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ESTIMATING NON-LINEAR FUNCTIONS OF THE SPECTRAL DENSITY, USING A DATA-TAPER*

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Abstract. Let $f(\omega)$ be the spectral density of a Gaussian stationary process. Consider periodogram-based estimators of integrals of certain non-linear functions ζ of $f(\omega)$, like $H_T := \int_{-\pi}^{\pi} \Lambda(\omega)\zeta(I_T(\omega))d\omega$, where $\Lambda(\omega)$ is a bounded function of bounded variation, possibly depending on the sample size T. Then it is known that, under mild conditions on ζ, a central limit theorem holds for these statistics H_T if the non-tapered periodogram $I_T(\omega)$ is used. In particular, Taniguchi (1980, *J. Appl. Probab.*, **17**, 73–83) gave a consistent and asymptotic normal estimator of $\int_{-\pi}^{\pi} \Lambda(\omega)\Phi(f(\omega))d\omega$, choosing ζ to be a suitable transform of a given function Φ. In this work we shall generalize this result to statistics H_T where a taper-modified periodogram is used. We apply our result to the use of data-tapers in nonparametric peak-insensitive spectrum estimation. This was introduced in von Sachs (1994, *J. Time Ser. Anal.*, **15**, 429–452) where the performance of this estimator was shown to be substantially improved by using a taper.

Key words and phrases: Gaussian stationary process, spectral density, periodogram, data-taper, peak-insensitive spectral estimator.

1. Introduction

Using data-tapers in periodogram-based spectrum estimation is a well-known remedy to reduce leakage effects (see e.g. Tukey (1967), Bloomfield (1976)), in particular if the spectrum contains high peaks. The usual asymptotic properties which hold for non-tapered statistics carry over to the taper-modified estimates, if these depend *linearly* on the periodogram (like smoothed periodograms, integrated periodograms, estimators of the spectral distribution function), see e.g. Brillinger (1981) and Dahlhaus (1983, 1985, 1988). So it is of general interest whether this is still true for a kind of estimators with a *non-linear* functional dependence on the periodogram ordinates, i.e. like those considered in Taniguchi (1980) for the non-tapered case:

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Given a real-valued Gaussian stationary time series X with mean 0 and spectral density $f(\omega)$, and given a sample $X_1, \ldots, X_T$ of size T, consider the following statistics:

$$\int_{-\pi}^{\pi} \Lambda(\omega)\zeta(I_T(\omega))d\omega, \tag{1.1}$$

where ζ is a certain non-linear function, $I_T(\omega)$ denotes the periodogram of $X_1, \ldots, X_T$ and $\Lambda(\omega)$ is a bounded function of bounded variation, possibly depending on the sample size T. Note that this allows to consider both kernel estimators and integrated $\zeta(I_T(\omega))$ (i.e. $\Lambda(\omega) \equiv 1$ or $\Lambda(\omega) = 1_{[-\pi,\lambda]}(\omega)$ for fixed $\lambda \in (-\pi, \pi]$). Then it is known that, under mild conditions on ζ, a central limit theorem holds for these statistics (1.1) if the non-tapered periodogram $I_T(\omega)$ is used. In particular, Taniguchi (1980) gave a consistent and asymptotic normal estimator of $\int \Lambda(\omega)\Phi(f(\omega))d\omega$, choosing ζ to be a suitable transform (inverse Laplace) of an appropriate given function Φ (to introduce asymptotic unbiasedness of the resulting estimator).

In this work we shall generalize this result to statistics of the form (1.1) where a taper modified periodogram is used (cf. Dahlhaus (1985), for linear statistics). Again, as mentioned above, the resulting estimator will profit from reduced leakage, e.g. leading to a faster decay of the bias for spectra with high peaks (see in particular Dahlhaus (1988)). Concerning the proof of this central limit theorem (CLT), introducing data-tapers essentially changes the covariance structure of the periodogram ordinates, such that, e.g., the proof of Taniguchi (1980) does not work any longer without essential modifications (cf. also Remark 5.1 in Section 5).

As application, we give the proof of asymptotic normality (including consistency) of the tapered peak-insensitive nonparametric spectral estimator, introduced in von Sachs (1994), henceforth referred to as vS. There, a particular choice of ζ (related to non-linear bounded Ψ-functions, which typically occur in M-estimation theory) yields a spectral density estimator that is insensitive to outliers in the frequency domain. In vS it was shown that the performance of this estimator is substantially improved by the use of a taper (and the proof of the central limit theorem, i.e. Theorem 3.2 of that reference, was postponed to this work).

Note that the proofs in both situations (tapered and non-tapered) only work with Gaussian time series data as they heavily depend on the use of the normal distribution function. (It is still an open problem how to treat the non-Gaussian case though it might be possible to carry over some of these ideas.)

Our paper is organized as follows: In the next section we introduce the notation for tapered nonparametric spectrum estimation and state the assumptions which are to hold throughout this work. Section 3 gives the main line for proving the central limit theorem, where lemmas and auxiliary results as well as their proofs are postponed to the appendix, which is Section 5. Section 4 finally deals with applying our results to the specific problems as in Taniguchi (1980), henceforth referred to as TG, and in vS.

2. Preliminaries

Let $X_1, \ldots, X_T$ be a sample of a stationary real-valued Gaussian process $\{X_t\}$ with $EX_t = 0$ and continuous spectral density $f(\omega)$ which is bounded away from zero. Let

$$I_T(\omega) = (H_{2,T})^{-1} \cdot \left| \sum_{t=1}^{T} h_t \cdot X_t e^{-i\omega t} \right|^2, \qquad -\pi \leq \omega \leq \pi,$$

denote the periodogram of the sample based on the tapered data $(h_t \cdot X_t)_{t=1,\ldots,T}$. Note that $I_T(\omega) = (H_{2,T})^{-1} \cdot d_T(\omega) d_T(-\omega)$, where $d_T(\omega) = \sum_{t=1}^{T} h_t \cdot X_t e^{-i\omega t}$ denotes the finite Fourier transform of the tapered data. Here $h_t = h(t/T)$, $t = 1, \ldots, T$, with a bounded real function $h(x)$ of bounded variation and $h(x) = 0$ for $x \notin (0,1)$. $H_{2,T} = \sum_{t=1}^{T} h_t^2$ denotes the appropriate normalizing factor, for which we assume $H_{2,T} \sim T$ (we follow the notation of Dahlhaus (1988)).

In order to restrict the dependence structure of the process X we impose

Assumption 1. $\sum |uc(u)| < \infty$, where $c(u) = E[X(t)X(t+u)]$ denotes the autocovariance function of X. Let further $H_T := \int_{-\pi}^{\pi} \Lambda_T(\omega)\zeta(I_T(\omega))d\omega$, with $\Lambda_T(\omega) = \eta_T \Lambda(\omega)$, where η_T is a sequence of positive numbers with $T\eta_T^{-1} \to \infty$, as $T \to \infty$, and where $\Lambda(\omega)$, $\omega \in \Pi := (-\pi, \pi]$, is a real-valued bounded function of bounded variation.

For the function $\zeta(x)$ on the real line we impose the following assumptions:

Assumption 2. $\zeta(x)$ is a continuous function for $x \in (0, \infty)$.

Assumption 3. For every positive integer m, $\int_0^\infty |\zeta(x)|^m \exp\{-ux\}dx$ exists and is continuous on $(0, \infty)$ with respect to u.

Note that Assumption 3 is met if we impose boundedness of ζ (which will be done in the context of peak-insensitive spectrum estimation, in vS).

Finally, only for technical reasons of proving the CLT by the chosen method of cumulants, we need an additional assumption on the taper function $h(x)$, for which we give a more general definition:

Let $0 < \rho \leq 1$ denote the proportion of the data which are going to be tapered. In order to introduce a suitable kind of symmetry property of $h(x)$, let

$$h_\rho(x) := u(x/\rho)1_{[0,\rho/2)}(x) + 1_{[\rho/2, 1-\rho/2]}(x) + v((1-x)/\rho)1_{(1-\rho/2, 1]}(x), \tag{2.1}$$

where u and v are some real-valued functions of bounded variation on $[0,1]$ that are symmetric around $1/2$, with $u(1/2) = v(1/2) = 1$. Note that, with this definition (2.1), $h(x) := h_1(x)$, and that with ρ tending to zero $h_\rho(x)$ converges to $1_{[0,1]}(x)$.

Assumption 4. Let ρ depend on T like $\rho_T = cT^{-\beta}$ with $0 < \beta$ and some constant c which we set to unity without loss of generality.

So we assume the taper to vanish asymptotically with rate $T^{-\beta}$. Note that for $\beta \geq 1$ it is not necessary to give a new proof of asymptotic normality in the tapered case: we can immediately proceed as in TG, as in this case, in the resulting covariance matrix Σ_{2m} (see Section 3), the covariances are of order $O(T^{-1})$, as it is in TG. Thus the only situation of interest is $0 < \beta < 1$ (cf. also Remark 5.1). However, here no further assumption on the rate of decay will be needed for the proofs such that β is allowed to be arbitrarily small! So Assumption 4 is not restrictive at all. Moreover, an asymptotically vanishing taper reflects the practical situation where, with an increasing number T of observations, it is less important to taper the series.

3. Asymptotic normality

Let $H := \lim_{T\to\infty} EH_T$ and $\gamma_T := T\eta_T^{-1}$. Then the considered CLT states as follows:

THEOREM 3.1. *Under Assumptions* 1 *to* 4,

$$\gamma_T^{1/2}(H_T - H) \xrightarrow{D} \mathcal{N}(0, V),$$

where V denotes the appropriate asymptotic variance.

Note that, by the asymptotic unbiasedness, this CLT includes consistency of the statistical estimation functional H_T. Note also that the precise form of both H and V depends on the kind of considered problem, i.e. the choice of the functions ζ and Λ_T (cf. Remark 3.1): Examples are our two concrete applications, which can be found in Section 4.

For proving the CLT 3.1 we shall approximate the integral H_T by the respective integral approximation z_T of H_T: Let

$$z_T := T^{-1} \sum_{k=-N}^{N} \Lambda_T(\omega_k)\zeta(I_T(\omega_k)),$$

where $\omega_k = \frac{2\pi k}{T}$, $k = -N, \ldots, N$, $N = [T/2]$.

Then the following approximation holds:

LEMMA 3.1. $|z_T - H_T| = o_p(\gamma_T^{-1/2})$.

With this lemma, we prove Theorem 3.1 by showing

THEOREM 3.2. *Under the assumptions of Theorem* 3.1,

$$\gamma_T^{1/2}(z_T - H) \xrightarrow{D} \mathcal{N}(0, V).$$

In the following we give the main lines of the proof of this theorem. The following propositions immediately yield Theorem 3.2 by the method of cumulants:

PROPOSITION 3.1. $\lim_{T\to\infty} \gamma_T^{1/2}(Ez_T - H) = 0$.

PROPOSITION 3.2. $\lim_{T\to\infty} \gamma_T \operatorname{Var}\{z_T\} = V$.

PROPOSITION 3.3. $\lim_{T\to\infty} \gamma_T^{L/2} \operatorname{cum}_L\{z_T\} = 0$ *for all* $L \geq 3$.

To show these propositions we introduce the following setup: Let

$$A_j := A_T(\omega_j) := f(\omega_j)^{-1/2} 2^{1/2} (H_{2,T})^{-1/2} \cdot \operatorname{Re}(d_T(\omega_j))$$

and

$$B_j := B_T(\omega_j) := f(\omega_j)^{-1/2} 2^{1/2} (H_{2,T})^{-1/2} \cdot \operatorname{Im}(d_T(\omega_j)).$$

Then $1/2(A_j^2 + B_j^2) = f(\omega_j)^{-1} I_T(\omega_j)$, and we describe the m-th cumulants of $\zeta(I_T(\omega_k))$, for fixed order $m \geq 1$, by means of integrals of ζ w.r.t. the joint probability density function (p.d.f.) of the Gaussian vector $(A_T(\omega_{j_i}), B_T(\omega_{j_i}))_{i=1,\ldots,m}$ with $\omega_{j_i} = 2\pi j_i/T$, $j_i = -N, \ldots, N$: it is distributed as $\mathcal{N}(0_{2m}, \Sigma_{2m})$ with $2m \times 2m$-matrix $\Sigma_{2m} = (\Sigma_T(\omega_{j_i}, \omega_{j_k}))_{i,k=1,\ldots,m}$, where

$$\Sigma_T(\omega_{j_i}, \omega_{j_k}) = \begin{pmatrix} \operatorname{Cov}(A_T(\omega_{j_i}), A_T(\omega_{j_k})) & \operatorname{Cov}(A_T(\omega_{j_i}), B_T(\omega_{j_k})) \\ \operatorname{Cov}(B_T(\omega_{j_i}), A_T(\omega_{j_k})) & \operatorname{Cov}(B_T(\omega_{j_i}), B_T(\omega_{j_k})) \end{pmatrix}.$$

The corresponding p.d.f. writes as $\phi_{2m}(\underline{y}) = (2\pi)^{-m}(\det \Sigma_{2m})^{-1/2} \exp -(\underline{y}, \Sigma_{2m}^{-1}\underline{y})/2d\underline{y}$, where $\underline{y} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_m}) = (y_{11}, y_{12}; y_{21}, y_{22}; \ldots; y_{m1}, y_{m2}) \in \boldsymbol{R}^{2m}$, with $\boldsymbol{y_i} = (y_{i1}, y_{i2}) \in \boldsymbol{R}^2$. The basic estimates for the elements of $\Sigma := \Sigma_{2m}$ are Lemmas 5.1 and 5.2 in the Appendix, which yield that $\lim_{T\to\infty} \Sigma_{2m} = I_{2m}$ and $\lim_{T\to\infty} \det \Sigma_{2m} = 1$. (To be more precise the rates of convergence of $\det \Sigma_{2m}$ are of the same order as those which will be derived in the sequel for the convergence of the matrix elements of Σ_{2m}.) Hence, in the sequel, we shall omit $(\det \Sigma_{2m})^{-1/2}$ in all occurring Gaussian p.d.f.'s (cf. equation (3.8)).

Now we introduce the characteristic function $\chi(\underline{t})$, $\underline{t} = (t_1, t_2, \ldots, t_L)$, of a random vector that consists of L components with elements in $\{\zeta(I_T(\omega_{j_i}))\}_{i=1,\ldots,m}$, with $j_i \neq \pm j_k$ for $i \neq k$, and with $1 \leq m \leq L$ (see Definition 3.1 below):

$$\chi(\underline{t}) = E\left[\exp\left\{i \sum_{i=1}^{L} t_i \zeta_i\right\}\right] = \int_{R^{2m}} \prod_{i=1}^{L} \exp\{i t_i \zeta_i\} \phi_{2m}(\underline{y}) d\boldsymbol{y},$$

where $\zeta_i := \zeta(f(\omega_{j_i}) \|\boldsymbol{y_i}\|^2/2) = \zeta(f(\omega_{j_i})(y_{i1}^2 + y_{i2}^2)/2)$. Moreover, for $1 \leq m \leq L$, define the set Π_m of indices j_i as follows:

DEFINITION 3.1. $(j_1, \ldots, j_L) \in \Pi_m :\Leftrightarrow \exists\, p_1, \ldots, p_m > 0$ with $p_1 + \cdots + p_m = L$ and $0 < j_1 = \cdots = j_{p_1} < j_{p_1+1} = \cdots = j_{p_1+p_2} < \cdots < j_{L-p_m+1} = \cdots = j_L \leq N$, where in the last line j_i stands for $|j_i|$, i.e. we do not distinguish between j_i and $-j_i$!

With this separation introduced

$$\text{(3.1)}\quad \mathrm{cum}_L\{z_T\} = \mathrm{cum}_L\left\{T^{-1}\sum_{j=-N}^{N}\Lambda_T(\omega_j)\zeta(I_T(\omega_j))\right\}$$
$$= \sum_{m=1}^{L} A(L,m) \sum_{(j_1,\dots,j_L)\in\Pi_m} \alpha_{j_1}\cdots\alpha_{j_L}\cdot \mathrm{cum}\{\zeta_{j_1},\dots,\zeta_{j_L}\}_m,$$

where $\alpha_j := T^{-1}\Lambda_T(\omega_j)$, with $\alpha_j = O(\gamma_T^{-1})$, uniformly in j, and where

$$\mathrm{cum}\{\zeta_{j_1},\dots,\zeta_{j_L}\}_m = i^{-L}\partial^L/(\partial t_1\cdots\partial t_L)\log\chi(t_1,t_2,\dots,t_L)|_{t_1=\cdots=t_L=0}$$
$$= i^{-L}\partial^L/\partial \underline{t}\log\left(\int_{R^{2m}}\prod_{i=1}^{L}\exp\{it_i\zeta_i\}\phi_{2m}(\underline{\boldsymbol{y}})d\underline{\boldsymbol{y}}\right).$$

(In the sequel, we omit in the notation that the logarithmic derivative is always taken at the point $\underline{t} = \underline{0}$!) Note that $\{\zeta_{j_1},\dots,\zeta_{j_L}\}_m$ is used for $\{\zeta_{j_1},\dots,\zeta_{j_L}\}$ if $(j_1,\dots,j_L)\in\Pi_m$.

Finally $A(L,m)$ is used as a constant not depending on T, to give the right number of terms of the sum for each m.

We proceed proving Theorem 3.2 by first presenting the proof of Proposition 3.3 which includes part of the proof of Proposition 3.2, where $L=2$, namely the one with $m=2$ in $\mathrm{cum}_2\{z_T\}$:

Proposition 3.4. *For $L\geq 3$, $1\leq m\leq L$, and for $L=2=m$,*

$$\lim_{T\to\infty}\gamma_T^{L/2}\sum_{(j_1,\dots,j_L)\in\Pi_m}\alpha_{j_1}\cdots\alpha_{j_L}\cdot\mathrm{cum}\{\zeta_{j_1},\dots,\zeta_{j_L}\}_m = 0.$$

3.1 *Proof of Proposition* 3.4

Let, for given $\beta>0$, $L\geq 2$ be fixed. We consider the respective terms of (3.1) labeled by m. The simplest case occurs for $L\geq 3$, $m=1$ (i.e. $j_1=\cdots=j_L$): By (5.6) in Lemma 5.4, with

$$\mathrm{cum}\{\zeta_{j_1},\dots,\zeta_{j_L}\}_1 = (-i)^L\partial^L/\partial t^L\log\left(\int_{R^2}\exp\{it\zeta_1\}\phi_2(\boldsymbol{y_1})d\boldsymbol{y_1}\right) = O(1),$$
$$\sum_{(j_1,\dots,j_L)\in\Pi_1}\alpha_{j_1}\cdots\alpha_{j_L}\cdot\mathrm{cum}\{\zeta_{j_1},\dots,\zeta_{j_L}\}_1 = o(\gamma_T^{-L/2})\quad\text{for}\quad L\geq 3.$$

For all the other cases we proceed, for each fixed L (≥ 2) and for any m ($1<m\leq L$), as follows:

For given L choose $K=K(L,\beta)$ such that $\gamma_T^{L/2}T^{-K\beta}\to 0$. Let $\Delta := I-\Sigma = I_{2m}-\Sigma_{2m}$. Then

$$\text{(3.2)}\qquad \Sigma^{-1} = (I-\Delta)^{-1} = I + \sum_{k=1}^{K-1}\Delta^k + R_K$$

$$\text{(3.3)}\qquad =: I + \sum_{k=1}^{K-1}(D_k+N_k) + R_K =: H + N + R_K,$$

where D_k is the part of Δ^k which consists only in 2×2-diagonal blocks and $H := I + \sum_{k=1}^{K-1} D_k$, N_k is the remaining matrix part of Δ^k consisting only in the off-diagonal blocks and $N := \sum_{k=1}^{K-1} N_k$. Finally $R_K := \sum_{k \geq K} \Delta^k$ denotes the remainder of order K.

LEMMA 3.2. $(\underline{y}, R_K \underline{y}) = O(\|\underline{y}\|^2 T^{-K\beta})$.

With the expansion (3.3),

$$\exp -(\underline{y}, \Sigma_{2m}^{-1} \underline{y}) = \exp -(\underline{y}, H\underline{y}) \exp -(\underline{y}, N\underline{y}) \exp -(\underline{y}, R_K \underline{y}), \tag{3.4}$$

where $\exp -(\underline{y}, H\underline{y}) = \prod_{i=1}^m \exp -(\boldsymbol{y}_i, H_i \boldsymbol{y}_i)$, with H_i being the i-th 2×2-diagonal block, $\exp -(\underline{y}, N\underline{y}) = 1 + \sum_{l=1}^{K-1} c_l (\underline{y}, N\underline{y})^l + \theta_K (\underline{y}, N\underline{y})^K$, with $c_l := (-1)^l / l!$ and $\theta_K = \theta_K(\underline{y})$ and $(\underline{y}, N\underline{y})^K = O(\|\underline{y}\|^{2K} T^{-K\beta})$ (see Lemma 5.3(a)), and where finally $\exp -(\underline{y}, R_K \underline{y}) = 1 + O((\underline{y}, R_K \underline{y})) = 1 + O(\|\underline{y}\|^2 T^{-K\beta})$. Now, given fixed m, we introduce

$$\prod_{i=1}^{L} \exp\{it_i \zeta_i\} = \prod_{i=1}^{m} \exp i \left\{ \zeta_i \sum_{n=p_1+\cdots+p_{i-1}+1}^{p_1+\cdots+p_i} t_n \right\} =: \prod_{i=1}^{m} Q_i(t_{(i)}; \zeta_i) =: Q(\underline{t}; \underline{\zeta}),$$

where, for $1 \leq i \leq m$, $t_{(i)} := (t_{p_1+\cdots+p_{i-1}+1}, \ldots, t_{p_1+\cdots+p_i})$, and where $\underline{\zeta} := (\zeta_1, \ldots, \zeta_L)$. With (omitting the determinant $\det \Sigma_{2m}$),

$$\begin{aligned}
\chi(\underline{t}) &= \int_{R^{2m}} \prod_{i=1}^{L} \exp\{it_i \zeta_i\} \phi_{2m}(\underline{y}) d\underline{y} \qquad \text{and expansion (3.4)} \\
&= \int_{R^{2m}} \prod_{i=1}^{m} Q_i(t_{(i)}; \zeta_i) \exp -(\underline{y}, H\underline{y})/2 \exp -(\underline{y}, N\underline{y})/2 \exp -(\underline{y}, R_K \underline{y})/2 \, d\underline{y} \\
&= \int_{R^{2m}} Q(\underline{t}; \underline{\zeta}) \exp -(\underline{y}, H\underline{y})/2 \left[1 + \sum_{l=1}^{K-1} c_l (\underline{y}, N\underline{y})^l + \theta_K (\underline{y}, N\underline{y})^K \right] \\
&\quad \cdot [1 + O((\underline{y}, R_K \underline{y}))] d\underline{y}.
\end{aligned}$$

As both $\theta_K (\underline{y}, N\underline{y})^K$ and $(\underline{y}, R_K \underline{y})$ are bounded by some $MT^{-K\beta}$ with an integrable constant $M = M(\|\underline{y}\|^2)$,

$$\log(\chi(\underline{t})) = \log(L_0(\underline{t})) + \log(1 + x(\underline{t})) + O(T^{-K\beta}), \tag{3.5}$$

where

$$L_0(\underline{t}) := \prod_{i=1}^{m} \left(\int_{R^2} Q_i(t_{(i)}; \zeta_i)(\exp -(\boldsymbol{y}_i, H_i \boldsymbol{y}_i)/2) d\boldsymbol{y}_i \right) =: \prod_{i=1}^{m} L_0^{(i)}(t_{(i)}),$$

$$L_1(\underline{t}) := \int_{R^{2m}} Q(\underline{t}; \underline{\zeta}) \sum_{l=1}^{K-1} c_l (\underline{y}, N\underline{y})^l \exp -(\underline{y}, H\underline{y})/2 \, d\underline{y} =: \sum_{l=1}^{K-1} L_1^{(l)}(\underline{t})$$

and

$x(\underline{t}) := L_0(\underline{t})^{-1}L_1(\underline{t})$.

We have the following two lemmas:

LEMMA 3.3. *$\partial^L/\partial\underline{t}\log(L_0(\underline{t})) = 0$. Moreover,*

$$\operatorname{cum}\{\zeta_{j_1},\ldots,\zeta_{j_L}\}_m = i^{-L}\partial^L/\partial\underline{t}\log(1+x(\underline{t})) + O(T^{-K\beta}).$$

Expanding further,

$$\log(1+x(\underline{t})) = \sum_{n\geq 1}(-1)^{n+1}x(\underline{t})^n/n = \sum_{n=1}^{K-1}(-1)^{n+1}x(\underline{t})^n/n + RL_k. \tag{3.6}$$

LEMMA 3.4. *In expansion* (3.6) *$RL_K = O(x(\underline{t})^K)$ with $x(\underline{t})^K = O(T^{-K\beta})$.*

Now it remains to deal with the derivative w.r.t. $\underline{t}$ of the first $K-1$ terms of the expansion (3.6) of $\log(1+x(\underline{t}))$. We first consider ($n=1$) $\partial^L/\partial\underline{t}(x(\underline{t})) = \partial^L/\partial\underline{t}(L_0(\underline{t})^{-1}\sum_{l=1}^{K-1}L_1^{(l)}(\underline{t}))$ and recall that

$$L_1^{(l)}(\underline{t}) = \int_{R^{2m}} Q(\underline{t};\underline{\zeta})c_l(\underline{\boldsymbol{y}},N\underline{\boldsymbol{y}})^l \exp-(\underline{\boldsymbol{y}},H\underline{\boldsymbol{y}})/2d\underline{\boldsymbol{y}}.$$

LEMMA 3.5. *$\partial^L/\partial\underline{t}(L_0(\underline{t})^{-1}\sum_{l=1}^{m-1}L_1^{(l)}(\underline{t})) = 0$.*

With Lemma 3.5 it is sufficient to bound $L_1^{(l)}(\underline{t})$, $l\geq m$, from above:

LEMMA 3.6. *For $l\geq m$ the following estimate for $L_1^{(l)}(\underline{t})$ is holding:*

$$|L_1^{(l)}(\underline{t})| \leq \sum_{i_1\neq k_1}^{m}\cdots\sum_{i_{m/2}\neq k_{m/2}}^{m} M_{i_1k_1\cdots i_{m/2}k_{m/2}}(\underline{t}) \cdot (|j_{i_1}-j_{k_1}|^{-2}\cdots|j_{i_{m/2}}-j_{k_{m/2}}|^{-2}\wedge T^{-m\beta}),$$

where the sum is over indices i_u and k_u, $1\leq u\leq m/2$, being elements of $\{1,\ldots,m\}$, which are mutually different from each other, and where $M = M(\underline{t})$ denotes the finite integrals depending on these specific permutations of the indices i_u and k_u. (Note that for odd m this holds with $l\geq m+1$, with a slightly different upper bound being of similar form.)

As the form of the bound of Lemma 3.6 does not depend on $M(\underline{t})$, this yields the following estimate, up to terms of higher order:

LEMMA 3.7.

$$\gamma_T^{L/2}\sum_{(j_1,\ldots,j_L)\in\Pi_m}\alpha_{j_1}\cdots\alpha_{j_L}\cdot\partial^L/\partial\underline{t}(x(\underline{t})) = O(\gamma_T^{m/2-L/2}T^{-(m/2)\beta}).$$

Now, to finish the treatment of $\log(1 + x(\underline{t}))$, we should consider $x(\underline{t})^n$ for $n \geq 2$, cf. expansion (3.6). But these terms have the same behavior as the one treated by Lemmas 3.6 and 3.7 for $n = 1$. Hence, with (3.6) we have the following estimate:

LEMMA 3.8.

$$\gamma_T^{L/2} \sum_{(j_1,\ldots,j_L)\in\Pi_m} \alpha_{j_1} \cdots \alpha_{j_L} \cdot \partial^L/\partial\underline{t} \log(1 + x(\underline{t})) \\ = O(\gamma_T^{m/2-L/2} T^{-(m/2)\beta}) + O(\gamma_T^{L/2} T^{-K\beta}).$$

Finally, we put together all estimates and remainders of $\mathrm{cum}\{\zeta_{j_1}, \ldots, \zeta_{j_L}\}_m$ in Lemma 3.3 and insert them in $\gamma_T^{L/2} \mathrm{cum}_L\{z_T\}$. The proof of Proposition 3.4 ends by considering $m = L$ in the bound of Lemma 3.8, as the convergence of $\gamma_T^{L/2} \mathrm{cum}_L\{z_T\}$ is determined by the term $\mathrm{cum}\{\zeta_{j_1}, \ldots, \zeta_{j_L}\}_m$ with $m = L$ (in the sum (3.1)):

PROPOSITION 3.5. *Up to terms of higher order,*

$$\gamma_T^{L/2} \mathrm{cum}_L\{z_T\} = O(T^{-(L/2)\beta}) + O(\gamma_T^{L/2} T^{-K\beta}) = o(1),$$

by the choice of $K = K(L, \beta)$ such that $\gamma_T^{L/2} T^{-K\beta} \to 0$.

3.2 *Proof of Proposition* 3.1

We want to show that

$$\lim_{T\to\infty} \gamma_T^{1/2}(Ez_T - H) = 0.$$

Noting that this proof will work without using Assumption 4, we first recall that

$$E[z_T] = T^{-1} \sum_{j=-N}^{N} \Lambda_T(\omega_j) E[\zeta_j] = \sum_{j=-N}^{N} \alpha_j E[\zeta_j] \quad \text{with}$$
$$E[\zeta_j] = (2\pi)^{-1} (\det \Sigma_2)^{-1/2} \int_{R^2} \zeta(f(\omega_j) \|\boldsymbol{y}\|^2/2) \exp -(\boldsymbol{y}, \Sigma_2^{-1}(\omega_j)\boldsymbol{y})/2 \, d\boldsymbol{y}.$$

Now, with $\Sigma := \Sigma_2(\omega_j)$, the matrix expansion of Σ^{-1}, analogously to (3.2), writes as

$$\Sigma^{-1} = I + \Delta + R_2, \tag{3.7}$$

for which the following lemma holds (cf. Remark 5.1):

LEMMA 3.9. *In the expansion* (3.7) *the elements of Δ are bounded from above by some $C_j = O(|j|^{-1})$. Hence, $(\boldsymbol{y}, \Delta\boldsymbol{y}) = O(\|\boldsymbol{y}\|^2 C_j) = O(\|\boldsymbol{y}\|^2 |j|^{-1})$, and $(\boldsymbol{y}, R_2\boldsymbol{y}) = O(\|\boldsymbol{y}\|^2 C_j^2) = O(\|\boldsymbol{y}\|^2 |j|^{-2})$.*

Hence, $\exp-(\boldsymbol{y},\Sigma^{-1}\boldsymbol{y}) = \exp-\|\boldsymbol{y}\|^2\cdot\exp-(\boldsymbol{y},\Delta\boldsymbol{y})\cdot\exp-(\boldsymbol{y},R_2\boldsymbol{y})$, and expanding further,

$$\begin{aligned}&\exp-(\boldsymbol{y},\Sigma^{-1}\boldsymbol{y})/2\\ &\quad=\exp-\|\boldsymbol{y}\|^2/2\cdot[1-(\boldsymbol{y},\Delta\boldsymbol{y})/2+O((\boldsymbol{y},\Delta\boldsymbol{y}))^2]\cdot[1+O((\boldsymbol{y},R_2\boldsymbol{y}))].\end{aligned}$$

The leading term of $E[\zeta_j]$ equals

$$E_j := B(\zeta(f(\omega_j))) := (2\pi)^{-1}\int_{R^2}\zeta(f(\omega_j)\|\boldsymbol{y}\|^2/2)\exp-\|\boldsymbol{y}\|^2/2d\boldsymbol{y}, \tag{3.8}$$

as $(\det\Sigma_2)^{-1/2}=1+RD_T$, with $RD_T=O(C_j^2)$ and by (3.10) below. Elementary calculations show that

$$\begin{aligned}B(\zeta(f(\omega_j))) &= \int_0^\infty \zeta(f(\omega_j)x)e^{-x}dx\\ &= \int_0^\infty \zeta(z)f(\omega_j)^{-1}\exp\{-f(\omega_j)^{-1}z\}dz.\end{aligned}\tag{3.9}$$

Turning to the remainders of $E[\zeta_j]$, we proceed as follows:

With $\Delta_{11}=1-\mathrm{Cov}(A_j,A_j)=-\Delta_{22}$ and $\Delta_{12}=\mathrm{Cov}(A_j,B_j)=\Delta_{21}$ (see Lemma 5.1),

$$\begin{aligned}&\int_{R^2}\zeta(f(\omega_j)\|\boldsymbol{y}\|^2/2)(\boldsymbol{y},\Delta\boldsymbol{y})/2\,\exp-\|\boldsymbol{y}\|^2/2\,d\boldsymbol{y}\\ &\quad=\Delta_{11}\int_{R^2}\zeta(f(\omega_j)\|\boldsymbol{y}\|^2/2)(y_{11}^2-y_{22}^2)/2\,\exp-\|\boldsymbol{y}\|^2/2\,d\boldsymbol{y}\\ &\qquad+\Delta_{12}\int_{R^2}\zeta(f(\omega_j)\|\boldsymbol{y}\|^2/2)(y_{11}y_{12})\exp-\|\boldsymbol{y}\|^2/2\,d\boldsymbol{y}=0,\end{aligned}$$

due to symmetry arguments. Furthermore, with Lemma 3.9, the final remainder of $E[\zeta_j]$ is bounded from above by

$$\int_{R^2}\zeta(f(\omega_j)\|\boldsymbol{y}\|^2/2)[(\boldsymbol{y},\Delta\boldsymbol{y})^2+(\boldsymbol{y},R_2\boldsymbol{y})]\exp-\|\boldsymbol{y}\|^2/2\,d\boldsymbol{y}=O(C_j^2)=O(|j|^{-2}),$$

such that any of the remainders of $E[z_T]=\sum_{j=-N}^N\alpha_jE[\zeta_j]$ can be treated similarly to

$$\sum_{j=-N}^{N}\alpha_jC_j^2\le\sum_{j=-N}^{N}\alpha_j|j|^{-2}=O(\gamma_T^{-1})=o(\gamma_T^{-1/2}),\qquad\text{as}\quad\alpha_j=O(\gamma_T^{-1}).\tag{3.10}$$

The leading term of $E[z_T]$, finally, which is $\sum_{j=-N}^N\alpha_jB(\zeta(f(\omega_j)))$, tends to H with rate of order $O(\gamma_T^{-1})$, where we note that $H=\int_{-\pi}^{\pi}\lim_{T\to\infty}\Lambda_T(\omega)B(\zeta(f(\omega)))d\omega$:

$$T^{-1}\sum_{j=-N}^{N}\Lambda_T(\omega_j)B(\zeta(f(\omega_j)))-H=O(\gamma_T^{-1}).\tag{3.11}$$

This can be shown by the same arguments as in the proof of Lemma 3.1 for integral approximations by sums (analogously to Brillinger (1981), Theorem 5.10.1), as B is a function of bounded variation of ω, and so is Λ. This ends the proof of Proposition 3.1.

3.3 *Proof of Proposition* 3.2 (*remaining part*)
It remains to treat the term of the sum (3.1) with $L = 2$, $m = 1$:

LEMMA 3.10. $\lim_{T\to\infty} \gamma_T \sum_{j\in\Pi_1} \alpha_j^2 \operatorname{var}\{\zeta_j\} = V$.

PROOF OF LEMMA 3.10.
As $\sum_{j\in\Pi_1} \alpha_j^2 \operatorname{var}\{\zeta_j\} = \sum_{j=-N}^{N} \alpha_j^2 (E[\zeta_j^2] - \{E[\zeta_j]\}^2)$, where

$$E[\zeta_j^2] = (2\pi)^{-1}(\det \Sigma_2)^{-1/2} \int_{R^2} \zeta^2(f(\omega_j)\|\boldsymbol{y}\|^2/2) \exp -(\boldsymbol{y}, \Sigma_2^{-1}(\omega_j)\boldsymbol{y})/2\, d\boldsymbol{y},$$

we can proceed quite analogously to the proof of Proposition 3.1, where the only modification arises from dealing with weights α_j^2 instead of α_j:

Concerning the leading term of $E[\zeta_j^2]$ which is

$$\begin{aligned}(3.12)\quad V_j := V(\zeta(f(\omega_j))) &:= (2\pi)^{-1} \int_{R^2} \zeta^2(f(\omega_j)\|\boldsymbol{y}\|^2/2) \exp -\|\boldsymbol{y}\|^2/2\, d\boldsymbol{y} \\ &= \int_0^\infty \zeta^2(z) f(\omega_j)^{-1} \exp\{-f(\omega_j)^{-1}z\} dz,\end{aligned}$$

we note that $\gamma_T \sum_{j\in\Pi_1} \alpha_j^2 V_j$ tends to $\int_{-\pi}^{\pi} \Lambda^2(\omega) V(\zeta(f(\omega_j))) d\omega$ with rate of order γ_T^{-1}, analogously to (3.11), whereas, for the leading term of $\{E[\zeta_j]\}^2$ we note that $\gamma_T \sum_{j\in\Pi_1} \alpha_j^2 E_j^2$ tends to $\int_{-\pi}^{\pi} \Lambda^2(\omega) B^2(\zeta(f(\omega))) d\omega$, again with rate of order γ_T^{-1}. Concerning the remainders of $E[\zeta_j^2]$ everything runs quite similarly to the ones of $E[\zeta_j]$, as they are determined by exactly the same p.d.f. as occurring in (3.12). □

To end this section on the proof of the CLT 3.1, we add the following remark on the form of both H and V:

Remark 3.1. In Theorem 3.1,

$$H = \int_{-\pi}^{\pi} \lim_{T\to\infty} \Lambda_T(\omega) B(\zeta(f(\omega))) d\omega,$$

where $B(\zeta(f(\omega))) = \lim_{T\to\infty} E[\zeta(I_T(\omega))] = \int_0^\infty \zeta(f(\omega)x) e^{-x} dx$ (see equation (3.9)). Note that the limiting value of $\Lambda_T(\omega) = \eta_T \Lambda(\omega)$ depends on the behavior of η_T (e.g. for kernel estimates $\Lambda_T(\omega) = K(\omega/b_T)/b_T$, and $\int_{-\pi}^{\pi} \lim_{T\to\infty} \Lambda_T(\omega) d\omega = \int_{-\pi}^{\pi} K(\omega) d\omega = \text{const.}$). Analogously,

$$V = \int_{-\pi}^{\pi} \Lambda^2(\omega) \{V(\zeta(f(\omega))) - B^2(\zeta(f(\omega)))\} d\omega,$$

where $V(\zeta(f(\omega))) = \lim_{T\to\infty} E[\zeta^2(I_T(\omega))] = \int_0^\infty \zeta^2(f(\omega)x) e^{-x} dx$ (see equation (3.12)).

4. Applications

Now we shall apply the CLT 3.1 to our specific situations. First, we recall the situation of TG, using his notation with $\Lambda_T(\omega) = \Psi(\omega)$ (i.e. $\gamma_T = T$) for any continuous function Ψ, and $\zeta(x) = \mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}}$, for an arbitrary function Φ, such that $\zeta(x)$ fulfills Assumptions 2 and 3, where $\mathcal{L}^{-1}\{G(u)\}_{\{x\}}$ denotes the Laplace inverse transform of $G(u)$ at argument x. Note that

$$\mathcal{L}\{F(x)\}_{\{u\}} = \int_0^\infty F(x)\exp\{-ux\}dx,$$

denotes the Laplace transform of $F(x)$ at argument u, whereas the inverse writes as

$$\mathcal{L}^{-1}\{G(u)\}_{\{x\}} = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} G(u)\exp\{ux\}du,$$

where σ is greater than the abscissa of absolute convergence (cf. TG, p. 75). Note that this transform is used to introduce asymptotic unbiasedness of the resulting estimator: With $\mathcal{L}[\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}}]_{\{u\}} = \Phi(1/u)1/u$, for $u \in (0,\infty)$, and $\zeta(I_T(\omega)) = \mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{I_T(\omega)\}}$,

$$\begin{aligned}\lim_{T\to\infty} E[\zeta(I_T(\omega))] &= \int_0^\infty \mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}} f(\omega)^{-1}\exp\{-f(\omega)^{-1}x\}dx \\ &= f(\omega)^{-1}\cdot\mathcal{L}[\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}}]_{\{f(\omega)^{-1}\}} = \Phi(f(\omega))\end{aligned}$$

(cf. also TG, Lemma 1).

With this, $H_T = \int_{-\pi}^{\pi}\Psi(\omega)\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{I_T(\omega)\}}d\omega$ is a consistent and asymptotic normal estimator for $H = \int \Psi(\omega)\Phi(f(\omega))d\omega$, and the resulting CLT (which, for the non-tapered case, is Taniguchi's Theorem 2) states as

THEOREM 4.1.

$$T^{1/2}\int_{-\pi}^{\pi}\Psi(\omega)[\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{I_T(\omega)\}} - \Phi(f(\omega))]d\omega \xrightarrow{D} \mathcal{N}(0,V)$$

with asymptotic variance

$$\begin{aligned}V = &\int_{-\pi}^{\pi}\Psi^2(\omega)\\ &\cdot\left\{\int_0^\infty[\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}}]^2 f(\omega)^{-1}\exp\{-f(\omega)^{-1}x\}dx - \Phi^2(f(\omega))\right\}d\omega.\end{aligned}$$

Some examples for useful Φ-functions, which satisfy Assumptions 2 and 3, are $\Phi(x) = x^n$, $0 < n < \infty$, where $\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}} = (\Gamma(n+1))^{-1}x^n$, or $\Phi(x) = \log x$, where $\mathcal{L}^{-1}\{\Phi(1/t)1/t\}_{\{x\}} = \log \alpha x$, with $\alpha = \exp\gamma$ ($\gamma \approx 0.57721$, Euler's constant), which are Taniguchi's examples 1 and 2, respectively (for further details see therein).

Second, we turn to the situation of peak-insensitive kernel spectral estimation, as introduced in vS:

There we are dealing with a nonparametric spectral estimator $f_T(\alpha)$ for a fixed $\alpha \in \Pi$, which is defined as the root, pointwise in α, of the following equation in $s > 0$

$$H_T(\alpha, s) := T^{-1} \sum_{k=-N}^{N} K_b(\alpha - \omega_k)\Psi\left(\frac{I_T(\omega_k)}{s} - 1\right) = 0.$$

Here $K_b(\vartheta) := b^{-1}K(\vartheta/b)$ with a smooth kernel function K with compact support, with smoothing parameter (bandwidth) $b = b_T \to 0$ and $Tb \to \infty$, as $T \to \infty$, and Ψ is some smooth bounded function with $\int_0^\infty \Psi(x-1)e^{-x}dx = 0$.

In order to show asymptotic normality of this estimator $f_T(\alpha)$ the main step is to prove a CLT for $H_T(\alpha) = H_T(\alpha, f(\alpha))$ (cf. vS, Theorem 3.4) in the tapered case. Then, it is a standard technique to carry over to $f_T(\alpha)$, by e.g. the δ-method (i.e. expanding $H_T(\alpha, f_T(\alpha))$ around $H_T(\alpha, f(\alpha))$ and using stochastic convergence of the denominator in the resulting ratio for $f_T(\alpha) - f(\alpha)$). To match with the notation here, $\Lambda_T(\omega) = (2\pi)^{-1}K((\alpha - \omega)/b_T)/b_T$ (i.e. $\eta_T = b_T^{-1}$ and $\gamma_T = Tb_T$). Further, $\zeta(x) = \Psi(x/f(\alpha) - 1)$, such that, by the above assumption on Ψ, $B(\zeta(f(\alpha))) = \lim_{T\to\infty} E\zeta(I_T(\alpha)) = 0$. So

$$H_T = H_T(\alpha) = (2\pi)^{-1} \int_{-\pi}^{\pi} b_T^{-1}K((\alpha - \omega)/b_T)\Psi(I_T(\omega)/f(\alpha) - 1)d\omega$$

and $H = \lim_{T\to\infty} EH_T(\alpha) = (2\pi)^{-1} \int K(\beta)d\beta \cdot \int \Psi(x-1)e^{-x}dx = 0$, as with (3.9), $B(\zeta(f(\omega))) = \int_0^\infty \zeta(f(\omega)x)e^{-x}dx = \int_0^\infty \Psi(f(\omega)x/f(\alpha) - 1)e^{-x}dx$, and because $\int_{-\pi}^{\pi} K_b(\alpha - \omega)B(\zeta(f(\omega)))d\omega$ tends to $\int K(\beta)d\beta \cdot \int \Psi(f(\alpha)x/f(\alpha) - 1)e^{-x}dx = 0$ (note that $K_b(\vartheta)$ is an approximate convolution identity). We finally end up with an asymptotically unbiased estimator $f_T(\alpha)$, if in addition,

$$EH_T(\alpha) = o(\gamma_T^{-1/2}) = o((Tb_T)^{-1/2}).$$

For this it is sufficient to assume a kernel K of second order, a twice continuously differentiable spectrum f and the existence of a Lipschitz Ψ', together with a common bandwidth condition in kernel estimation theory, namely $Tb^5 \to 0$ (see vS, Theorem 3.2 and Proposition 3.5).

This leads to the following resulting CLT (which, for the non-tapered case, is Theorem 3.4 in vS):

THEOREM 4.2. $(Tb)^{1/2} \cdot H_T(\alpha, f(\alpha)) \xrightarrow{D} \mathcal{N}(0, V)$, *with asymptotic variance* $V = (2\pi)^{-1} \int_{-\pi}^{\pi} K^2(\beta)d\beta \cdot \int_0^\infty \Psi^2(x-1)e^{-x}dx$.

It remains to note that, in vS a slightly different assumption on the underlying process $\{X_t\}$ was given, where X was assumed to be a general linear process of the form

$$X_t = \sum_{k=-\infty}^{\infty} a_k \epsilon_{t-k}, \qquad -\infty < t < \infty,$$

where ϵ_t, $-\infty < t < \infty$, are i.i.d. random variables with mean zero and finite variance σ_ϵ^2, and where $\sum |ka_k| < \infty$. But this is equivalent to Assumption 1, as, on one hand, Assumption 1 is met by any general linear process of this kind, and on the other hand, under Assumption 1 $\{X_t\}$ admits a representation as a (one-sided) linear process of the above form. Of course, for the CLT 4.2 to hold, the ϵ_t have to be Gaussian.

5. Appendix: Proofs

PROOF OF LEMMA 3.1. First, we show that

$$\operatorname{var}\{z_T - H_T\} = o(\gamma_T^{-1}), \tag{5.1}$$

which is

$$\operatorname{Cov}\left\{ \int_{-\pi}^{\pi} G_T(\lambda)d\lambda - T^{-1} \sum_{j=-N}^{N} G_T(\omega_j), \right.$$
$$\left. \int_{-\pi}^{\pi} G_T(\mu)d\mu - T^{-1} \sum_{k=-N}^{N} G_T(\omega_k) \right\} = o(\gamma_T^{-1}),$$

with $G_T(\lambda) := \Lambda_T(\lambda)\zeta(I_T(\lambda))$. This can be shown by arguments similar to Dahlhaus ((1983), Lemmas 4 and 6), and making use of the properties of $(H_{2,T})^{-1} \cdot |H_2^T(\alpha)|^2$ as an approximate identity (see Lemma 3 therein). The main step is to show that each of the four resulting covariance terms tends to the same limiting quantity,

$$V = \int_{-\pi}^{\pi} \Lambda^2(\omega)\{V(\zeta(f(\omega))) - B^2(\zeta(f(\omega)))\}d\omega,$$

which is the asymptotic variance, as in the proof of Proposition 3.2. Secondly,

$$EH_T - H = o(\gamma_T^{-1/2}). \tag{5.2}$$

A continuous analogue to equation (3.10) makes use of the so-called $L^{(T)}$-function, introduced by Dahlhaus (1983): Let

$$L_0^{(T)}(\omega) := \begin{cases} T & \text{for} \quad |\omega| \le T^{-1} \\ |\omega|^{-1} & \text{for} \quad T^{-1} < |\omega| \le \pi. \end{cases}$$

Then, with $|H_2^T(\omega)| \le K L_0^{(T)}(\omega)$,

$$(H_{2,T})^{-2} \int_{-\pi}^{\pi} \Lambda_T(\omega)|H_2^T(\omega)|^2 d\omega \le (H_{2,T})^{-2} \cdot \text{const.}\, \eta_T T = O(\gamma_T^{-1}) = o(\gamma_T^{-1/2}),$$

as $\int_{-\pi}^{\pi} |L_0^{(T)}(\omega)|^2 d\omega \le \text{const.}\, T$, with Lemma 1 of Dahlhaus (1983). The proof ends by noting that (5.2), together with Proposition 3.1, yields $E[z_T - H_T] = o(\gamma_T^{-1/2})$, and this is, together with (5.1) the assertions of Lemma 3.1. □

LEMMA 5.1. *Let j, $k \in \{j_i\}_{i=1,\dots,m}$, and let $H_2^T(\omega) := \sum_{t=1}^T h_t^2 \exp\{-i\omega t\}$.*

Let $F(\lambda,\mu) := f(\lambda)^{-1/2} f(\mu)^{-1/2} f_2(\lambda,\mu)$, where $f_2(\lambda,\mu)$ denotes the second order cumulant spectrum as introduced in Brillinger ((1981), equation (2.6.2)). Note that for $\lambda = \pm\mu$, $f_2(\lambda,\mu)$ equals the common spectral density $f_2(\lambda)$. Then, up to remainders which are of order $O((H_{2,T})^{-1}) = O(T^{-1})$, uniformly in j and k,

$$\begin{aligned}
\operatorname{Cov}(A_j, A_k) &= (H_{2,T})^{-1} \cdot F(\omega_j,\omega_k)\operatorname{Re}\{H_2^T(\omega_j+\omega_k) + H_2^T(\omega_j-\omega_k)\},\\
\operatorname{Cov}(A_j, B_k) &= (H_{2,T})^{-1} \cdot F(\omega_j,\omega_k)\operatorname{Im}\{H_2^T(\omega_j+\omega_k) - H_2^T(\omega_j-\omega_k)\},\\
\operatorname{Cov}(B_j, A_k) &= (H_{2,T})^{-1} \cdot F(\omega_j,\omega_k)\operatorname{Im}\{H_2^T(\omega_j-\omega_k) + H_2^T(\omega_j+\omega_k)\},\\
\operatorname{Cov}(B_j, B_k) &= (H_{2,T})^{-1} \cdot F(\omega_j,\omega_k)\operatorname{Re}\{H_2^T(\omega_j-\omega_k) - H_2^T(\omega_j+\omega_k)\}.
\end{aligned}$$

In particular, as $F(\omega_j,\omega_j) = 1$, and $H_2^T(0) = H_{2,T}$,

$$\begin{aligned}
\operatorname{Cov}(A_j, A_j) &= 1 + (H_{2,T})^{-1} \cdot \operatorname{Re}\{H_2^T(2\omega_j)\} + O(T^{-1}),\\
\operatorname{Cov}(B_j, B_j) &= 1 - (H_{2,T})^{-1} \cdot \operatorname{Re}\{H_2^T(2\omega_j)\} + O(T^{-1}), \qquad \text{and}\\
\operatorname{Cov}(A_j, B_j) &= (H_{2,T})^{-1} \cdot \operatorname{Im}\{H_2^T(2\omega_j)\} + O(T^{-1}),
\end{aligned}$$

where for $\omega_k = -\omega_j$ analogous expressions are holding.

PROOF OF LEMMA 5.1. We mainly use Brillinger ((1981), Theorem 4.3.2):

$$\operatorname{Cov}(d_T(\lambda), d_T(\mu)) = H_2^T(\lambda+\mu) f_2(\lambda) + O(1),$$

where the remainder is uniform in λ, μ. The assertions of Lemma 5.1 follow immediately as, e.g.,

$$\begin{aligned}
&\operatorname{Cov}(A_T(\omega_j), A_T(\omega_k))\\
&\quad = 2 f(\omega_j)^{-1/2} f(\omega_k)^{-1/2} (H_{2,T})^{-1} \cdot \operatorname{Cov}(\operatorname{Re} d_T(\omega_j), \operatorname{Re} d_T(\omega_k)),
\end{aligned}$$

and $\operatorname{Cov}(\operatorname{Re} d_T(\omega_j), \operatorname{Re} d_T(\omega_k)) = \operatorname{Cov}(d_T(\omega_j), d_T(\omega_k)) + \operatorname{Cov}(d_T(\omega_j), d_T(-\omega_k)) + \operatorname{Cov}(d_T(-\omega_j), d_T(\omega_k)) + \operatorname{Cov}(d_T(-\omega_j), d_T(-\omega_k))$. □

LEMMA 5.2. (a) *$|(H_{2,T})^{-1} H_2^T(\omega_j - \omega_k)| = O(|j-k|^{-1})$ for $j \neq k$ and $|(H_{2,T})^{-1} H_2^T(\omega_j + \omega_k)| = O(|j+k|^{-1})$ for $j \neq -k$,*

$$\textit{in particular} \quad |(H_{2,T})^{-1} H_2^T(\omega_j)| = O(|j|^{-1}) \quad \textit{for} \quad j \neq 0.$$

Moreover, with Assumption 4 holding, i.e. $\rho_T = T^{-\beta}$ with $0 < \beta < 1$:
(b) *$|(H_{2,T})^{-1} H_2^T(\omega_j \pm \omega_k)| = O(|j \pm k|^{-1} \wedge T^{-\beta})$ for $j \neq \mp k$.*

PROOF OF LEMMA 5.2. (a) For all assertions it is sufficient to proceed as follows: With $j \neq 0$,

$$|H_2^T(\omega_j)| \le \left| \sum_{t=1}^T (h_t^2 - h_{t+1}^2) \sum_{u=1}^t \exp\{-i\omega_j u\} \right| \qquad (\text{with } h_{T+1} := 0)$$

$$\leq \left| \sum_{t=1}^{T} (h_t^2 - h_{t+1}^2) \exp\{-i\omega_j\}(1 - \exp\{-i\omega_j t\})/(1 - \exp\{-i\omega_j\}) \right|$$
$$\leq \operatorname*{var}_x h^2(x)\, 2\, |2\sin(\omega_j/2)|^{-1}$$
$$\Big(\text{where } \operatorname*{var}_x \text{ denotes the total variation}\Big)$$
$$\leq C_h \pi / |\omega_j| \qquad (\text{as } \sin|\alpha| \geq 2|\alpha|/\pi \text{ for } |\alpha| \leq \pi/2)$$
$$= T C_h |2j|^{-1},$$

and the assertion follows, as $H_{2,T} \sim T$.

(b) As extension of part (a) the modification for the proof of (b), showing $|(H_{2,T})^{-1} H_2^T(\omega_j)| = O(|j|^{-1} \wedge T^{-\beta})$ for $j \neq 0$, runs as follows: Let $T' := \rho_T T = T^{1-\beta}$. First we have to show that still $H_{2,T} = c_h T$ (i.e. $H_{2,T} \sim T$): With (2.1) and using the symmetry of v,

$$\begin{aligned} H_{2,T} &= \sum_{t=1}^{T} h_\rho^2(t/T) \\ &= \sum_{t=1}^{T/2} u^2(t/T') 1_{[0,\rho/2)}(t/T) + \sum_{t=T/2+1}^{T-T'/2} 1_{[\rho/2,1-\rho/2]}(t/T) \\ &\quad + \sum_{t=1}^{T/2} v^2(t/T') 1_{[0,\rho/2)}(t/T) \\ &= c_u T'/2 + T - T' + c_v T'/2 = T + c_h' T' = T(1 + \text{const.}\, \rho_T) = c_h T, \end{aligned}$$

with $c_h = 1 + O(T^{-\beta})$. Second, again using the symmetry of both v and $\exp\{-i\omega_j t\}$,

$$\begin{aligned} H_2^T(\omega_j) &= \sum_{t=1}^{T} h_\rho^2(t/T) \exp\{-i\omega_j t\} \\ &= \sum_{t=1}^{T'/2} u^2(t/T') \exp\{-i\omega_j t\} + \sum_{t=1}^{T'/2} v^2(t/T') \exp\{i\omega_j t\} \\ &\quad + \sum_{t=T'/2+1}^{T/2} (\exp\{-i\omega_j t\} + \exp\{i\omega_j t\}) \\ &= (1) + (2). \end{aligned}$$

Calculating (2) we proceed with

$$\begin{aligned} &\sum_{t=T'/2+1}^{T/2} (\exp\{-i\omega_j t\} + \exp\{i\omega_j t\}) \\ &\quad = \sum_{t=1}^{T} \exp\{-i\omega_j t\} - \sum_{t=1}^{T'/2} \exp\{-i\omega_j t\} - \sum_{t=1}^{T'/2} \exp\{i\omega_j t\}. \end{aligned}$$

First, $\sum_{t=1}^{T} \exp\{-i\omega_j t\} = 0$, and secondly

$$\left|\sum_{t=1}^{T'/2} \exp\{-i\omega_j t\}\right| \le \min\{(\pi/4)T', T|2j|^{-1}\}, \qquad \text{as :}$$

$$\sum_{t=1}^{T'/2} \exp\{-i\omega_j t\} = \exp\{-i\omega_j\}/(1-\exp\{-i\omega_j\}) \cdot (1-\exp\{-i\omega_j T'/2\}),$$

where $|\exp\{-i\omega_j\}/(1-\exp\{-i\omega_j\})| \le T|4j|^{-1}$, and $|(1-\exp\{-i\omega_j T'/2\})| \le \min\{\pi\rho_T|j|, 2\}$. The same holds for $\sum_{t=1}^{T'/2} \exp\{i\omega_j t\}$, such that $|(2)| \le \min\{(\pi/2)T', T|j|^{-1}\}$. For (1) we can use nearly the same estimates which can be derived analogously to part (a):

$$\left|\sum_{t=1}^{T'/2} u^2(t/T') \exp\{-i\omega_j t\} + \sum_{t=1}^{T'/2} v^2(t/T') \exp\{i\omega_j t\}\right|$$
$$\le \operatorname*{var}_x h^2(x) \cdot \min\{(\pi/2)T', T|j|^{-1}\}. \qquad \square$$

Remark 5.1. By Lemma 5.2, the off-diagonal elements of the i-th diagonal block of Σ, i.e., the elements of the i-th diagonal block of $\Delta = I - \Sigma$, are bounded from above by some $C_{j_i} = O(|j_i|^{-1})$. Moreover, the bounds for the (i,j)-th off-diagonal block elements of Σ are $C_{j_i j_k} = O(|j_i - j_k|^{-1} + |j_i + j_k|^{-1})$, both up to terms of order $O(T^{-1})$.

Summarizing, under the additional Assumption 4, the off-diagonal elements of Σ are bounded from above by some $C(\omega_{j_i}, \omega_{j_k})$ which, for $j, k \in \{j_i\}_{i=1,\ldots,m}$, are of order

$$O([|j+k|^{-1}(1-\delta_{j,-k}) + |j-k|^{-1}(1-\delta_{j,k})] \wedge T^{-\beta}) + O(T^{-1}).$$

This is, with $\beta < 1$, the essential difference to the non-tapered situation of TG, where these elements are of order $O(T^{-1})$, uniformly in ω_{j_i}, ω_{j_k}.

PROOF OF LEMMA 3.2. We want to show that for the remainder $R_K := \sum_{k\ge K} \Delta^k$ of the expansion (3.2), the following estimate holds:

$$|(\underline{\boldsymbol{y}}, R_K \underline{\boldsymbol{y}})| \le \|\underline{\boldsymbol{y}}\|^2 |\lambda_{\max}(R_K)| \le \|\underline{\boldsymbol{y}}\|^2 \lambda^K, \tag{5.3}$$

where $\lambda := |\lambda_{\max}(\Delta)|$ denotes the modulus of the largest eigenvalue $\lambda_{\max}$ of Δ, for which, under Assumption 4, $\lambda = O(T^{-\beta})$, i.e. $\lambda^K = O(T^{-K\beta})$. Note that, by the spectral decomposition for a symmetric matrix A:

$$\sup_{\boldsymbol{x}}((\boldsymbol{x}, A\boldsymbol{x})/\|\boldsymbol{x}\|^2) = \lambda_{\max}(A) \qquad \text{for} \qquad \boldsymbol{x} \in \boldsymbol{R}^n.$$

Thus, it is sufficient to show that

$$|\lambda_{\max}(R_K)| \le \lambda^K. \tag{5.4}$$

This holds due to well-known expansion theory for geometric series, applied to matrices with eigenvalues bounded above from unity:

$$\lambda_{\max}(R_K) = \lambda_{\max}((I-\Delta)^{-1}\{I-(I-\Delta^K)\}) = \lambda_{\max}(\Sigma^{-1})\lambda_{\max}(\Delta^K),$$

where $\lambda_{\max}(\Sigma^{-1}) = O(1)$ and $|\lambda_{\max}(\Delta^K)| \le \lambda^K$. Finally, $\lambda = O(T^{-\beta})$, as $\lambda^2 \le \sum_{i,k=1}^{m}\sum_{r,s=1}^{2}([\Delta]_{i,k})_{rs}^2$, where $[\Delta]_{i,k}$ denotes the (i,k)-th block of dimension 2 of Δ, with $([\Delta]_{i,k})_{rs} = O(T^{-\beta})$. □

At this place we give an additional estimate for the individual elements of the matrix N:

LEMMA 5.3. (a) $(\underline{\boldsymbol{y}}, N\underline{\boldsymbol{y}}) = O(\|\underline{\boldsymbol{y}}\|^2 T^{-\beta})$.

(b) *The elements of the (i,k)-th off-diagonal block of N are bounded from above by*

$$C_{i,k}^{[N]} := M_{i,k}(K,m)(|j_i - j_k|^{-1} + |j_i + j_k|^{-1}) \wedge CT^{-\beta} + O(T^{-1}), \tag{5.5}$$

with a constant $M_{i,k}(K,m)$ that depends on both K and the dimension $(2m)$ of the matrices under consideration.

PROOF OF LEMMA 5.3. (a) is a direct consequence of Lemma 3.2. For part (b) we first note that the off-diagonal blocks of N_1 are the ones of Σ (where we derived this bounds in Lemmas 5.1 and 5.2). Second, it can be shown by matrix multiplication and induction on k, that the same kind of bounds holds for each N_k, $k \ge 2$: By Remark 5.1, the elements of the i-th diagonal block of Δ are bounded from above by $C_{j_i} = O(|j_i|^{-1})$. The bounds for the (i,k)-th off-diagonal block elements of Δ are $C_{j_i j_k} = O(|j_i - j_k|^{-1} + |j_i + j_k|^{-1})$, both up to terms of order $O(T^{-1})$. In addition, both kind of elements are bounded from above by some $O(T^{-\beta})$, due to Assumption 4. □

Further, we give some summation properties that help to proceed with the remainders of $\mathrm{cum}_L\{z_T\}$:

LEMMA 5.4.

$$\sum_{(j_1,\dots,j_L)\in\Pi_m} \alpha_{j_1}\cdots\alpha_{j_L} = O(\gamma_T^{m-L}) \qquad (1 \le m \le L). \tag{5.6}$$

Let $2 \le m \le L$. For even m,

$$\sum_{(j_1,\dots,j_L)\in\Pi_m} \alpha_{j_1}\cdots\alpha_{j_L}\cdot(C_{j_{i_1}j_{k_1}})^2\cdots(C_{j_{i_{m/2}}j_{k_{m_2}}})^2 = O(\gamma_T^{m/2-L}T^{-(m/2)\beta}), \tag{5.7}$$

where $C_{j_{i_\nu}j_{k_\nu}} := C_{i_\nu,k_\nu}^{[N]}$, $\nu = 1,\dots,m/2$ (see (5.5)), and where i_ν and k_ν are elements of $\{1,\dots,m\}$, which are mutually different from each other, leading to a sum only over mutually different indices $(j_1,\dots,j_L)\in\Pi_m$. For odd m,

$$\sum_{(j_1,\dots,j_L)\in\Pi_m} \alpha_{j_1}\cdots\alpha_{j_L}\cdot(C_{j_{i_1}j_{k_1}})^2\cdots(C_{j_{i_{m'}}j_{k_{m'}}})^2 = O(\gamma_T^{(m-1)/2-L}T^{-(m'/2)\beta}), \tag{5.8}$$

with $m' := (m+1)/2$ and the indices i_u and k_u, $1 \le u \le m'$, being different from each other.

PROOF OF LEMMA 5.4. First we show (5.6): For $m = 1$ (i.e. $j_1 = \cdots = j_L$),

$$\sum_{(j_1,\ldots,j_L)\in\Pi_1} \alpha_{j_1}\cdots\alpha_{j_L} \le \gamma_T^{-L} \sum_{j=-\gamma_T/2}^{\gamma_T/2} M_j = O(\gamma_T^{-(L-1)}),$$

as $\alpha_j = O(\gamma_T^{-1})$ uniformly in j, and where M_j denotes a constant which includes the uniformly bounded parts of the α_{j_i} which do not depend on T. For $1 < m \le L$ we have $j_1 = \cdots = j_{p_1}$ up to $j_{L-p_m+1} = \cdots = j_L$:

$$\sum_{(j_1,\ldots,j_L)\in\Pi_m} \alpha_{j_1}\cdots\alpha_{j_L} = O(\gamma_T^{-(p_1-1)})\cdots O(\gamma_T^{-(p_m-1)}) = O(\gamma_T^{-(L-m)}),$$

as we can repeat the same argumentation as for $m = 1$: In each of the m groups of p_i $(1 \le i \le m)$ equal indices we proceed with p_i instead of L, to derive an upper bound of order $O(\gamma_T^{-(p_i-1)})$ for each group of them. Secondly, we restrict to prove (5.7) for the case of $L = m = 2$, i.e. we show, with $j \ne \pm k$,

$$\sum_{j=-N}^{N}\sum_{k=-N}^{N} \alpha_j\alpha_k(C_{jk})^2 = O(\gamma_T^{-1}T^{-\beta}). \tag{5.9}$$

The assertions for $m > 2$ can be shown quite analogously.

It remains to show (5.9): Without loss of generality, by Lemma 5.3(b), we estimate $C_{jk} = \min\{O(|j-k|^{-1}), O(T^{-\beta})\}$ up to terms of higher order, due to reasons of symmetry. For $\gamma_T \ge T^\beta$ (otherwise (5.9) is fulfilled trivially as the sum is always of order $O(T^{-2\beta})$),

$$\begin{aligned}\sum_{j\ne k=-N}^{N} \alpha_j\alpha_k(C_{jk})^2 \le{}& M_1\gamma_T^{-2} \sum_{j=-\gamma_T/2}^{\gamma_T/2}\ \sum_{|j-k|>T^\beta}^{\gamma_T} |j-k|^{-2} \\ &+ M_2\gamma_T^{-2} \sum_{j=-\gamma_T/2}^{\gamma_T/2}\ \sum_{|j-k|\le T^\beta} T^{-2\beta},\end{aligned}$$

where M_1 and M_2 are constants used analogously to the proof of (5.6). The first term of the sum behaves like $O(\gamma_T^{-1}) \cdot \sum_{u>T^\beta}^{\gamma_T} |u|^{-2} = O(\gamma_T^{-1}T^{-\beta})$, where the second one like $O(\gamma_T^{-1}) \cdot O(T^\beta) \cdot O(T^{-2\beta})$. This proves (5.9). □

PROOF OF LEMMA 3.3.

$$\begin{aligned} i^L \operatorname{cum}\{\zeta_{j_1},\ldots,\zeta_{j_L}\}_m &= \partial^L/\partial\underline{t}\log(\chi(\underline{t})) \\ &= \partial^L/\partial\underline{t}\log(L_0(\underline{t})) + \partial^L/\partial\underline{t}\log(1+x(\underline{t})) + RC_K, \end{aligned}$$

with (3.5) and where, as in (3.5), RC_K is bounded by some $MT^{-K\beta}$ with an integrable constant $M = M(\|\underline{y}\|^2)$, which does not depend on $\underline{t}$. Further, with $L_0^{(i)}(t_{(i)})$ as in (3.5),

$$\begin{aligned}(5.10) \qquad \partial^L/\partial\underline{t}\log(L_0(\underline{t})) &= \partial^L/\partial\underline{t}\log\left(\prod_{i=1}^m L_0^{(i)}(t_{(i)})\right) \\ &= \sum_{i=1}^m \partial^L/\partial\underline{t}\log(L_0^{(i)}(t_{(i)})) = 0.\end{aligned}$$

(Note that, for $m > 1$, no $L_0^{(i)}(t_{(i)})$ depends on *all* of the t_j, $j = 1, \ldots, L$.) □

PROOF OF LEMMA 3.4. With $x(\underline{t}) = L_0(\underline{t})^{-1}L_1(\underline{t})$ and the definition of $L_1(\underline{t})$, $x(\underline{t}) = O(T^{-\beta})$ follows immediately by $(\underline{y}, N\underline{y}) = O(\|\underline{y}\|^2 T^{-\beta})$ (see Lemma 5.3). □

PROOF OF LEMMA 3.5. Note that in $L_1(\underline{t})$ it is sufficient only to consider terms with even l due to symmetry arguments, which can be seen by straightforward calculations:

$$(5.11) \qquad \int_{R^{2m}} \prod_{i=1}^m Q_i(t_{(i)};\zeta_i)(\underline{y}, N\underline{y})^l \prod_{i=1}^m (\exp -(\boldsymbol{y}_i, H_i\boldsymbol{y}_i)/2)d\underline{y} = 0$$

for all odd l. Now, for even l, each $(\underline{y}, N\underline{y})^l$ leads to terms of a sum where for $l < m$, any of the resulting terms of this sum is of one of the following two kinds: either there is at least one of the $\{\boldsymbol{y}_i\}_{i=1,\ldots,m}$ missing, such that factorization leads to terms which cancel, similarly to the leading term (5.10) of $\text{cum}\{\zeta_{j_1}, \ldots, \zeta_{j_L}\}_m$, i.e. $\partial^L/\partial\underline{t}\log(L_0(\underline{t}))$. Or, secondly, if no $\boldsymbol{y}_i$ is missing, which may be the case only for $m/2 \le l < m$, then the resulting integral expressions are zero due to the same symmetry arguments as they were given in (5.11) above (i.e. for odd l). □

PROOF OF LEMMA 3.6. It is sufficient to consider $l = m$ (in case of even m). As in the proof of (5.9) we estimate $\boldsymbol{Z}_{ik} := (\boldsymbol{y}_i, [N]_{i,k}\boldsymbol{y}_k)_{i,k=1,\ldots,m}$, up to terms of higher order, using Lemma 5.3(b):

$$(\boldsymbol{Z}_{ik})^2 \le c(\boldsymbol{y}_i, \boldsymbol{y}_k)(C_{i,k}^{[N]})^2 = \min\{O(|j_i - j_k|^{-2}), O(T^{-2\beta})\},$$

with $c(\boldsymbol{y}_i, \boldsymbol{y}_k) := [\sum_{r,s=1}^2 |y_{ir}y_{ks}|]^2$, and as each $|([N]_{i,k})_{rs}| \le C_{i,k}^{[N]}$, by Lemma 5.3(b). By the argumentation in the proof of Lemma 3.5, for $(\underline{y}, N\underline{y})^m$ we only have to consider terms which are of the described form

$$\sum_{i_1 \ne k_1}^m \cdots \sum_{i_{m/2} \ne k_{m/2}}^m (\boldsymbol{Z}_{i_1 k_1})^2 \cdots (\boldsymbol{Z}_{i_{m/2} k_{m/2}})^2,$$

where the summation is over those indices that fulfill the restrictions of Lemma 3.6:

$$(\boldsymbol{Z}_{i_1k_1})^2\cdots(\boldsymbol{Z}_{i_{m/2}k_{m/2}})^2 \le c(\underline{\boldsymbol{y}})(C^{[N]}_{i_1,k_1})^2\cdots(C^{[N]}_{i_{m/2},k_{m/2}})^2$$
$$= O(|j_{i_1}-j_{k_1}|^{-2}\cdots|j_{i_{m/2}}-j_{k_{m/2}}|^{-2}\wedge T^{-m\beta}),$$

where $c(\underline{\boldsymbol{y}})$ is used to denote a constant that is analogous to the one defined above.

For odd m we have to replace m by $m+1$, and, after applying Lemma 3.7, we end up with the analogous bound (5.8). □

PROOF OF LEMMA 3.7. We use Lemma 3.6, together with (5.7), i.e., with $C_{j_{i_\nu}j_{k_\nu}} := C^{[N]}_{i_\nu,k_\nu}$, $\nu=1,\dots,m/2$, as in the proof of Lemma 3.6:

$$\sum_{(j_1,\dots,j_L)\in\Pi_m}\alpha_{j_1}\cdots\alpha_{j_L}\cdot(C_{j_{i_1}j_{k_1}})^2\cdots(C_{j_{i_{m/2}}j_{k_{m/2}}})^2 = O(\gamma_T^{m/2-L}T^{-(m/2)\beta}). \quad \square$$

PROOF OF LEMMA 3.8. For the terms $x(\underline{t})^n$, $n=2,\dots,K-1$, we have the same behavior of symmetry and factorization as for $n=1$. However, now, as non-vanishing terms we have to consider the following n-th powers of $L_1(\underline{t})$, for which $l_1+\cdots+l_n=m$ (for even m, and $\Sigma l_i = m+1$ for odd m):

$$(L_1(\underline{t}))^n = \left(\sum_{l=1}^{K-1}L_1^{(l)}(\underline{t})\right)^n = \sum_{l_1=1}^{K-1}\cdots\sum_{l_n=1}^{K-1}L_1^{(l_1)}(\underline{t})\cdots L_1^{(l_n)}(\underline{t}).$$

This is generalizing the case $l=m$ ($=m+1$, resp.) for $n=1$, as a product of the "subcases" $l_1=m_1,\dots,l_n=m_n$, with $m_1+\cdots+m_n=m$, again with only even l_i. □

PROOF OF LEMMA 3.9. For the first assertion note that the elements of Δ are bounded from above by $C_j = O(|j|^{-1})$ (by Remark 5.1). For the second part, let again $\lambda := |\lambda_{\max}(\Delta)|$ denote the modulus of the largest eigenvalue of Δ (as in proof of Lemma 3.2). Now

$$R_2 = \sum_{k\ge 2}\Delta^k \quad \text{with} \quad |(\underline{\boldsymbol{y}},R_2\underline{\boldsymbol{y}})| \le \|\underline{\boldsymbol{y}}\|^2|\lambda_{\max}(R_2)| \le \|\underline{\boldsymbol{y}}\|^2\lambda^2,$$

with $\lambda = O(C_j) = O(|j|^{-1})$ (cf. (5.3), (5.4) and the derivations of an upper bound for λ in the proof of Lemma 3.2). □

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