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THE EXACT DENSITY FUNCTION OF THE RATIO OF TWO DEPENDENT LINEAR COMBINATIONS OF CHI-SQUARE VARIABLES

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Abstract. A computable expression is derived for the raw moments of the random variable $Z = N/D$ where $N = \sum_1^n m_i X_i + \sum_{n+1}^s m_i X_i$, $D = \sum_{n+1}^s l_i X_i + \sum_{s+1}^r n_i X_i$, and the X_i's are independently distributed central chi-square variables. The first four moments are required for approximating the distribution of Z by means of Pearson curves. The exact density function of Z is obtained in terms of sums of generalized hypergeometric functions by taking the inverse Mellin transform of the h-th moment of the ratio N/D where h is a complex number. The case $n = 1$, $s = 2$ and $r = 3$ is discussed in detail and a general technique which applies to any ratio having the structure of Z is also described. A theoretical example shows that the inverse Mellin transform technique yields the exact density function of a ratio whose density can be obtained by means of the transformation of variables technique. In the second example, the exact density function of a ratio of dependent quadratic forms is evaluated at various points and then compared with simulated values.

Key words and phrases: Exact density, approximate density, moments, ratios of quadratic forms, Mellin transform.

1. Introduction

We are considering the random variable

$$Z = \frac{\sum_1^n m_i X_i + \sum_{n+1}^s m_i X_i}{\sum_{n+1}^s l_i X_i + \sum_{s+1}^r n_i X_i} \tag{1.1}$$

where the m_i's are real numbers, $l_i > 0$, $i = n+1, \dots, s$, $n_i > 0$, $i = s+1, \dots, r$, and

$$X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}, \qquad i = 1, \dots, r,$$

that is, the X_i's are independently distributed chi-square variables having r_i degree of freedom.

Such a structure arises in a variety of contexts. Chaubey and Nur Enayet Talukder ((1983), equation 2.1) obtained the moments of the quantity Q_1/Q_2 for the case where $Q_1 = \sum a_i X_i + \sum c_i Z_i$, $Q_2 = \sum b_i Y_i + \sum d_i Z_i$, and $Q_1 + Q_2$ is distributed as a chi-square variable; a representation of the raw moments of the ratio Q_1/Q_2 was obtained in closed forms by Morin-Wahhab (1985). Statistics having the structure of Z also appear in Lauer and Han ((1972), p. 255) where probabilities of certain ratios of chi-square variables are examined; in von Neumann ((1941), p. 369) where the ratio of the mean square successive difference to the variance is studied; in Toyoda and Ohtani ((1986), equation 8) where a statistic involved in a two-stage test is considered; and in Provost ((1986), p. 291) where a statistic is derived in connection with tests on the structural coefficients of a multivariate linear functional relationship model.

One may also use the random variable Z in the case of statistics expressed as ratios of sums of gamma variables (by selecting the coefficients of the X_i's in (1.1) accordingly and by assigning positive real values to the degrees of freedom) or as ratios of quadratic forms in central normal variables where some variables may be common to the numerator and the denominator (by diagonalizing the matrices of the quadratic forms).

The technique of the inverse Mellin transform (see for instance, Springer (1979) or Mathai and Saxena (1978)) is used in Section 3 to obtain in closed form a representation of the exact density of Z. Many researchers have utilized this technique in order to solve various distributional problems. For example Mathai and Tan (1977) obtained the distribution of the likelihood ratio criterion for testing the hypothesis that the covariance matrix in a multivariate distribution is diagonal; Pederzoli and Rathie (1983) derived the exact distribution of Bartlett's criterion for testing the equality of covariance matrices; Bagai (1972) obtained the distribution of a statistic used to test the hypotheses of equality of two dispersion matrices, equality of the multidimensional mean vectors, and the independence between a p set and a q set of variates. This technique was also used by Gupta and Rathie (1982) who considered the distribution of the likelihood ratio criterion for testing the hypothesis of equality of variances in k-normal populations.

A representation of the raw moments of Z is obtained in Section 2. The exact density of Z is derived for a particular case in Section 3; it is also shown that the approach used applies in the general case. Two examples are provided in Section 4.

2. The raw moments of Z

One needs the first four moments of Z in order to approximate the distribution of Z by means of Pearson curves. We derive in this section the h-th moment of Z for any positive integer h.

Let $Z = N/D$ and h be positive integer; then $Z^h = N^h/D^h$ where

$$N^h = \left(\sum_{i=1}^{s} m_i X_i\right)^h = \sum h! \left(\prod_{i=1}^{s} (m_i X_i)^{h_i}/h_i!\right), \tag{2.1}$$

the unindexed summation sign denoting a sum over the nonnegative integers

$h_1, \ldots, h_s$ such that $h_1 + \cdots + h_s = h$, and

$$D^{-h} = \left(\sum_{i=n+1}^{s} l_i X_i + \sum_{i=s+1}^{r} n_i X_i \right)^{-h} \tag{2.2}$$
$$= \Gamma(h)^{-1} \int_0^\infty t^{h-1} e^{-t(\sum_{i=n+1}^{s} l_i X_i + \sum_{i=s+1}^{r} n_i X_i)} dt.$$

Let $f_i(x_i)$ denotes the pdf of the random variable X_i, then the h-th moment of Z is

$$\begin{aligned} E(Z^h) = \sum \mu \prod_{i=1}^{r} (2^{r_i/2} \Gamma(r_i/2))^{-1} \\ \times \int_0^\infty t^{h-1} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} m_i^{h_i} x_i^{h_i + r_i/2 - 1} e^{-x_i/2} \\ \times \prod_{i=n+1}^{s} m_i^{h_i} x_i^{h_i + r_i/2 - 1} e^{-x_i(l_i t + 1/2)} \\ \times \prod_{i=s+1}^{r} x_i^{r_i/2 - 1} e^{-x_i(n_i t + 1/2)} dx_1 \cdots dx_r dt \end{aligned}$$

where

$$\mu = h! \left(\prod_{i=1}^{s} h_i! \right)^{-1} (\Gamma(h))^{-1}.$$

Noting that

$$\int_0^\infty \frac{e^{-x_i(n_i t + 1/2)} x_i^{r_i/2 - 1}}{\Gamma(r_i/2)} dx_i = (n_i t + 1/2)^{-r_i/2} \quad \text{for} \quad i = s+1, \ldots, r,$$
$$\int_0^\infty \frac{e^{-x_i/2} x_i^{h_i + r_i/2 - 1}}{\Gamma(h_i + r_i/2)} dx_i = (1/2)^{-(h_i + r_i/2)} \quad \text{for} \quad i = 1, \ldots, n,$$

and that

$$\int_0^\infty \frac{e^{-x_i(l_i t + 1/2)} x_i^{h_i + r_i/2 - 1}}{\Gamma(h_i + r_i/2)} dx_i = (l_i t + 1/2)^{-(h_i + r_i/2)} \qquad \text{for} \quad i = n+1, \ldots, s,$$

one has

$$E(Z^h) = \sum \rho \int_0^\infty t^{h-1} \prod_{i=n+1}^{s} \left(l_i t + \frac{1}{2} \right)^{-(h_i + r_i/2)} \prod_{i=s+1}^{r} \left(n_i t + \frac{1}{2} \right)^{-r_i/2} dt \tag{2.3}$$

where

$$\rho = \mu \prod_{i=1}^{n} m_i^{h_i} \Gamma(h_i + r_i/2) 2^{h_i} [\Gamma(r_i/2)]^{-1} \times \prod_{i=n+1}^{s} m_i^{h_i} \Gamma(h_i + r_i/2) 2^{-r_i/2} [\Gamma(r_i/2)]^{-1} \times \prod_{i=s+1}^{r} 2^{-r_i/2}. \tag{2.4}$$

The integral in (2.3) is evaluated using the following lemma.

LEMMA 2.1. *Let* $\eta = -h + \sum_r^q k_i + \sum_{q+1}^p a_i$, *then*

$$\int_0^\infty t^{h-1} \prod_{i=q+1}^{p} (1+c_i t)^{-a_i} \prod_{i=r}^{q} (\theta_i t + 1/b_i)^{-k_i} dt = \prod_{i=r}^{q} \theta_i^{-k_i} \prod_{i=q+1}^{p} c_i^{-a_i} [\Gamma(\eta)\Gamma(h)(\Gamma(\eta+h))^{-1}] \times \sum_{j=0}^{\infty} \sum_{j_r+\cdots+j_p=j} \frac{(\eta)_j}{(\eta+h)_j \prod_{i=r}^{p} j_i!} \times \prod_{i=q+1}^{p} (a_i)_{j_i} \prod_{i=r}^{q} (k_i)_{j_i} \prod_{i=q+1}^{p} (1-1/c_i)^{j_i} \prod_{i=r}^{q} (1-1/(\theta_i b_i))^{j_i} \tag{2.5}$$

provided

(2.6) $|1 - 1/(\theta_i b_i)| < 1$ *for* $i = r, \ldots, q$; $|1 - 1/c_i| < 1$ *for* $i = q+1, \ldots, p$; $R(\eta) > 0$ *and* $R(h) > 0$.

PROOF. The right-hand side of (2.5) can be expressed as

$$C \int_0^\infty t^{h-1} \prod_{i=q+1}^{p} (t + 1/c_i)^{-\alpha_i} \prod_{i=r}^{q} (t + 1/(\theta_i b_i))^{-k_i} dt = I_2 \qquad \text{(say)},$$

where $C = \prod_{i=r}^{q} \theta_i^{-k_i} \prod_{i=q+1}^{p} c_i^{-a_i}$.

Letting $u = 1/(1+t)$, i.e., $t = (1-u)/u$ so that $|dt/du| = u^{-2}$,

$$I_2 = C \int_0^1 \left(\frac{1-u}{u}\right)^{h-1} \prod_{i=q+1}^{p} \left(\frac{(1-u)}{u} + \frac{1}{c_i}\right)^{-a_i} \times \prod_{i=r}^{q} \left(\frac{(1-u)}{u} + \frac{1}{\theta_i b_i}\right)^{-k_i} u^{-2} du$$

$$= C \int_0^1 u^{\eta-1} (1-u)^{h-1} \prod_{i=q+1}^{p} (1 - u(1-1/c_i))^{-a_i} \times \prod_{i=r}^{q} (1 - u(1 - 1/(\theta_i b_i)))^{-k_i} du$$

where $\eta = -h + \sum_{i=q+1}^{p} a_i + \sum_{i=r}^{q} k_i$. Therefore

$$I_2 = C\frac{\Gamma(\eta)\Gamma(h)}{\Gamma(\eta+h)} F_D(\eta; a_{q+1}, \dots, a_p, k_r, \dots, k_q; \eta+h; (1-1/c_{q+1}), \dots, (1-1/c_p), (1-1/(\theta_r b_r)), \dots, (1-1/(\theta_q b_q)))$$

provided $R(\eta) > 0$ and $R(h) > 0$ where $F_D(\cdot)$ is Lauricella's type D hypergeometric function (see Mathai and Saxena (1978), p. 162 for a series representation and p. 163 for an integral representation) whose series representation leads to (2.5) provided the conditions specified in (2.6) are satisfied.

The h-th moment given in (2.3) may now be expressed in terms of series as follows:

$$E(Z^h) = \sum_{h_1+\cdots+h_s=h} \rho B(\eta, \alpha-\eta) \sum_{\nu=0}^{\infty} \sum_{\nu} [(\eta)_\nu/(\alpha)_\nu] \times \prod_{j=n+1}^{s} \gamma_j^{\nu_j} \frac{(h_j + r_j/2)_{\nu_j}}{\nu_j!} l_i^{-(h_i+r_i/2)} \times \prod_{j=s+1}^{r} \gamma_j^{\nu_j} \frac{(r_j/2)_{\nu_j}}{\nu_j!} n_i^{-(r_i/2)}, \tag{2.7}$$

where the last sum $\sum_\nu$ is over the nonnegative ν_j's such that $\sum_{j=n+1}^{r} \nu_j = \nu$, $(\epsilon)_\nu = \Gamma(\epsilon+\nu)/\Gamma(\epsilon)$, and ρ is given in (2.4); $\gamma_i = 1 - 1/(2l_i)$, $i = n+1, \dots, s$, $\gamma_i = 1 - 1/2n_i$, $i = s+1, \dots, r$, $\alpha = \sum_{n+1}^{s}(h_i + r_i/2) + \sum_{s+1}^{r} r_i/2$, and $\alpha - h = \eta$.

If the conditions specified in (2.6) are not satisfied, both the numerator and the denominator of Z^h may be multiplied by a scalar quantity b^h chosen so that $2bl_i > \frac{1}{2}$ for $i = 1, \dots, s$, and $2bn_i > \frac{1}{2}$ for $i = s+1, \dots, r$. Then the h-th moment of Z can be evaluated using (2.7) as the conditions given in (2.6) are satisfied.

After evaluating the first four moments of Z, one can select the Pearson curve which best approximates the exact density of Z. For a complete development of the curves and the associated rules, see for example Elderton and Johnson (1969).

It is worth noting that the representation (2.7) still holds whether the coefficients m_i are positive or negative since the expansion of the numerator of Z given in (2.1) applies to sums of positive and negative numbers. However, in view of (2.2), none of the terms in the denominator of Z may be negative.

3. The exact density of Z

First, an expression for the h-th moment of Z, where h is a complex number, is derived for the case $n = 1$, $s = 2$ and $r = 3$. A representation of the exact density of Z is then obtained as the inverse Mellin transform of the h-th moment for this particular case. It is finally shown that the same technique also applies to the general case.

Let

$$Z = \frac{m_1 X_1 + m_2 X_2}{l X_2 + n X_3} \tag{3.1}$$

where

$$X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}, \qquad i = 1, 2, 3.$$

Then the h-th moment of Z where h is a complex number, is

$$E(Z^h) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{(m_1x_1 + m_2x_2)^h}{(lx_2 + nx_3)^h} \prod_{i=1}^{3} \left(\frac{x_i^{r_i/2-1}}{2^{r_i/2}} \frac{e^{-x_i/2}}{\Gamma(r_i/2)} dx_i \right). \tag{3.2}$$

Letting

$$(lx_2 + nx_3)^{-h} = \int_0^\infty \frac{t^{h-1} e^{-(lx_2+nx_3)t}}{\Gamma(h)} dt, \tag{3.3}$$

one can integrate out x_3 as follows:

$$\int_0^\infty \frac{x_3^{r_3/2-1} e^{-x_3(nt+1/2)}}{2^{r_3/2}\Gamma(r_3/2)} dx_3 = \frac{\Gamma(r_3/2)\left(nt + \dfrac{1}{2}\right)^{-r_3/2}}{\Gamma(r_3/2)2^{r_3/2}} = (2nt+1)^{-r_3/2}. \tag{3.4}$$

Then, letting $\psi = m_1x_1 + m_2x_2$ and $x_2 = x_2$, one has

$$\begin{aligned} &\int_0^\infty x_2^{r_2/2-1} e^{-(lt+1/2)x_2} \int_0^\infty (m_1x_1 + m_2x_2)^h x_1^{r_1/2-1} e^{-x_1/2} dx_1 dx_2 \\ &= \int_0^\infty x_2^{r_2/2-1} e^{-(lt+1/2)x_2} \int_{m_2x_2}^\infty \frac{1}{m_1} \psi^h \left(\frac{1}{m_1}(\psi - m_2x_2) \right)^{r_1/2-1} \\ &\quad \times e^{-(\psi - m_2x_2)/2m_1} d\psi dx_2 \\ &= I_1(t) \qquad \text{(say)}. \end{aligned} \tag{3.5}$$

Note that all the integrands in this paper are measurable functions. We may therefore change the order of integration by Fubini's theorem as was done for the integral with respect to t. In order to express

$$E(Z^h) = \int_0^\infty \frac{t^{h-1}(2nt+1)^{-r_3/2}}{\Gamma(h)\Gamma(r_1/2)\Gamma(r_2/2)2^{r_1/2+r_2/2}} I_1(t) dt$$

in terms of series, one needs the following lemma which can be proved using identity 3.383,4. of Gradshteyn and Ryzhik (1980).

LEMMA 3.1.

$$\begin{aligned} &\int_u^\infty x^{\nu-1}(x-u)^{\mu-1} e^{-\beta x} dx \\ &= \sum_{r=0}^\infty \frac{(1-\nu)_r}{(2-\mu-\nu)_r} \cdot \frac{1}{r!} \Gamma(\mu+\nu-1) \beta^{r+1-\mu-\nu} u^r e^{-\beta u} \\ &\quad + \sum_{r=0}^\infty \frac{(\mu)_r}{(\mu+\nu)_r} \cdot \frac{1}{r!} B(\mu, 1-\mu-\nu) \beta^r u^{\mu+\nu+r-1} e^{-\beta u} \end{aligned} \tag{3.6}$$

where $B(a,b) = (\Gamma(a)\Gamma(b))/\Gamma(a+b)$ *and* $(a)_b = \Gamma(a+b)/\Gamma(a)$.

Using Lemma 3.1, $I_1(t)$ becomes

$$
\begin{aligned}
(3.7)\quad & \int_0^\infty x_2^{r_2/2-1} e^{-(lt+1/2)x_2} \left(\frac{1}{m_1}\right)^{r_1/2} \Gamma(r_1/2+h)(2m_1)^{r_1/2+h} \\
&\quad \times \left(\sum_{r=0}^\infty \frac{(-h)_r}{(1-r_1/2-h)_r} \cdot \frac{1}{r!}\left(\frac{m_2x_2}{2m_1}\right)^r\right) dx_2 \\
&\quad + \int_0^\infty x_2^{r_2/2+r_1/2+h-1} e^{-(lt+1/2)x_2} \left(\frac{1}{m_1}\right)^{r_1/2} \\
&\quad \times B(r_1/2, -(r_1/2+h)) \\
&\quad \times \left(\sum_{r=0}^\infty \frac{(r_1/2)_r}{(r_1/2+h+1)_r} \cdot \frac{1}{r!}\left(\frac{m_2x_2}{2m_1}\right)^r\right) dx_2 \\
&= \left\{\sum_{r=0}^\infty \frac{(-h)_r}{(1-r_1/2-h)_r}\right. \\
&\quad \times \frac{1}{r!}\Gamma(r_1/2+h)(2m_1)^{r_1/2+h-r}(m_2)^r \left(\frac{1}{m_1}\right)^{r_1/2} \\
&\quad \left. \times \Gamma(r_2/2+r)\left(lt+\frac{1}{2}\right)^{-(r_2/2+r)}\right\} \\
&\quad + \left\{\sum_{r=0}^\infty \frac{(r_1/2)_r}{(r_1/2+h+1)_r}\right. \\
&\qquad \times \frac{1}{r!}B(r_1/2, -(r_1/2+h))(2m_1)^{-r} m_1^{-r_1/2} m_2^{r_1/2+h+r} \\
&\qquad \left. \times \Gamma(r_1/2+r_2/2+h+r)\left(lt+\frac{1}{2}\right)^{-(r_1/2+r_2/2+h+r)}\right\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(3.8)\quad E(Z^h) = & \sum_{r=0}^\infty \frac{(-h)_r}{(1-r_1/2-h)_r} \\
& \times \frac{\Gamma(r_1/2+h)(2m_1)^{r_1/2+h-r} m_2^r \Gamma(r_2/2+r)}{r! 2^{r_2/2+r_1/2} m_1^{r_1/2}\Gamma(r_1/2)\Gamma(r_2/2)\Gamma(h)} \\
& \times \int_0^\infty t^{h-1}(1+2nt)^{-r_3/2}\left(lt+\frac{1}{2}\right)^{-(r_2/2+r)} dt \\
& + \sum_{r=0}^\infty \frac{\Gamma(r_1/2)\Gamma(-(r_1/2+h))}{\Gamma(-h)} \\
& \times \frac{(r_1/2)_r (2m_1)^{-r} m_2^{r_1/2+h+r}\Gamma(r_1/2+r_2/2+h+r)}{(r_1/2+h+1)_r r! 2^{r_2/2+r_1/2} m_1^{r_1/2}\Gamma(r_1/2)\Gamma(r_2/2)\Gamma(h)}
\end{aligned}
$$

$$\times \int_0^\infty t^{h-1}(1+2nt)^{-r_3/2}\left(lt+\frac{1}{2}\right)^{-(r_1/2+r_2/2+h+r)} dt.$$

Using Lemma 2.1, we have

$$\text{(3.9)} \quad E(Z^h) = \left\{ \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j} K_1 \frac{\Gamma(r-h)\Gamma(1-\frac{r_1}{2}-h)\Gamma(\frac{r_1}{2}+h)\Gamma(j+\frac{r_2}{2}+\frac{r_3}{2}+r-h)}{(2m_1)^{-h}\Gamma(-h)\Gamma(r+1-\frac{r_1}{2}-h)} \right\}$$
$$+ \left\{ \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+n=j} K_2 \frac{m_2^h}{l^h} \frac{\Gamma(-\frac{r_1}{2}-h)\Gamma(\frac{r_1}{2}+1+h)}{\Gamma(-h)\Gamma(r+\frac{r_1}{2}+1+h)} \times \frac{\Gamma(\lambda+\frac{r_1}{2}+\frac{r_2}{2}+r+h)}{\Gamma(j+\frac{r_1}{2}+\frac{r_2}{2}+\frac{r_3}{2}+r+h)} \right\}$$

where

$$K_1 = \frac{\Gamma(r_3/2+u)\Gamma(r_2/2+r+\lambda)m_2^r(1-1/(2n))^u(1-1/(2l))^\lambda}{r!2^{r_1/2+r_2/2}\Gamma(r_1/2)\Gamma(r_2/2)\Gamma(r_3/2)m_1^{r_1/2}(2m_1)^{-r_1/2+r}l^{r_2/2+r}(2n)^{r_3/2}} \times \frac{1}{\Gamma(r_2/2+r_3/2+r+j)\lambda!u!}$$

and

$$K_2 = \frac{\Gamma(r_1/2+r)\Gamma(r_3/2+u)\Gamma(r_1/2+r_2/2+r_3/2+r+j)m_2^{r_1/2+r}}{r!2^{r_1/2+r_2/2}(2m_1)^r m_1^{r_1/2}\Gamma(r_1/2)\Gamma(r_2/2)\Gamma(r_3/2)} \times \frac{(1-1/(2n))^u(1-1/(2l))^\lambda}{l^{r_1/2+r_2/2+r}(2n)^{r_3/2}\lambda!u!}.$$

Note that K_1 and K_2 do not involve h.

The inverse Mellin transform of the moment expression in (3.9) yields the density of Z:

$$\text{(3.10)} \quad f(z) = \frac{1}{2\pi i}\int_C E(Z^h)z^{-(h+1)}dh$$
$$= \frac{1}{2\pi i}z^{-1}\int_C E(Z^{-s})z^s ds \qquad (\text{letting } s=-h)$$
$$= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j}\left\{ z^{-1}\left[\frac{K_1}{2\pi i}\int_C \Gamma(r+s)\Gamma(1-r_1/2+s) \times \frac{\Gamma\left(j+\frac{r_2}{2}+\frac{r_3}{2}+r+s\right)\Gamma\left(\frac{r_1}{2}-s\right)\left(\frac{z}{2m_1}\right)^s}{\Gamma(s)\Gamma\left(r+1-\frac{r_1}{2}+s\right)}ds \right.\right.$$
$$+ \frac{K_2}{2\pi i}\int_C \Gamma(-r_1/2+s)$$

$$\times \frac{\Gamma\left(\frac{r_1}{2}+1-s\right)\Gamma\left(\lambda+\frac{r_1}{2}+\frac{r_2}{2}+r-s\right)\left(\frac{lz}{m_2}\right)^s}{\Gamma\left(r+\frac{r_1}{2}+1-s\right)\Gamma\left(j+\frac{r_1}{2}+\frac{r_2}{2}+\frac{r_3}{2}+r-s\right)\Gamma(s)}ds\Bigg]\Bigg\}$$

where C is a suitable contour in the complex plane (for a definition of the Mellin transform and its inverse, see for instance Springer (1979) or Mathai (1992)).

The density given in (3.10) can be expressed in terms of the H-function which is defined below as an inverse Mellin transform:

$$H_{pq}^{mn}\left[x\,\middle|\,\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right]=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}M(s)x^{-s}ds \tag{3.11}$$

where

$$M(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)},$$

m, n, p, q are nonnegative integers such that $0\le n\le p$, $1\le m\le q$; A_j, $j=1,\ldots,p$, B_j, $j=1,\ldots,q$ are positive numbers, a_j, $j=1,\ldots,p$, b_j, $j=1,\ldots,q$ are complex numbers such that

$$-A_j(b_h+v)\neq B_h(1-a_j+\lambda) \tag{3.12}$$

for $v,\lambda=0,1,2,\ldots$; $h=1,\ldots,m$; $j=1,\ldots,n$ and $(c-i\infty,c+i\infty)$ is the Bromwich path that separates the poles of $\Gamma(b_j+B_js)$, $j=1,\ldots,m$, from the poles of $\Gamma(1-a_j-A_js)$, $j=1,\ldots,n$. Hence one must have that

$$\max_{1\le j\le m}\mathrm{Re}\{-b_j/B_j\}<c<\min_{1\le j\le n}\mathrm{Re}\{(1-a_j)/A_j\}. \tag{3.13}$$

When $A_j=B_h=1$ for $j=1,\ldots,p$ and $h=1,\ldots,q$, the H-function reduces to Meijer's G-function. For a definition of the H-function, a description of the conditions for its existence, and series representations, the reader is referred to Mathai and Saxena (1978).

In view of (3.13), it is seen that the poles of the first integrand in (3.10) are separated for any number c satisfying

$$-r_1/2<c<1-r_1/2$$

and the poles of the second integrand are separated for any number c_1 satisfying

$$-1-r_1/2<c_1<-r_1/2.$$

Taking C as the left Bromwich contour corresponding to the path $(c_1-i\infty,c_1+\infty)$, the right-hand side of (3.10) may be expressed in terms of H-functions as follows:

$$\begin{aligned}f(z)=z^{-1}\Bigg\{&\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j}\\&\left(K_1\left(H_{33}^{13}\left[\frac{z}{2m_1}\,\middle|\,\begin{matrix}(1-r,1),(r_1/2,1),(1-j-(r_2+r_3)/2-r,1)\\(r_1/2,1),(1,1),(r_1/2-r,1)\end{matrix}\right]\right.\right.\\&\left.\left.-\mathrm{Res}\left(-\frac{r_1}{2}\right)\right)+K_2\cdot H_{33}^{31}\right.\\&\left.\times\left[\frac{lz}{m_2}\,\middle|\,\begin{matrix}(1+r_1/2,1),(r+r_1/2+1,1),(j+(r_1+r_2+r_3)/2+r,1)\\(r_1/2+1,1),(\lambda+r_1/2+r_2/2+r,1),(1,1)\end{matrix}\right]\right)\Bigg\}\end{aligned} \tag{3.14}$$

where $\mathrm{Res}(-\frac{r_1}{2})$ is the residue of the first integrand in (3.10) at the pole $(-\frac{r_1}{2})$. The residue of a function $g(\cdot)$ at the simple pole x_0 is defined as follows:

$$\mathrm{Res}_g(x_0) = \lim_{x \to x_0} (x - x_0) g(x).$$

Note that in view of condition (3.13), the first integral in (3.10) cannot be expressed as an H-function when the Bromwich path, $(c_1 - i\infty, c_1 + \infty)$, is used. Hence, one must use the Bromwich path, $(c - i\infty, c + \infty)$, in order to obtain a representation of this first integral in terms of an H-function "which is equal to the sum of the residues of the integrand at the poles located to the left of the Bromwich path, $(c - i\infty, c + \infty)$" and then subtract the additional residue evaluated at the only one of these poles located to the right of the Bromwich path, $(c_1 - i\infty, c_1 + \infty)$, initially chosen to define the contour C.

The two H-functions in (3.14) exist for $0 < |\frac{z}{2m_1}| < 1$ and $0 < |\frac{lz}{m_2}| < 1$, respectively; the following relationship

$$H_{pq}^{mn}\left[x \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right] = H_{qp}^{nm}\left[\frac{1}{x} \,\middle|\, \begin{matrix}(-b_q, B_q)\\(1 - a_p, A_p)\end{matrix}\right]$$

allows us to compute them when the arguments are greater than one in absolute value.

The technique described above may also be used for determining the exact density of Z in its general form:

$$Z = \frac{\sum_1^n m_i X_i + \sum_{n+1}^s m_i X_i}{\sum_{n+1}^s l_i X_i + \sum_{s+1}^r n_i X_i}.$$

The h-th moment of Z is

$$E(Z^h) = \int_0^\infty \cdots \int_0^\infty z^h \prod_{i=1}^r f(x_i) dx_i. \tag{3.15}$$

Making use of the following identities

$$\left(\sum_{n+1}^s l_i x_i + \sum_{s+1}^r n_i x_i\right)^{-h} = \int_0^\infty \frac{t^{h-1}}{\Gamma(h)} \cdot e^{-(\sum_{n+1}^s l_i x_i + \sum_{s+1}^r n_i x_i)t} dt \tag{3.16}$$

and

$$\prod_{i=s+1}^r \int_0^\infty \frac{x_i^{r_i/2-1} e^{-x_i(t+1/2)}}{2^{r_i/2}\Gamma(r_i/2)} dx_i = \prod_{i=s+1}^r (2t+1)^{-r_i/2}, \tag{3.17}$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=n+1}^s x_i^{r_i/2-1} e^{-(l_i t + 1/2)x_i}$$

$$\times \int_0^\infty \cdots \int_0^\infty \left(\sum_1^s m_i x_i\right)^h \prod_{i=1}^n (x_i^{r_i/2-1} e^{-x_i/2}) dx_1 dx_2 \cdots dx_s$$

can be written as

$$\begin{aligned}(3.18)\qquad &\int_0^\infty \psi_s^{r_s/2-1} e^{-\psi_s(b_s-b_{s-1})}\\ &\times \int_{\psi_s}^\infty \cdots \int_{\psi_3}^\infty \prod_{i=2}^{s-1}((\psi_i-\psi_{i+1})^{r_i/2-1} e^{-\psi_i(b_i-b_{i-1})})\\ &\times \int_{\psi_2}^\infty \psi_1^h(\psi_1-\psi_2)^{r_1/2-1} e^{-\psi_1 b_1} d\psi_1 d\psi_2\cdots d\psi_s,\end{aligned}$$

letting $\psi_i = m_i x_i + \cdots + m_s x_s$, $i = 1,\ldots,s$, i.e., $m_i x_i = \psi_i - \psi_{i+1}$, $i = 1,\ldots,s-1$ and $m_s x_s = \psi_s$, and $b_i = (1/2 + \xi_i t)$ where $\xi_i = l_i$ for $i = n+1,\ldots,s$ and $\xi_i = 0$ for $i = 1,\ldots,n$.

Each integral in (3.18) can be evaluated successively using Lemma 3.1 except the last one (over ψ_s) whose integrand is seen to be proportional to a gamma probability density function, as was the case in (3.7).

The sums thus obtained will be of the type of those appearing in (3.8) except that the integral over t will be of the general type given in Lemma 2.1. With the appropriate substitutions, the resulting expression will involve multiple sums of the type appearing in (3.9). Hence, the only functions involving h will be gamma functions and constants raised to the power h.

The exact density of Z can therefore be expressed in terms of sums of G-functions by taking the inverse Mellin transform of $E(Z^h)$ as was done in (3.10) and (3.14). Then it can be evaluated using the series representations of the G-functions as illustrated in the second example presented in the next section.

4. Examples

First, a theoretical example shows that the inverse Mellin transform technique yields the exact density of a ratio whose density can be obtained by means of the transformation of variables technique. In the second example, the exact distribution function is evaluated at various points covering the range of the variable and then compared with values obtained by simulation.

In the first example, the following ratio of independently distributed chi-square variables is considered:

$$(4.1)\qquad Z = \frac{\frac{1}{2}X_1 + X_2}{X_3}$$

where $X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}$, $r_1 = 4$, $r_2 = 2$ and $r_3 = 6$.

Using the approach described in Section 3, the h-th moment of Z is found to be equal to

$$\begin{aligned}E(Z^h) = 2\Gamma(-h+3)\Gamma(h+1) - \Gamma(h+2)\Gamma(-h+3)\left(\frac{1}{2}\right)^h\\ - 2\Gamma(h+1)\Gamma(-h+3)\left(\frac{1}{2}\right)^h.\end{aligned}$$

The inverse Mellin transform of the moment expression gives the following representation of the density of Z:

$$f(z) = 2G_{11}^{11}\left[z \,\middle|\, \begin{matrix} -3 \\ 0 \end{matrix}\right] - G_{11}^{11}\left[2z \,\middle|\, \begin{matrix} -3 \\ 1 \end{matrix}\right] - 2G_{11}^{11}\left[2z \,\middle|\, \begin{matrix} -3 \\ 0 \end{matrix}\right]. \tag{4.2}$$

The first G-function can be expressed as the following series

$$G_{11}^{11}\left[z \,\middle|\, \begin{matrix} -3 \\ 0 \end{matrix}\right] = \sum_{k=0}^{\infty} \Gamma(4+k)\frac{(-1)^k}{k!}z^k = \frac{6}{(1+z)^4},$$

for $0 < z < 1$, by considering the residues at the poles $0, -1, -2, \ldots,$ and

$$G_{11}^{11}\left[z \,\middle|\, \begin{matrix} -3 \\ 0 \end{matrix}\right] = \sum_{k=0}^{\infty} \Gamma(4+k)\frac{(-1)^k}{k!}z^{-k-4} = \frac{6}{(1+z)^4}$$

for $z > 1$, by considering the residues at the poles at $4, 5, 6, \ldots$. The infinite series are simplified by applying the general binomial theorem. The following identities are obtained similarly:

$$\begin{aligned} G_{11}^{11}\left[2z \,\middle|\, \begin{matrix} -3 \\ 1 \end{matrix}\right] &= \frac{48z}{(1+2z)^5}, \quad \text{for} \quad z > 0, \\ G_{11}^{11}\left[2z \,\middle|\, \begin{matrix} -3 \\ 0 \end{matrix}\right] &= \frac{6}{(1+2z)^4}, \quad \text{for} \quad z > 0. \end{aligned} \tag{4.3}$$

This yields

$$\begin{aligned} f(z) &= 2\frac{6}{(1+z)^4} - \frac{48z}{(1+2z)^5} - 2\frac{6}{(1+2z)^4} \\ &= \frac{312z^5 + 660z^4 + 480z^3 + 120z^2}{(2z+1)^5(z+1)^4} \quad \text{for} \quad z > 0, \end{aligned} \tag{4.4}$$

which is the expression obtained with the transformation of variables technique.

The next example involves a ratio of linear combinations of chi-square variables whose numerator and denominator are not independently distributed. Let

$$Z = \frac{X_1 + \frac{1}{2}X_2}{\frac{1}{2}X_2 + X_3} \tag{4.5}$$

where $X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}$, $r_1 = 4$, $r_2 = 1$ and $r_3 = 6$. We multiply both the numerator and denominator by the parameter θ in order to accelerate the convergence of the series expression representing the density function. The h-th moment of Z calculated from (3.9) is

$$\begin{aligned} E(Z^h) = &\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j} K_1(2\theta)^h \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(3.5+j+r-h)}{\Gamma(-h)\Gamma(r-1-h)} \\ &+ \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j} K_2 \frac{\Gamma(-2-h)\Gamma(3+h)\Gamma(2.5+j-u+r+h)}{\Gamma(-h)\Gamma(r+3+h)\Gamma(5.5+j+r+h)} \end{aligned}$$

where

$$K_1 = \frac{(u+1)(u+2)\Gamma(0.5+r+j-u)\left(1-\frac{1}{2\theta}\right)^u\left(1-\frac{1}{\theta}\right)^{j-u}}{r!(2\theta)^{r+3}2\theta^{0.5}\Gamma(0.5)\Gamma(3.5+r+j)(j-u)!}$$

and

$$K_2 = \frac{(1+r)(u+1)(u+2)\Gamma(5.5+r+j)\left(1-\frac{1}{2\theta}\right)^u\left(1-\frac{1}{\theta}\right)^{j-u}}{(2\theta)^{r+5}2\theta^{0.5}\Gamma(0.5)(j-u)!}.$$

The density function of Z obtained from (3.14) is

$$(4.6)\qquad f(z) = z^{-1}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j}\left\{K_1\cdot G_{33}^{13}\left[\frac{z}{2\theta}\,\middle|\,\begin{matrix}1-r,2,-2.5-j-r\\2,1,2-r\end{matrix}\right]\right.$$
$$\left.-\operatorname{Res}(-2)+K_2\cdot G_{33}^{21}\left[z\,\middle|\,\begin{matrix}3,r+3,5.5+j+r\\3,2.5+j-u+r,1\end{matrix}\right]\right\}.$$

By applying the method of residues (see Springer (1979)), the G-functions can be expressed in terms of series as follow:

$$G_{33}^{13}\left[\frac{z}{2\theta}\,\middle|\,\begin{matrix}1-r,2,-2.5-j-r\\2,1,2-r\end{matrix}\right]$$
$$=\sum_{v=2}^{\infty}(r+v-1)\Gamma(3.5+v+j+r)\frac{(-1)^v}{(v-1)!}\left(\frac{z}{2\theta}\right)^v \qquad \text{for} \quad \frac{z}{2\theta}<1,$$

$$G_{33}^{13}\left[\frac{z}{2\theta}\,\middle|\,\begin{matrix}1-r,2,-2.5-j-r\\2,1,2-r\end{matrix}\right]$$
$$=\sum_{v=0}^{\infty}(4.5+v+j)\Gamma(4.5+v+j+r)\frac{(-1)^v}{v!}\left(\frac{z}{2\theta}\right)^{-(3.5+r+j+v)}$$
$$\text{for} \quad \frac{z}{2\theta}>1,$$

$$G_{33}^{21}\left[z\,\middle|\,\begin{matrix}3,r+3,5.5+j+r\\3,2.5+j-u+r,1\end{matrix}\right]$$
$$=\sum_{v=3}^{r+2}\frac{\Gamma(2.5-u+j+r-v)}{\Gamma(5.5+j+r-v)\Gamma(r-v+3)}\frac{(-1)^{v-1}}{(v-1)!}z^v$$
$$+\sum_{v=0}^{u+2}\frac{\Gamma(-2.5-r-j+u-v)}{\Gamma(0.5-j+u-v)\Gamma(u-v+3)}\frac{(-1)^v}{v!}z^{v+j+r-u+2.5}$$
$$\text{for} \quad z<1,$$

$$G_{33}^{21}\left[z\,\middle|\,\begin{matrix}3,r+3,5.5+j+r\\3,2.5+j-u+r,1\end{matrix}\right]$$
$$=\frac{\Gamma(0.5+j-u+r)}{\Gamma(1+r)\Gamma(3.5+j+r)}z^2-\frac{\Gamma(1.5+j-u+r)}{\Gamma(2+r)\Gamma(4.5+j+r)}z \qquad \text{for} \quad z>1$$

and

Table 1.

z	0.1	0.2	0.4	0.8	1.0	1.6	2.0	4.2	4.4
$f(z)$	0.501	0.843	0.944	0.570	0.412	0.168	0.091	0.009	0.004
$f^*(z)$	0.502	0.842	0.944	0.577	0.421	0.170	0.098	0.010	0.008

$$\begin{aligned}\mathrm{Res}(-2) &= \lim_{h\to -2}(h+2)\left(\frac{2\theta}{z}\right)^h \\ &\quad\times \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(3.5+j+r-h)}{\Gamma(-h)\Gamma(r-1-h)} \\ &= (r+1)\Gamma(5.5+j+r)\left(\frac{z}{2\theta}\right)^2.\end{aligned}$$

Upon substituting in (4.6) the series expansions of the G-functions for the intervals $(0,1)$, $(1,2\theta)$ and $(2\theta,\infty)$, one can evaluate the probability density function of Z. For comparison purposes the density function of Z was simulated using a sample size of 1,000,000; the simulated values are denoted by $f^*(z)$. The numerical results are given in Table 1.

Note that this accuracy has been achieved by summing only the first 25 terms in the series representations of the G-functions.

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