that the equality (4.1) holds for some $s > 0$, then we want to prove $F = G$.

Define the distributions

$$\phi_1(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy \quad \text{and} \quad \phi_2(x) = \frac{1}{\mu}\int_0^x \bar{G}(y)dy \qquad \text{for} \quad x \geq 0.$$

Then $\phi_1 \geq \phi_2$ because $F \leq_c G$, and hence

$$0 < \phi_1^{-1}(t) \leq \phi_2^{-1}(t) \qquad \text{for} \quad t \in (0,1),$$

where $\phi_i^{-1}(t) = \inf\{x : \phi_i(x) \geq t\}$ for $t \in (0,1)$.

Using integration by parts we write

$$Ee^{-sX} = s\int_0^\infty e^{-sx}F(x)dx \tag{4.2}$$

and similarly

$$Ee^{-sY} = s\int_0^\infty e^{-sx}G(x)dx. \tag{4.3}$$

It follows from (4.1), (4.2) and (4.3) that

$$\int_0^\infty e^{-sx}F(x)dx = \int_0^\infty e^{-sx}G(x)dx,$$

or, equivalently,

$$\int_0^\infty e^{-sx}\bar{F}(x)dx = \int_0^\infty e^{-sx}\bar{G}(x)dx. \tag{4.4}$$

In terms of ϕ_i we can rewrite (4.4) as follows:

$$\int_0^1 \exp(-s\phi_1^{-1}(t))dt = \int_0^1 \exp(-s\phi_2^{-1}(t))dt. \tag{4.5}$$

Since $\phi_1^{-1} \leq \phi_2^{-1}$, the equality (4.5) implies that $\phi_1^{-1}(t) = \phi_2^{-1}(t)$ a.e. on $(0,1)$ and hence $F = G$. The proof is complete.

COROLLARY 4.1. *Let F be an HNBUE distribution with mean $\mu \in (0, \infty)$, and be obeyed by a r.v. X. Then the equality $Ee^{-sX} = (1 + \mu s)^{-1}$ holds for some $s > 0$ if and only if F is exponential.*

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