Ann. Inst. Statist. Math.
Vol. 46, No. 3, 537–555 (1994)

NONPARAMETRIC ESTIMATION OF COMPOUND DISTRIBUTIONS WITH APPLICATIONS IN INSURANCE

S. M. PITTS

Department of Statistical Science, University College London, Gower Street, London WC1E 6BT, U.K.

(Received May 6, 1993; revised September 7, 1993)

Abstract. A nonparametric estimator of the distribution function G of a random sum of independent identically distributed random variables, with distribution function F, is proposed in the case where the distribution of the number of summands is known and a random sample from F is available. This estimator is found by evaluating the functional that maps F onto G at the empirical distribution function based on the random sample. Strong consistency and asymptotic normality of the resulting estimator in a suitable function space are established using appropriate continuity and differentiability results for the functional. Bootstrap confidence bands are also obtained. Applications to the aggregate claims distribution function and to the probability of ruin in the Poisson risk model are presented.

Key words and phrases: Compound distributions, nonparametric estimation, aggregate claims, probability of ruin.

1. Introduction

Compound distributions arise as quantities of interest in insurance mathematics, as well as in other areas of applied probability. Before giving an illustration from insurance, we give a description of the general set-up and introduce notation to be used throughout. Let $\{X_i\}_{i=1}^{\infty}$ be independent and identically distributed random variables with distribution function F, defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$, and let N be a random variable, defined on the same probability space, independent of $\{X_i\}_{i=1}^{\infty}$, and taking values in $\boldsymbol{N}_0 = \{0, 1, 2, \dots\}$. Let $p = (p_k)_{k \in \boldsymbol{N}_0}$ be given by $p_k = \boldsymbol{P}(N = k)$, so that p specifies the distribution of N. Write $S = \sum_{i=1}^{N} X_i$ when N is positive, and define S to be zero when N is zero. Then S, a random sum, has a compound distribution with distribution function

$$G(x) = \sum_{k=0}^{\infty} p_k F^{\star k}(x), \tag{1.1}$$

where for $k \geq 1$, $F^{\star k}(x) = \boldsymbol{P}(\sum_{i=1}^{k} X_i \leq x)$ is the k-fold convolution of F, and $F^{\star 0}(x) = I_{[0,\infty)}(x)$ (where $I_{[0,\infty)}$ is the indicator function of the set $[0, \infty)$).

Observe that we do not require F to be concentrated on $[0, \infty)$, although in many applications $F(0)$ is zero.

We give an example from insurance mathematics. Claims arrive at an insurance company, and the number of claims arriving in the period $(0, t]$ is a random variable $N(t)$. Let X_i be the size of the i-th claim. Assume that successive claim sizes are independent identically distributed random variables with distribution function F_1, and that the claim sizes are independent of $N(t)$. In this context one quantity of interest is the distribution of the *total claim amount* during $(0, t]$. Writing p_k for the probability that there are k claims in this period, we see that the total claim amount is a random variable having a compound distribution with distribution function G given by (1.1) with F equal to F_1.

In this paper we concentrate on the statistical problem of estimating G when F is unknown but p is known. We suppose that we have a random sample of observations on F. We establish asymptotic properties of a particular nonparametric estimator of G. We take a functional view of the stochastic model, as in Grübel (1989), and the estimator is defined in terms of the relevant functional as in Grübel and Pitts (1993). This means that we consider the functional that maps the input distribution function F onto the output distribution function G. By analogy with a procedure often followed in a parametric context, the estimator is constructed by evaluating the functional at a nonparametric estimator of the input distribution function based on the random sample from F. The particular input distribution function estimator considered here is the empirical distribution function. Statistical properties of the output estimator are then proved by combining statistical properties of the input estimator with analytical properties of the functional itself. The method by which the input properties are combined with the analytical properties of the functional to obtain strong consistency, asymptotic normality and asymptotic validity of bootstrap confidence bands is well known (see Gill (1989)). However application of these methods to a specific situation requires a detailed analysis of local properties of the particular functional in question. This we do in the present paper for the compound distribution functional.

The compound distribution function does not have an easy explicit form except for certain special cases for the input distribution function and for p, for example if F is exponential and p is geometric. Much attention in the literature has been concerned with approximations, as in, for example, Embrechts *et al.* (1985*a*) and von Chossy and Rappl (1983). In the insurance context, the tail of the compound distribution function is of interest and so several papers deal with the asymptotic behaviour of this quantity, see for example Embrechts *et al.* (1985*b*) and Willekens (1989). Csörgő and Teugels (1990) consider nonparametric estimation of an approximation to the asymptotic behaviour of the tail of a compound distribution function, i.e., to $1 - G(x)$ for fixed x, given a random sample of the claim sizes and assuming p is known. They obtain consistency and asymptotic normality of their estimators.

A second important example from insurance mathematics is the probability of ruin in the classical Poisson risk model. With notation as introduced for the total claim amount above, let $\rho = \lambda\mu/c$ where μ is $E(X_1)$, c is the premium rate (so that the premium income in $(0, t]$ is ct), and the claims are assumed to arrive in a Poisson process rate λ. We assume that the (relative) safety loading $\eta = 1/\rho - 1$ is positive so that the expected amount claimed per unit time is less than the premium income per unit time. The probability $R(x)$ of no eventual ruin, starting with initial capital $x > 0$, is given by the right hand side of (1.1) with $p_k = (1-\rho)\rho^k$, $k \in \boldsymbol{N}_0$, and $F(x) = (1/\mu)\int_0^x (1 - F_1(y))dy$ (see Embrechts and Veraverbeke (1982)), and is thus seen to fall into the general framework considered here. Nonparametric estimation of the probability of ruin in this classical risk model is studied in Croux and Veraverbeke (1990). They estimate $F^{\star k}(x)$ by a U-statistic and then calculate a finite sum approximation to (1.1), using ideas from Frees (1986*b*). Frees (1986*a*) proves consistency of finite and infinite time horizon ruin probability estimators obtained by sample re-use methods. Both Frees (1986*a*) and Croux and Veraverbeke (1990) estimate $1 - R(x)$ for a fixed value of x. Hipp (1989) takes a functional approach in discussing nonparametric estimators of quantities such as $1 - R(x)$, defined in the same way as the estimators in this paper, as the output of a functional when the input is the empirical distribution function. However we go beyond estimation of $1 - R(x)$ for a single x to consider nonparametric estimation of the curve $I_{[0,\infty)} - R$.

To this end, we estimate the compound distribution function as an element of a suitable function space and we regard the compound distribution functional as a map between function spaces. Section 2 gives definitions of the relevant spaces and collects various properties linking the topology and convolution. The choice of topology is closely related to the form of the bootstrap confidence region for the unknown compound distribution function. We aim to obtain non-constant width, rather than constant width, bootstrap confidence bands, since the former may well be more useful for the tails of distribution functions and interest in insurance often focuses on distribution tails. The weighted function spaces introduced in Section 2 provide one setting in which this aim can be achieved. These spaces are those considered in Grübel and Pitts (1993). In Section 3 we obtain a continuity result for the functional and combine this with an appropriate strong consistency result for the input estimator to derive a strong consistency result, Theorem 3.1, for the output estimator. In Section 4 we prove an asymptotic normality result, Theorem 4.1, for our estimator in terms of convergence in distribution to a Gaussian process, and in Theorem 4.2, we show that bootstrap (simultaneous) confidence bands give asymptotically correct coverage probabilities. Both these results depend on a differentiability result for the functional, proved in Proposition 4.1. In Section 5 we build on the results of earlier sections to obtain corresponding results for the probability of ruin in a classical Poisson risk model, and we give examples. We also discuss the case where p is unknown. The approach throughout this paper is that of Grübel and Pitts (1993).

2. Definitions

A more careful description of the compound distribution functional involves definition of the spaces forming its domain and codomain. We need these spaces to include distribution functions, and it is natural to consider the space D_∞ of all real-valued functions f on $[-\infty, \infty]$ that are right-continuous with left-hand limits, and left-continuous at $+\infty$. Let

$$\|f\|_\infty = \sup_{x\in[-\infty,\infty]} |f(x)|,$$

then $(D_\infty, \|\cdot\|_\infty)$ is a Banach space. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ with finite limits at $\pm\infty$ can be extended to an element of D_∞ by defining $f(+\infty) = \lim_{x\to\infty} f(x)$ and $f(-\infty) = \lim_{x\to-\infty} f(x)$. We write f for both the original and the extended function.

This space provides one setting for the continuity and differentiability of the functional. Such results are straightforward to prove and yield for our estimator strong $\|\cdot\|_\infty$-consistency and asymptotic normality in terms of convergence in distribution to a Gaussian process in D_∞. We would also obtain a theoretical justification for a constant width bootstrap confidence band. However, as noted above, we wish to explore the possibility of obtaining a non-constant width band through the use of weighted spaces of functions.

For β in $\boldsymbol{R}$ and a function f mapping $\boldsymbol{R}$ to $\boldsymbol{R}$, let

$$(T_\beta f)(x) = (1+|x|)^\beta f(x).$$

We define D_β to be the set of all functions $f : \boldsymbol{R} \to \boldsymbol{R}$ such that $T_\beta f$ is extendable to an element of D_∞. For f in D_β let $\|f\|_\beta = \|T_\beta f\|_\infty$. Then $(D_\beta, \|\cdot\|_\beta)$ is a (nonseparable) Banach space. Note that $D_\infty = D_0$. The requirement that $T_\beta f$ is extendable to an element of D_∞ means that $T_\beta f : \boldsymbol{R} \to \boldsymbol{R}$ is right-continuous with left-hand limits, with finite limits at $\pm\infty$. For such an f we have $\sup_x (1+|x|)^\beta |f(x)| < \infty$. On occasion we deal with functions f that, while not in D_β, still have $T_\beta f$ bounded. We write $\|f\|_\beta$ for $\sup_x |(T_\beta f)(x)|$ in this case. For a given distribution function F, let $C_\beta(F)$ be the set of all elements of D_β that are continuous at continuity points of F. We note that $C_\beta(F)$ is a separable subspace of D_β. Below we collect elementary properties of D_β-spaces for future reference.

It is easily checked that if F is a distribution function and $\beta \ge 0$ then

$$\|I_{[0,\infty)} - F\|_\beta \le \int (1+|x|)^\beta F(dx). \tag{2.1}$$

The following lemma relates convergence in D_β-spaces to convergence of moments, using Theorem 8.1.2 of Chow and Teicher (1988).

LEMMA 2.1. *Let $\alpha' > 0$ and $0 \le \alpha < \alpha'$. Let $\{F_n\}_{n\in\boldsymbol{N}}$ and F be distribution functions with $I_{[0,\infty)} - F_n$ and $I_{[0,\infty)} - F$ in $D_{\alpha'}$ for all n, and $\|F_n - F\|_{\alpha'} \to 0$ as $n \to \infty$. Then, as $n \to \infty$,*

$$\int |x|^\alpha F_n(dx) \to \int |x|^\alpha F(dx) \quad \text{and}$$
$$\int (1+|x|)^\alpha F_n(dx) \to \int (1+|x|)^\alpha F(dx).$$

We now consider when the output distribution function G satisfies $I_{[0,\infty)} - G$ is in D_β for $\beta > 0$. We begin by collecting results for the moments of $\sum_{i=1}^k X_i$

and for $\|I_{[0,\infty)} - F^{\star k}\|_\beta$. For random variables X and Y, and for $\beta > 0$, we have (see, for example, Grimmett and Stirzaker ((1982), Theorem 7.3.7))

$$E(|X+Y|^\beta) \le (2^{\beta-1} \vee 1)(E(|X|^\beta) + E(|Y|^\beta)), \tag{2.2}$$

where $x \vee y$ is the maximum of x and y. As an application of Minkowski's inequality for $\beta \ge 1$ and of (2.2) for $0 < \beta < 1$, we have, for $\{X_i\}$ independent and identically distributed with distribution function F and $\beta \ge 0$,

$$E\left(\left|\sum_{i=0}^{k} X_i\right|^\beta\right) \le k^{\beta\vee 1} E(|X_1|^\beta). \tag{2.3}$$

For future reference we note that, from (2.1), (2.2) and (2.3), for $\beta > 0$,

$$\|I_{[0,\infty)} - F^{\star k}\|_\beta \le (2^{\beta-1} \vee 1)\left(1 + k^{\beta\vee 1}\int |x|^\beta F(dx)\right). \tag{2.4}$$

We obtain from (2.3).

LEMMA 2.2. *Let $\beta > 0$. Assume that*

$$\int |x|^\beta F(dx) < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} k^{\beta\vee 1} p_k < \infty.$$

Then, for $G = \sum_{k=0}^\infty p_k F^{\star k}$, $\int |x|^\beta G(dx) < \infty$.

Writing Φ_p for the functional that maps a distribution function F onto $\sum_{k=0}^\infty p_k F^{\star k}$, we see that, under the conditions of Lemma 2.2, $I_{[0,\infty)} - \Phi_p(F)$ is in D_β.

For measurability purposes we give D_β its open ball (projection) σ-field. If $\{Y_n\}$ and Y are random elements of D_β then we say that Y_n converges in distribution to Y in D_β, and write $Y_n \to_d Y$ in D_β as $n \to \infty$, if $E(f(Y_n)) \to E(f(Y))$ as $n \to \infty$ for all bounded, continuous, measurable f that map D_β to $\boldsymbol{R}$. We write $X =_d Y$ if random elements X and Y of D_β have the same distribution. See Pollard ((1984), Chapter IV) for further details of these notions for convergence in distribution in nonseparable metric spaces.

The nonparametric estimator of the input distribution function is the empirical distribution function $\hat{F}_n$, based on a random sample $X_1, \ldots, X_n$ from F, where, for x in $\boldsymbol{R}$ and ω in Ω,

$$\hat{F}_n(x,\omega) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(X_i(\omega)).$$

Our estimator of the output compound distribution function is then

$$\hat{G}_n(x,\omega) = \sum_{k=0}^{\infty} p_k \hat{F}_n^{\star k}(x,\omega),$$

i.e., $\hat{G}_n = \Phi_p(\hat{F}_n)$. It is easy to check that $I_{[0,\infty)}(x) - \hat{G}_n(x,\cdot)$ is a random variable for each x, and so $I_{[0,\infty)} - \hat{G}_n$ is a random element of D_β.

Since our functional involves convolutions, and since we wish to prove continuity and differentiability results for this functional in D_β-spaces, we investigate the $\|\cdot\|_\beta$-norms of convolutions. Let H be a nonnegative and nondecreasing function from $\boldsymbol{R}$ to $\boldsymbol{R}$ so that the Lebesgue-Stieltjes measure ν_H can be defined by $\nu_H((a,b]) = H(b) - H(a)$, $a < b$. Suppose that f is a real-valued function on $\boldsymbol{R}$ such that $f(x-\cdot)$ is integrable with respect to ν_H for all x in $\boldsymbol{R}$. Then the convolution $f \star H$ is defined by $f \star H(x) = \int f(x-y)H(dy)$, x in $\boldsymbol{R}$.

LEMMA 2.3. *Let $\beta > 0$. Then*

$$\|f \star H\|_\beta \le 2^\beta \|f\|_\beta \left(\left\|I_{[0,\infty)}\|H\|_\infty - H\right\|_\beta + \|H\|_\infty\right).$$

PROOF. For x in $\boldsymbol{R}$,

$$\begin{aligned}(1+|x|)^\beta &|f \star H(x)| \\ &\le (1+|x|)^\beta \left(\int_{(-\infty,x/2]} |f(x-y)|H(dy) + \int_{(x/2,\infty)} |f(x-y)|H(dy)\right).\end{aligned}$$

For $x \ge 0$ this is at most

$$\begin{aligned}(1+|x|)^\beta &\left(\sup_{y \ge x/2} |f(y)|\|H\|_\infty + \|f\|_\infty(\|H\|_\infty - H(x/2))\right) \\ &\le 2^\beta \|f\|_\beta \left(\|H\|_\infty + \left\|I_{[0,\infty)}\|H\|_\infty - H\right\|_\beta\right).\end{aligned}$$

Using similar arguments for $x < 0$ we obtain the result. □

Observe that this lemma gives conditions for the existence of $f \star H(x)$. We shall also need the following lemma, which is proved using similar techniques to those employed in the proof of Lemma 2.3.

LEMMA 2.4. *Let $\beta > 0$. Let F and G be distribution functions with $I_{[0,\infty)} - F$ and $I_{[0,\infty)} - G$ in D_β. Then*

$$\|I_{[0,\infty)} - F \star G\|_\beta \le 2^\beta (\|I_{[0,\infty)} - F\|_\beta + \|I_{[0,\infty)} - G\|_\beta).$$

We shall need the following applications of these two lemmas for distribution functions $\{F_n\}_{n\in \boldsymbol{N}}$ and F with $I_{[0,\infty)} - F_n$ and $I_{[0,\infty)} - F$ in $D_{\beta'}$ and $\|F_n - F\|_{\beta'} \to 0$ as $n \to \infty$, where $\beta' > \beta \ge 0$. For k in $\boldsymbol{N}$, write $H_{n,k} = \sum_{i=0}^{k-1} F_n^{\star(k-1-i)} \star F^{\star i}$, and let f be in D_β. Write $c_1(\beta)$ for 2^β and $c_2(\beta)$ for $2^{\beta-1} \vee 1$. Then by Lemma 2.3

$$\begin{aligned}&\|f \star (F_n^{\star(k-1-i)} \star F^{\star i})\|_\beta \\ &\quad \le c_1(\beta)\|f\|_\beta(1 + \|I_{[0,\infty)} - F_n^{\star(k-1-i)} \star F^{\star i}\|_\beta),\end{aligned}$$

so that by (2.1) and Lemma 2.1, for all n large enough,

$$\|f \star H_{n,k}\|_\beta \le kc_1(\beta)\|f\|_\beta \left(1 + \left(2\int (1+|x|)^\beta F(dx)\right)^{k-1}\right). \tag{2.5}$$

From Lemma 2.4 we also have

$$\begin{aligned}&\|f \star (F_n^{\star(k-1-i)} \star F^{\star i})\|_\beta \\ &\quad \le c_1(\beta)\|f\|_\beta (1 + c_1(\beta)(\|I_{[0,\infty)} - F_n^{\star(k-1-i)}\|_\beta + \|I_{[0,\infty)} - F^{\star i}\|_\beta)),\end{aligned}$$

so that, by (2.4) and Lemma 2.1,

$$\begin{aligned}&(2.6)\ \|f \star H_{n,k}\|_\beta \\ &\quad \le kc_1(\beta)\|f\|_\beta \left(1 + 2c_1(\beta)c_2(\beta)\left(1 + 2(k-1)^{\beta\vee 1}\int |x|^\beta F(dx)\right)\right),\end{aligned}$$

for all n large enough.

3. Continuity and consistency

We are now in a position to prove the continuity of the functional Φ_p.

PROPOSITION 3.1. *Let $\beta' > \beta \ge 0$. Let $\{F_n\}_{n\in \boldsymbol{N}}$ and F be distribution functions with*

$$\int |x|^\beta F(dx) < \infty, \qquad \int |x|^\beta F_n(dx) < \infty \ \textit{for } n \in \boldsymbol{N} \qquad \textit{and}$$
$$\|F_n - F\|_{\beta'} \to 0 \quad \textit{as} \quad n \to \infty.$$

Let $p = (p_k)_{k\in \boldsymbol{N}_0}$ satisfy $\sum_{k=1}^\infty k^{\beta\vee 1} p_k < \infty$. Then

$$\|\Phi_p(F_n) - \Phi_p(F)\|_\beta \to 0 \quad \textit{as} \quad n \to \infty.$$

PROOF. By Lemma 2.2 $\Phi_p(F_n) - \Phi_p(F)$ is in D_β. For any M in $\boldsymbol{N}$, we have

$$\begin{aligned}\|\Phi_p(F_n) - \Phi_p(F)\|_\beta \le &\sum_{k>M} p_k\|I_{[0,\infty)} - F^{\star k}\|_\beta \\ &+ \sum_{k>M} p_k\|I_{[0,\infty)} - F_n^{\star k}\|_\beta \\ &+ \sum_{k=0}^{M} p_k\|F_n^{\star k} - F^{\star k}\|_\beta.\end{aligned} \tag{3.1}$$

By (2.4) the first term is at most

$$c_2(\beta)\left(\sum_{k>M} p_k + \int |x|^\beta F(dx) \sum_{k>M} k^{\beta\vee 1} p_k\right),$$

and the second term is at most

$$c_2(\beta)\left(\sum_{k>M} p_k + \int |x|^\beta F_n(dx) \sum_{k>M} k^{\beta\vee 1} p_k\right). \tag{3.2}$$

By Lemma 2.1, for n large enough, (3.2) is at most

$$c_2(\beta)\left(\sum_{k>M} p_k + 2\int |x|^\beta F(dx) \sum_{k>M} k^{\beta\vee 1} p_k\right),$$

and so, for n large enough, each of the first two terms can be made arbitrarily small by taking M large enough. For the third term on the right hand side of (3.1) we have $F_n^{\star k} - F^{\star k} = (F_n - F)\star H_{n,k}$ where $H_{n,k}$ is defined at the end of Section 2. By (2.5), for all n large enough, the third term is bounded by

$$c_1(\beta)\|F_n - F\|_\beta \sum_{k\le M} kp_k \left(1 + \left(2\int (1+|x|)^\beta F(dx)\right)^{k-1}\right),$$

and this is at most

$$c_1(\beta)\|F_n - F\|_\beta M^2 \left(1 + \left(2\int (1+|x|)^\beta F(dx)\right)^{M-1}\right).$$

This can be made arbitrarily small for fixed M by taking n large enough. □

To obtain a strong consistency result for our estimator, we combine this continuity result for the functional with a strong consistency result for the input estimator. This is given next.

LEMMA 3.1. *Let $\{X_i\}_{i\in N}$ be independent identically distributed random variables with distribution function F. Let $\hat{F}_n$ be the empirical distribution function based on $X_1,\dots,X_n$. Then, for $\alpha > 0$,*

$$E(|X_i|^\alpha) < \infty \Rightarrow \|\hat{F}_n - F\|_\alpha \to 0 \quad \text{as} \quad n\to\infty,$$

with probability one.

This is a weighted Glivenko-Cantelli result, and appears as Proposition 3.7 in Grübel and Pitts (1993). It is proved by rescaling by F the corresponding result (see Shorack and Wellner ((1986), 10.2)) for random variables uniformly distributed on $(0,1)$. Proposition 3.1 and Lemma 3.1 together give

THEOREM 3.1. *Let $\beta' > \beta \ge 0$. Let $\{X_i\}_{i\in N}$ be independent identically distributed random variables with distribution function F satisfying*

$$\int |x|^{\beta'} F(dx) < \infty.$$

Let $p = (p_k)_{k\in N_0}$ satisfy $\sum_{k=1}^\infty k^{\beta\vee 1} p_k < \infty$. Then with probability one

$$\|\Phi_p(\hat{F}_n) - \Phi_p(F)\|_\beta \to 0 \quad \text{as} \quad n\to\infty.$$

4. Differentiability and asymptotic normality

The derivative of the functional involves the quantity $H = \sum_{k=1}^{\infty} kp_k F^{\star(k-1)}$. Write m_p for $\sum_{k=1}^{\infty} kp_k$. The next lemma follows easily using (2.4).

LEMMA 4.1. *Let $\beta > 0$. Assume that $\sum_{k=1}^{\infty} k^{(1+\beta)\vee 2} p_k < \infty$ and $\int |x|^\beta F(dx) < \infty$. Then*

$$\|m_p I_{[0,\infty)} - H\|_\beta < \infty.$$

LEMMA 4.2. *Let $\alpha' > \alpha \geq 0$ and $\varepsilon > 0$. Suppose that g is in $D_{\alpha'}$. Then there exists g_l, a linear combination of indicator functions of the form $I_{[a,b)}$, $-\infty < a < b < \infty$, such that $\|g - g_l\|_\alpha < \varepsilon$.*

This is similar to Lemma 3.12 in Grübel and Pitts (1993).

LEMMA 4.3. *Let $\alpha > 0$ and let g_l be a linear combination of indicator functions of the form $I_{[a,b)}$, $-\infty < a < b < \infty$. Then there exists a constant $c(\alpha, g_l)$ such that, for distribution functions F and $\{F_n\}_{n\in \boldsymbol{N}}$ with $\|F_n - F\|_\alpha < \infty$ for all n,*

$$\|g_l \star F_n - g_l \star F\|_\alpha \leq c(\alpha, g_l)\|F_n - F\|_\alpha.$$

PROOF. We have

$$\begin{aligned}
&(1+|x|)^\alpha |I_{[a,b)} \star F_n(x) - I_{[a,b)} \star F(x)| \\
&\quad \leq (1+|x|)^\alpha (|F_n(x-a) - F(x-a)| + |F_n(x-b) - F(x-b)|) \\
&\quad \leq ((1+|a|)^\alpha + (1+|b|)^\alpha)\|F_n - F\|_\alpha.
\end{aligned}$$

The result now follows on using the triangle inequality. □

PROPOSITION 4.1. *Let $\beta' > \beta \geq 0$. Let F and $\{F_n\}_{n\in\boldsymbol{N}}$ be distribution functions with*

$$\int |x|^\beta F(dx) < \infty, \qquad \int |x|^\beta F_n(dx) < \infty \text{ for all } n \text{ in } \boldsymbol{N}, \qquad \text{and}$$
$$\sqrt{n}(F_n - F) \to g \quad \text{as} \quad n \to \infty \quad \text{in } D_{\beta'},$$

where $g \in D_{\beta'}$. Let $p = (p_k)_{k\in \boldsymbol{N}_0}$ satisfy $\sum_{k=1}^{\infty} k^{(1+\beta)\vee 2} p_k < \infty$. Then

$$\|\sqrt{n}(\Phi_p(F_n) - \Phi_p(F)) - g \star H\|_\beta \to 0 \quad \text{as} \quad n \to \infty.$$

PROOF. By Lemmas 2.3 and 4.1 we have that $\|g \star H\|_\beta < \infty$ and so

$$\|\sqrt{n}(\Phi_p(F_n) - \Phi_p(F)) - g \star H\|_\beta < \infty.$$

Observe that

$$
\begin{aligned}
(4.1)\qquad & \|\sqrt{n}(\Phi_p(F_n) - \Phi_p(F)) - g \star H\|_\beta \\
& \leq \left\| \sum_{k>M} p_k \sqrt{n}(F_n - F) \star H_{n,k} \right\|_\beta + \left\| g \star \sum_{k>M} k p_k F^{\star(k-1)} \right\|_\beta \\
& \quad + \left\| \sum_{k=1}^{M} p_k (\sqrt{n}(F_n - F) - g) \star H_{n,k} \right\|_\beta \\
& \quad + \left\| \sum_{k=1}^{M} p_k (g \star H_{n,k} - k g \star F^{\star(k-1)}) \right\|_\beta .
\end{aligned}
$$

For the first term on the right hand side of (4.1), by (2.6) we have

$$
\begin{aligned}
& \sum_{k>M} p_k \|\sqrt{n}(F_n - F) \star H_{n,k}\|_\beta \\
& \quad \leq c_1(\beta)(\|g\|_\beta + 1) \\
& \quad \cdot \sum_{k>M} k p_k \left(1 + 2c_1(\beta)c_2(\beta)\left(1 + 2(k-1)^{\beta \vee 1} \int |x|^\beta F(dx)\right)\right),
\end{aligned}
$$

for all n large enough, and this can be made arbitrarily small by taking M large enough.

By Lemma 2.3 and (2.4), the second term on the right hand side of (4.1) is at most

$$
c_1(\beta)\|g\|_\beta \sum_{k>M} \left(1 + c_2(\beta)\left(1 + (k-1)^{\beta \vee 1} \int |x|^\beta F(dx)\right)\right),
$$

and this can be made arbitrarily small by taking M large enough.

For fixed M, using (2.5), we have that the third term on the right hand side of (4.1) tends to zero as n tends to infinity.

For the fourth term, consider first

$$
g \star F_n^{\star(k-1-i)} \star F^{\star i} - g \star F^{\star(k-1)}.
$$

Write J_n for $F_n^{\star(k-1-i)} \star F^{\star i}$ and J for $F^{\star(k-1)}$. Using Lemma 4.2 we can find g_l, a linear combination of indicator functions of the form $I_{[a,b)}$, $-\infty < a < b < \infty$, with $\|g - g_l\|_\beta$ arbitrarily small. Then, by Lemmas 2.3 and 4.3, we have

$$
\begin{aligned}
(4.2)\qquad \|g \star J_n - g \star J\|_\beta \leq\ & c_1(\beta)\|g - g_l\|_\beta(\|I_{[0,\infty)} - J_n\|_\beta + 1) \\
& + c_1(\beta)\|g - g_l\|_\beta(\|I_{[0,\infty)} - J\|_\beta + 1) \\
& + c(\beta, g_l)\|J_n - J\|_\beta.
\end{aligned}
$$

Putting $H_{n,0}(x) = 0$ for all x, we see that

$$
\|J_n - J\|_\beta = \|(F_n - F) \star H_{n,k-1-i} \star F^{\star i}\|_\beta.
$$

Using Lemma 2.3, (2.4) and (2.5), we have

$$\begin{aligned}
&\|J_n - J\|_\beta \\
&\quad \le c_1^2(\beta)\|F_n - F\|_\beta (k-1-i)\left(1 + \left(2\int (1+|x|)^\beta F(dx)\right)^{k-2-i}\right) \\
&\qquad \times \left(1 + c_2(\beta)\left(1 + i^{\beta\vee 1}\int |x|^\beta F(dx)\right)\right).
\end{aligned}$$

Hence for $0 \le i < k \le M$, $\|J_n - J\|_\beta$ tends to zero as n tends to infinity. This implies that $\|I_{[0,\infty)} - J_n\|_\beta \le 2\|I_{[0,\infty)} - J\|_\beta$ for all n large enough. Thus we can make $\|g \star J_n - g \star J\|_\beta$ arbitrarily small by first choosing g_l close enough to g in D_β so that for all n large enough, the first two terms on the right hand side of (4.2) are small, and then taking n sufficiently large so that the third term is also small. This implies that, for fixed M, the fourth term on the right hand side of (4.1) can be made arbitrarily small by taking n large enough, and the assertion of the proposition then follows. □

Observe that $g \star H(x)$ tends to zero as x tends to $\pm\infty$. The above proposition gives an explicit form for the derivative $\Phi'_{p,F}$ of the functional Φ_p at F along certain curves of distribution functions; $\Phi'_{p,F}(g) = g \star H$.

The asymptotic normality of the input estimator is given in the following lemma. Let B be a standard Brownian bridge. We write $B \circ F$ for the process given by $B \circ F(x,\omega) = B(F(x),\omega)$. We call $B \circ F$ a Brownian bridge rescaled by F.

LEMMA 4.4. *Let $\gamma > 0$. Let $\{X_i\}_{i\in N}$ be independent identically distributed random variables with distribution function F, satisfying*

$$\int |x|^\gamma F(dx) < \infty.$$

Then for every β such that $0 \le \beta < \gamma/2$,

$$\sqrt{n}(\hat{F}_n - F) \to_d B \circ F \quad \text{as} \quad n \to \infty \quad \text{in } D_\beta.$$

This is obtained by rescaling by F the corresponding result (Shorack and Wellner ((1986), 3.7.1)) for random variables uniformly distributed on $(0,1)$, and appears as Proposition 3.8 in Grübel and Pitts (1993).

The last two results are combined using the delta method as explained in Gill (1989) to obtain asymptotic normality of our output estimator.

THEOREM 4.1. *Let $\beta' > \beta \ge 0$. Let $\{X_i\}_{i\in N}$ be independent identically distributed random variables with distribution function F satisfying*

$$\int |x|^{2\beta'} F(dx) < \infty.$$

Let $p = (p_k)_{k \in \boldsymbol{N}_0}$ *satisfy* $\sum_{k=1}^{\infty} k^{(1+\beta)\vee 2} p_k < \infty$. *Then*

$$\sqrt{n}(\Phi_p(\hat{F}_n) - \Phi_p(F)) \to_d Z \quad \text{as} \quad n \to \infty \quad \text{in } D_\beta,$$

where Z *is a Gaussian process with zero means and*

$$\operatorname{cov}(Z_s, Z_t) = \iint F[(s-x) \wedge (t-y)] H(dx) H(dy) - F \star H(s) F \star H(t).$$

PROOF. Choose α such that $\beta < \alpha < \beta'$. By Lemma 4.4,

$$\sqrt{n}(\hat{F}_n - F) \to_d B \circ F \quad \text{as} \quad n \to \infty \quad \text{in } D_\alpha.$$

Since $B \circ F$ concentrates on the separable subspace $C_\beta(F)$ of D_β, we can apply the Skorohod-Dudley-Wichura Theorem (see Shorack and Wellner ((1986), 2.3, Theorem 4)) there exists a probability space $(\Omega', \mathcal{F}', \boldsymbol{P}')$ with random elements $\{F_n'\}$ and B' defined on it, such that $F_n' =_d \hat{F}_n$, $B' =_d B$ and with $\boldsymbol{P}'$-probability one,

$$\sqrt{n}(F_n' - F) \to B' \circ F \quad \text{as} \quad n \to \infty \quad \text{in } D_\alpha. \tag{4.3}$$

Fix ω' in a set of $\boldsymbol{P}'$-probability one such that (4.3) holds. Write $g(x) = B'(F(x), \omega)$, so that it is in D_α. Applying Proposition 4.1 we have

$$\sqrt{n}(\Phi_p(F_n') - \Phi_p(F)) \to g \star H \quad \text{as} \quad n \to \infty \quad \text{in } D_\beta.$$

Write Z for the process obtained by applying the map that takes g to $g \star H$ to the sample paths of $B' \circ F$. This map is linear and bounded and so Z is Gaussian. Thus $\boldsymbol{P}'$-almost surely

$$\sqrt{n}(\Phi_p(F_n') - \Phi_p(F)) \to Z \quad \text{in } D_\beta,$$

which implies that

$$\sqrt{n}(\Phi_p(F_n') - \Phi_p(F)) \to_d Z \quad \text{in } D_\beta,$$

and this in turn implies the theorem since $\hat{F}_n =_d F_n'$, $B =_d B'$. It is easily checked that the process Z has zero means, and has the covariance structure shown in the statement of the theorem. □

To assess the quality of the estimator, we aim to give a confidence region for the unknown G. The development of such a confidence region follows that in Grübel and Pitts (1993).

For z in $\boldsymbol{R}$ let $R_n(z) = \boldsymbol{P}(\sqrt{n}\|\hat{G}_n - G\|_\beta \le z)$. If known it could be used to form a confidence region for G. Define, for z in $\boldsymbol{R}$, $R_Z(z) = \boldsymbol{P}(\|Z\|_\beta \le z)$ where Z is the limiting Gaussian process in Theorem 4.1. The continuous mapping theorem (see, for example, Pollard ((1984), IV.2.12)) implies that $R_n(z) \to R_Z(z)$ as $n \to \infty$ for all continuity points z of R_Z. Knowledge of the α-quantile $q(\alpha)$

of R_Z would lead to asymptotic confidence regions for G. However the structure of Z is complicated and it is not easy to see how $q(\alpha)$ can be obtained. In this situation we can consider bootstrap confidence regions constructed as follows. An estimator $\hat{R}_n$ of R_n is defined as the empirical distribution function for all n^n (not necessarily distinct) possible values of $\sqrt{n}\|\Phi_p(F_n^*) - \Phi_p(\hat{F}_n)\|_\beta$ where F_n^* runs through the empirical distribution functions obtained by taking samples of size n from $\hat{F}_n$. Let $\hat{q}_n(\alpha)$ be the α-quantile of $\hat{R}_n$. The next theorem shows that confidence regions constructed from $\hat{q}_n(\alpha)$ give asymptotically correct coverage probabilities.

THEOREM 4.2. *Let $\beta' > \beta \geq 0$. Let F be a distribution function satisfying $\int |x|^{2\beta'} F(dx) < \infty$. Then*

$$\boldsymbol{P}(\sqrt{n}\|\Phi_p(\hat{F}_n) - \Phi_p(F)\|_\beta \leq \hat{q}_n(\alpha)) \to \alpha \qquad as \qquad n \to \infty.$$

The proof of this theorem proceeds as the proof of Theorem 2.3 in Grübel and Pitts (1993), which is based on that in Gill (1989). It uses the differentiability of Φ_p to show that $\hat{R}_n \to_d R_Z$ in D_∞, and thus, loosely, we may use $\hat{R}_n$ as an approximation to R_n for n large enough. In practice, $\hat{R}_n$ and hence $\hat{q}_n(\alpha)$ are approximated by Monte-Carlo methods.

5. Discussion and applications

5.1 *Example*

As an example we consider the case where p is Poisson with mean 10 and F is exponential with mean 1. A sample of size 300 is generated from F and this is used to construct our estimate of the resulting compound distribution function, and a bootstrap confidence band. The tail estimate is shown in Fig. 1. The estimate and the "true" G are calculated using the fast Fourier transform algorithm. The non-constant width (simultaneous) 90% bootstrap confidence band corresponds to $\beta = 1$ in Theorem 4.2. The band is calculated from 300 bootstrap repetitions.

5.2 *The total claim amount*

In this subsection and the next one we relate the results about compound distributions to applications in insurance. Here we consider the total claim amount. Two important forms for the distribution p in the insurance context are Poisson and negative binomial, For both these distributions the conditions on p in Theorems 3.1, 4.1 and 4.2 are satisfied for any $\beta > 0$. Thus, assuming p known, we obtain strong consistency and asymptotic normality of the nonparametric estimator of the total claim amount distribution function G, together with a bootstrap confidence band for G. The estimate in the previous subsection may be interpreted as an estimate of the tail of the total claim amount distribution function for a classical risk model.

5.3 *Unknown claim number distribution*

The case where p is unknown and must be estimated from data covers various situations reflecting different assumptions on the claim arrivals, such as whether

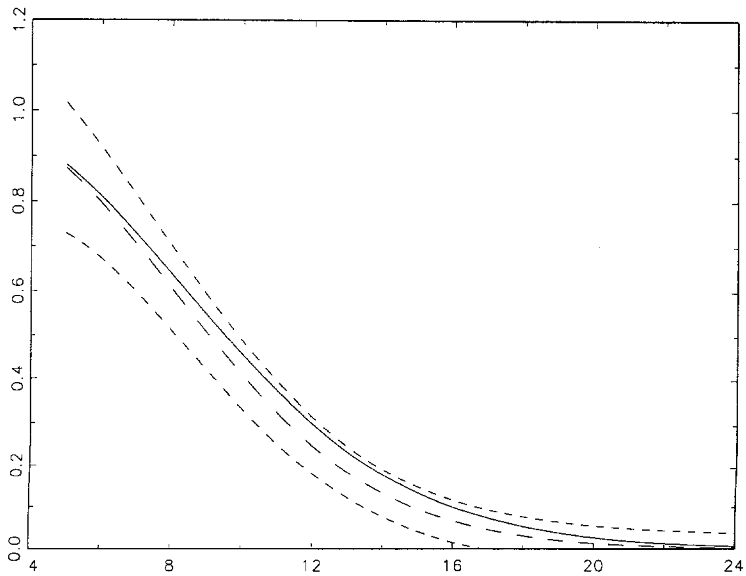

Fig. 1. Estimate (large dashes), true $I - G$ (solid line) and bootstrap confidence band (small dashes) for Example in 5.1.

p has a particular parametric form, or whether the claims arrive in a renewal process. There are also different possibilities for the nature of the data available for estimation of G. This is an area for future investigation, and it should be possible to extend the methods of this paper to derive statistical properties of estimators of G in some of the above situations.

As a straightforward example of the ideas involved, assume that the claims arrive in a Poisson process with rate λ, so that, taking $t = 1$ for simplicity, $p = p(\lambda) = (p_k(\lambda))_{k \in \boldsymbol{N}_0}$ where $p_k(\lambda) = e^{-\lambda}\lambda^k/k!$. For definiteness, assume now that λ is estimated by $\hat{\lambda}_n$ based on a sample of n inter-claim-arrival times independent of the claim sizes (for example, $\hat{\lambda}_n$ could be the reciprocal of their mean). Assume that $\hat{\lambda}_n \to \lambda$ with probability one and $\sqrt{n}(\hat{\lambda}_n - \lambda) \to_d N(0, \sigma^2)$ for some σ^2 as n tends to infinity. Assume that $\hat{F}_n$ and F are as in Sections 3 and 4. Then we would aim to combine the above properties of $\hat{\lambda}_n$ with continuity and differentiability results for the map taking the pair (λ, F) to $G = \sum_{k=0}^{\infty} p_k(\lambda) F^{\star k}$, to yield statistical properties of $\hat{G}_n = \sum_{k=0}^{\infty} p_k(\hat{\lambda}_n)\hat{F}_n^{\star k}$ as an estimator of G.

5.4 *The probability of ruin for the classical risk model*

The quantity of interest here is the probability of ruin $\psi(x)$ starting from initial capital $x > 0$ in the classical risk model. The ruin function ψ is related to the function R introduced in Section 1 by

$$\psi = 1 - R = 1 - \Phi_p(F),$$

where $p_k = (1 - \rho)\rho^k$ for k in $\boldsymbol{N}_0$, and $F(x) = (1/\mu)\int_0^x (1 - F_1(y))dy$, F_1 is the claim amount distribution function and $\mu = \int x F_1(dx)$. Here we suppose that

$F_1(0) = 0$. We consider the situation where the claim amount distribution function is unknown but a random sample $X_1, \ldots, X_n$ from F_1 is available. We may then wish to estimate ψ for different safety loadings or, equivalently, for different values of ρ. For a particular known value of ρ, the map Ψ taking the claim amount distribution function onto ψ can be decomposed

(5.1) $$F_1 \mapsto F \overset{\Phi_p}{\mapsto} R \mapsto \psi;$$

the resulting nonparametric estimator $\hat{\psi}_n$ of ψ is given by $\hat{\psi}_n = \Psi(\hat{F}_{1,n})$ where $\hat{F}_{1,n}$ is the empirical distribution function based on $X_1, \ldots, X_n$. We observe that there are other situations where different ruin functionals would be appropriate, for example, if data are also available for the inter-claim-arrival times and λ is to be estimated as well. This is not considered here, although the methods of this paper can be extended to construct an estimator for ψ in this case (see Subsection 5.3).

Returning to the fixed ρ example, in order to derive strong consistency and asymptotic normality results for $\hat{\psi}_n$, we first establish continuity and differentiability of Ψ. In addition to Propositions 3.1 and 4.1, we need these properties for the first and last stages in (5.1). The last stage is trivial.

Since F_1, F and ψ are all zero on $(-\infty, 0)$, we consider weighted spaces on $[0, \infty)$ and write now $D_\beta = \{f|_{[0,\infty)} : T_\beta f \in D_\infty\}$, with $\|\cdot\|_\beta$ correspondingly defined. Write I for the function that is identically 1 on $[0, \infty)$. Below we omit reference to the embedding necessary in order to apply Propositions 3.1 and 4.1 to the compound distribution functional as a map between these new D_β-spaces.

For f such that $cI - f$ is in D_β, $\beta > 1$, for some c in $\boldsymbol{R}$, write $f(\infty)$ for c, and let

$$(\Phi_1 f)(x) = \int_x^\infty (f(\infty) - f(y))dy \qquad x \in [0, \infty).$$

Writing Φ for the first map in (5.1), the decomposition becomes

$$F_1 \mapsto \Phi(F_1) = I - \frac{\Phi_1(F_1)}{\Phi_1(F_1)(0)} \overset{\Phi_p}{\mapsto} R \mapsto \psi = I - R.$$

Let $\{F_n\}_{n \in \boldsymbol{N}_0}$ be distribution functions with, for all n in $\boldsymbol{N}_0$, $F_n(0) = 0$.

LEMMA 5.1. *Let $\beta > 1$. Assume that $\int x^\beta F_n(dx) < \infty$ and*

$$\|F_n - F_0\|_\beta \to 0 \qquad as \qquad n \to \infty.$$

Then, for $0 \le \gamma \le \beta - 1$,

$$\|\Phi(F_n) - \Phi(F_0)\|_\gamma \to 0 \qquad as \qquad n \to \infty.$$

PROOF. The result follows easily on noting first that

$$\begin{aligned} &\left\| \frac{\Phi_1(F_n)}{\Phi_1(F_n)(0)} - \frac{\Phi_1(F_0)}{\Phi_1(F_0)(0)} \right\|_\gamma \\ &\quad \le \frac{1}{\Phi_1(F_n)(0)} \|\Phi_1(F_n) - \Phi_1(F_0)\|_\gamma \\ &\qquad + \|I - \Phi_1(F_0)\|_\gamma \left| \frac{1}{\Phi_1(F_n)(0)} - \frac{1}{\Phi_1(F_0)(0)} \right|. \end{aligned}$$

Then use

$$\begin{aligned}
&\|F_n - F_0\|_\beta \to 0 \Rightarrow \Phi_1(F_n)(0) \to \Phi_1(F_0)(0) \quad \text{as} \quad n \to \infty, \qquad \text{and} \\
&(1+x)^\gamma |\Phi_1(F_n)(x) - \Phi_1(F_0)(x)| \\
&\qquad \le (1+x)^\gamma \int_x^\infty |F_n(y) - F_0(y)| dy \\
&\qquad \le (1+x)^\gamma \|F_n - F_0\|_\beta \int_x^\infty (1+y)^{-\beta} dy
\end{aligned}$$

with $\gamma \le \beta - 1$. □

Now suppose that $\beta' > 0$, and that $\{F_n\}_{n \in \mathbf{N}_0}$ are distribution functions with $F_n(0) = 0$, $\int x^{1+\beta'} F_n(dx) < \infty$ and $\|F_n - F_0\|_{1+\beta'} \to 0$ as $n \to \infty$. From the above lemma, we have

$$\|\Phi(F_n) - \Phi(F_0)\|_{\beta'} \to 0 \quad \text{as} \quad n \to \infty.$$

We also have

$$\begin{aligned}
\int_{(0,\infty)} x^{\beta'} (\Phi F_n)(dx) &= \int_0^\infty x^{\beta'} \frac{(1 - F_n(x))}{(\Phi_1 F_n)(0)} dx \\
&= \frac{1}{(1+\beta')(\Phi_1 F_n)(0)} \int_0^\infty x^{1+\beta'} F_n(dx) \\
&< \infty,
\end{aligned}$$

on integrating by parts. This means that $\{\Phi(F_n)\}_{n \in \mathbf{N}_0}$ satisfy the conditions of Proposition 3.1. The conditions required of p are satisfied. Thus, for $0 \le \beta < \beta'$,

$$\|\Psi(F_n) - \Psi(F_0)\|_\beta = \|\Phi_p(\Phi(F_n)) - \Phi_p(\Phi(F_0))\|_\beta$$

and this last quantity tends to zero as n tends to infinity. This continuity result for the ruin functional Ψ leads to a strong consistency result for the ruin function estimator $\hat{\psi}_n$.

THEOREM 5.1. *Let $\beta' > \beta \ge 0$. Let F be a distribution function with $F(0) = 0$ and $\int x^{1+\beta'} F(dx) < \infty$. With notation as above, with probability one,*

$$\|\hat{\psi}_n - \psi\|_\beta \to 0 \quad \textit{as} \quad n \to \infty.$$

For the differentiability of Φ we have the following lemma.

LEMMA 5.2. *Let $\beta > 1$. Assume that $\int x^\beta F_n(dx) < \infty$ and*

$$\sqrt{n}(F_n - F_0) \to g \quad \textit{as} \quad n \to \infty \quad \textit{in } D_\beta,$$

where $g \in D_\beta$. Then for all $0 \le \gamma \le \beta - 1$

$$\sqrt{n}(\Phi(F_n) - \Phi(F_0)) \to \Phi'_{F_0}(g) \quad \textit{as} \quad n \to \infty \quad \textit{in } D_\gamma,$$

where $\Phi'_{F_0}(g) = -\Phi_1(g)/\Phi_1(F_0)(0) + (\Phi_1(g)(0)/\Phi_1(F_0)(0)^2)\Phi_1(F_0)$.

PROOF. Note first that

$$\sqrt{n}(\Phi_1(F_n)(0) - \Phi_1(F_0)(0)) - \Phi_1(g)(0) \to 0, \qquad \text{and so}$$
$$\sqrt{n}\left(\frac{1}{\Phi_1(F_n)(0)} - \frac{1}{\Phi_1(F_0)(0)}\right) + \frac{\Phi_1(g)(0)}{(\Phi_1(F_0)(0))^2} \to 0$$

as n tends to infinity. Furthermore

$$\sqrt{n}(\Phi_1(F_n) - \Phi_1(F_0)) - \Phi_1(g) \to 0 \quad \text{as} \quad n \to \infty \quad \text{in } D_\gamma.$$

The result then follows on using arguments similar to those used in proving a product rule for derivatives. □

We combine Lemma 5.2 with Proposition 4.1 to obtain differentiability of the ruin functional Ψ along certain curves.

Suppose that $\beta' > 0$ and that $\{F_n\}_{n\in \boldsymbol{N}_0}$ are distribution functions with $F_n(0) = 0$ and $\int x^{1+\beta'} F_n(dx) < \infty$ for all n in $\boldsymbol{N}_0$. Assume that

$$\sqrt{n}(F_n - F_0) \to g \quad \text{as} \quad n \to \infty \quad \text{in } D_{1+\beta'},$$

where $g \in D_{1+\beta'}$. By Lemma 5.2, $\{\Phi(F_n)\}_{n\in \boldsymbol{N}_0}$ satisfy the conditions of Proposition 4.1. Thus we obtain for $0 \le \beta < \beta'$

$$\begin{aligned}&\|\sqrt{n}(\Psi(F_n) - \Psi(F_0)) + \Phi'_{p,\Phi(F_0)}(\Phi'_{F_0}(g))\|_\beta \\ &\quad = \|\sqrt{n}(\Phi_p \circ \Phi(F_n) - \Phi_p \circ \Phi(F_0)) - \Phi'_{p,\Phi(F_0)}(\Phi'_{F_0}(g))\|_\beta\end{aligned}$$

and this tends to zero as n tends to infinity. We note that

$$\Psi'_{F_0}(g) = \Phi'_{F_0}(g) \star \sum_{k=1}^{\infty} kp_k(\Phi(F_0))^{\star(k-1)}.$$

The above leads to an asymptotic normality result for $\hat{\psi}_n$.

THEOREM 5.2. *Let $\beta' > \beta \ge 0$. Let $\{X_i\}_{i\in \boldsymbol{N}_0}$ be independent identically distributed random variables with distribution function F satisfying $F(0) = 0$ and $\int x^{2(1+\beta')} F(dx) < \infty$. With the above notation*

$$\sqrt{n}(\hat{\psi}_n - \psi) \to_d Z' \quad \textit{as} \quad n \to \infty \quad \textit{in } D_\beta,$$

where Z' is a Gaussian process obtained by applying Ψ'_F to the sample paths of $B \circ F$, where B is a standard Brownian bridge.

By analogy with the discussion preceding Theorem 4.2, define $R_n(z)$ to be $\boldsymbol{P}(\sqrt{n}\|\hat{\psi}_n - \psi\|_\beta \le z)$ and let $\hat{q}_n(\alpha)$ be the α-quantile of the bootstrap estimator $\hat{R}_n$ of R_n.

THEOREM 5.3. *Let $\beta' > \beta \geq 0$. Let F be a distribution function satisfying $\int x^{2(1+\beta')} F(dx) < \infty$. Then*

$$\boldsymbol{P}(\sqrt{n}\|\Psi(\hat{F}_n) - \Psi(F)\|_\beta \leq \hat{q}_n(\alpha)) \to \alpha \quad \text{as} \quad n \to \infty.$$

As an example, we consider the (grouped) data for 799 fire claims in Ramlau-Hansen (1988). The data used for our purposes are obtained from the grouped data by distributing the observations within a class at random uniformly over the class-interval. Figure 2 shows the resulting ruin function estimate $\hat{\psi}_n$, and approximate 90% bootstrap confidence region, with $\beta = 1$, and $\rho = 1/3$, for the unknown ψ.

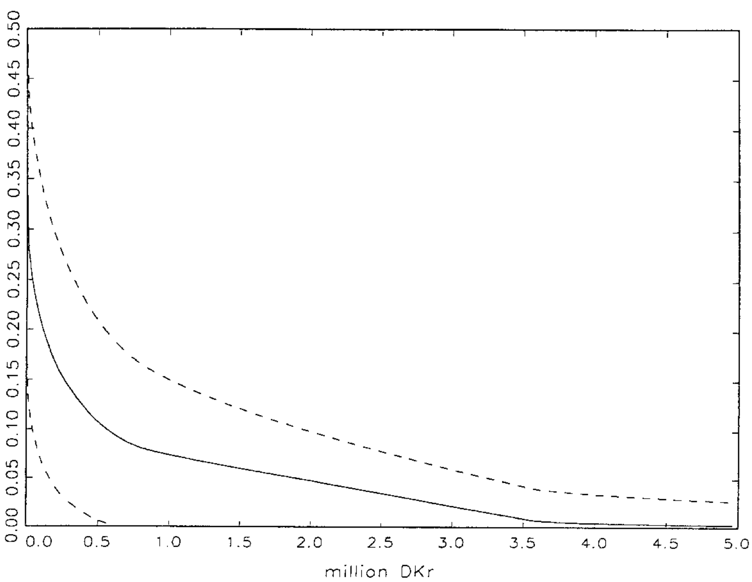

Fig. 2. Estimate of probability of ruin (solid line), and bootstrap confidence band (broken lines) for fire claims data.

It should be noted that this data set consists of 798 observations smaller than 8×10^5 DKr and one observation of 3.6×10^6 DKr, and that the estimator (and the probability of ruin function calculated for a bootstrap sample) is sensitive to the inclusion of the largest observation in the sample. However, this sensitivity, arising in a situation where we deal with heavy-tailed claim size distributions, is not specific to the estimator considered here.

The choice of β in a particular application reflects the tension between on the one hand the desirability of confidence limits that decrease rapidly at the far right of the compound distribution function tail (or of ψ), and on the other the moment assumptions that we are prepared to make about F.

The estimator in Frees (1986a) is defined when we also have observations on the inter-claim-arrival times, and hence is not directly applicable in this context.

For the estimator in Croux and Veraverbeke (1990), the comments in Section 4.2 of Grübel and Pitts (1993), concerning a similarly defined estimator for the renewal function proposed in Frees (1986*b*), are relevant. As mentioned in the introduction, the estimators studied here are in the same spirit as those of Hipp (1989), but we consider the estimates as elements of a function space and furthermore we investigate simultaneous, rather than pointwise, bootstrap confidence bands, establishing the asymptotic validity of such bands.

Acknowledgements

I would like to thank Professor Asmussen for the data used in Subsection 5.4. I would also like to thank Rudolf Grübel for his comments on a first version of this paper.

References

Chow, Y. S. and Teicher, H. (1988). *Probability Theory: Independence, Interchangeability, Martingales*, 2nd ed., Springer, New York.

Croux, K. and Veraverbeke, N. (1990). Nonparametric estimators for the probability of ruin, *Insurance Math. Econom.*, **9**, 127–130.

Csörgő, S. and Teugels, J. L. (1990). Empirical Laplace transforms and approximation of compound distributions, *J. Appl. Probab.*, **27**, 88–101.

Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims, *Insurance Math. Econom.*, **1**, 55–72.

Embrechts, P., Jensen, J. L., Maejima, M. and Teugels, J. L. (1985*a*). Approximations for compound poisson and pólya processes, *Adv. in Appl. Probab.*, **17**, 623–637.

Embrechts, P., Maejima, M. and Teugels, J. L. (1985*b*). Asymptotic behaviour of compound distributions, *ASTIN Bulletin*, **15**, 45–47.

Frees, E. W. (1986*a*). Nonparametric estimation of the probability of ruin, *ASTIN Bulletin*, **16S**, 81–90.

Frees, E. W. (1986*b*). Nonparametric renewal function estimation, *Ann. Statist.*, **14**, 1366–1378.

Gill, R. D. (1989). Non- and semi-parametric maximum likelihood estimators and the von Mises method (Part 1), *Scand. J. Statist.*, **16**, 97–128.

Grimmett, G. and Stirzaker, D. (1982). *Probability and Random Processes*, Clarendon Press, Oxford.

Grübel, R. (1989). Stochastic models as functionals: Some remarks on the renewal case, *J. Appl. Probab.*, **26**, 296–303.

Grübel, R. and Pitts, S. M. (1993). Nonparametric estimation in renewal theory I: The empirical renewal function, *Ann. Statist.* (to appear).

Hipp, C. (1989). Estimators and bootstrap confidence intervals for ruin probabilities, *ASTIN Bulletin*, **19**, 57–69.

Pollard, D. (1984). *Convergence of Stochastic Processes*, Springer, New York.

Ramlau-Hansen, H. (1988). A solvency study in non-life insurance: Part 1. Analyses of fire, windstorm and glass claims, *Scand. Actuar. J.*, 3–34.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*, Wiley, New York.

von Chossy, R. and Rappl, G. (1983). Some approximation methods for compound distributions, *Insurance Math. Econom.*, **2**, 251–270.

Willekens, E. (1989). Asymptotic approximations of compound distributions and some applications, *Bull. Soc. Math. Belg. Ser. B*, **41**, 55–61.