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ON SOME MULTIVARIATE GAMMA-DISTRIBUTIONS CONNECTED WITH SPANNING TREES

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Abstract. Any correlation matrix R can be mapped to a graph with edges corresponding to the non-vanishing correlations. In particular R is said to be of a "tree type" if the corresponding graph is a spanning tree. The tridiagonal correlation matrices belong to this class. If the accompanying correlation matrix R or its inverse is of a tree type, then some representations of the multivariate gamma distribution are obtained with a much simpler structure than the integral or series representations for the general case.

Key words and phrases: Multivariate gamma distribution, multivariate chi-square distribution, multivariate Rayleigh distribution.

1. Introduction

The following notations are used for matrices: If $A = (a_{ij})$ is a $p \times p$-matrix then $\dot{A}$ is defined as $A - \mathrm{Diag}(a_{11}, \dots, a_{pp})$, $\|A\|$ is the spectral norm of A, $|A|$ the determinant of A, $(a^{ij}) = A^{-1}$ and $A > 0$ means positive definiteness. A unit matrix is always denoted by I or $I_{p\times p}$. Formulas from the handbook of mathematical functions by Abramowitz and Stegun (1968) are cited by "A.S." and their number.

There are several papers on representations of the multivariate gamma distribution in the sense of Krishnamoorthy and Parthasarathy (1951) and the closely related multivariate Rayleigh and chi-square distribution. Most of the earlier work is found in the book of Miller (1964). Further papers are Miller and Sackrowitz (1967), Jensen (1970), Khatri *et al.* (1977) and Royen (1991*a*, 1991*b*, 1992).

An elementary formula for the multivariate Rayleigh density with a tridiagonal $R^{-1} = (r^{ij})$ was given by Blumenson and Miller (1963). With the univariate gamma density $g_\alpha(x) = x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ $(\alpha, x > 0)$ and the modified Bessel function

$$I_{\alpha-1}(x) = (x/2)^{\alpha-1}{}_0F_1(\alpha; x^2/4)/\Gamma(\alpha),$$

$$(1.1) \qquad {}_0F_1(\alpha; x) = \Gamma(\alpha)\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha+n)n!}$$

we obtain from their formula (2.1) the p-variate gamma density

$$(1.2)\quad f(x_1,\dots,x_p;\alpha,R)=\left(|R|\prod_{i=1}^{p}r^{ii}\right)^{-\alpha}\prod_{i=1}^{p}r^{ii}g_\alpha(r^{ii}x_i)\cdot\prod_{i<j}{}_0F_1(\alpha;r^{ij2}x_ix_j),$$

where 2α is not restricted to integer values (cf. Section 3).

No elementary formula is known for the corresponding density with a tridiagonal R. In Jensen (1970), Section 4, some series expansions are found for generalized multivariate Rayleigh distributions with several different tridiagonal correlation matrices, but unfortunately they are based on a formula for determinants of tridiagonal $m\times m$-matrices which does not hold for $m>3$ and leads to incorrect formulas for densities also in case of identical correlation matrices (cf. (4.12), (4.13) in Jensen's paper). The correct distribution function (d.f.) is contained in (3.9) as a special case.

For some multivariate simultaneous tests (or more general multiple test procedures) tail probabilities of $\max\{X_i \mid 1\le i\le p\}$ are required, where $(X_1,\dots,X_p)$ has a p-variate χ^2-distribution (or F-distribution, obtained by studentizing). E.g. the tridiagonal correlation structure is encountered in simultaneous comparisons of successive difference vectors $\theta_{i+1}-\theta_i$. Apart from multiple comparisons with a control, exact upper α-percentage points are difficult to compute. However very accurate (conservative) approximations are obtained by Bonferroni inequalities of third order containing three-variate χ^2-probabilities. Error bounds for the corresponding tail probabilities are given by sums of four-variate χ^2-probabilities. In particular, tables for the multivariate maximum range test procedure (Royen (1989, 1990)), originally obtained by a very large simulation, are computed rather accurately and more rapidly by use of analytical representations of multivariate χ^2-distributions. Among other types the tridiagonal correlation structure appears here too.

Besides the correlation matrices R with a tridiagonal R^{-1} and the one-factorial ones (i.e. $R=D+aa'$, diagonal $D>0$, real or imaginary column a) in Royen (1991a), it should be useful to know further types of R allowing much simpler integral or series representations of the multivariate gamma d.f. than the general case. In Section 3 such representations are derived for the class of the R with R^{-1} or R of a "tree type". Any covariance matrix $C_{p\times p}=(c_{ij})$ is said to be of a tree type if the graph $\mathcal{G}(C)$ with the vertices $1,\dots,p$ is a spanning tree containing the edge $[i,j]$ iff $c_{ij}\neq 0$. By definition a spanning tree is connected and has no cycles. Thus, it contains exactly $p-1$ edges and it holds for all "cyclic products" of C that

$$(1.3)\qquad c_{i_1i_2}c_{i_2i_3}\cdots c_{i_ki_1}=0\qquad(\{i_1,\dots,i_k\}\subseteq\{1,\dots,p\},\quad 3\le k\le p).$$

In particular R belongs to this class if R^{-1} or R is tridiagonal.

2. Some preliminary relations

Let $R_{p\times p} = (r_{ij})$ be any nonsingular correlation matrix and Q its standardized inverse with the elements $q_{ij} = r^{ij}/\sqrt{r^{ii}r^{jj}}$. With $T = \mathrm{Diag}(t_1, \ldots, t_p)$ the Laplace transform (L.t.) $\hat{f}(t_1, \ldots, t_p; \alpha, R)$ of the gamma density $f(x_1, \ldots, x_p; \alpha, R)$ is

$$|I+RT|^{-\alpha} = \begin{cases} \left(\prod_{i=1}^{p} z_i^{\alpha}\right) |I+\dot{R}U|^{-\alpha}, & (2.1) \\ \qquad \begin{pmatrix} z_i = (1+t_i)^{-1},\ u_i = 1 - z_i = t_i z_i \\ U = \mathrm{Diag}(u_1, \ldots, u_p) \end{pmatrix} & \\ \left(|Q|^{\alpha} \prod_{i=1}^{p} z_i^{\alpha}\right) |I+\dot{Q}Z|^{-\alpha}, & (2.2) \\ \qquad \begin{pmatrix} z_i = (1+t_i/r^{ii})^{-1},\ u_i = 1 - z_i \\ Z = \mathrm{Diag}(z_1, \ldots, z_p) \end{pmatrix}. & \end{cases}$$

For any symmetrical C and $Y = \mathrm{Diag}(y_1, \ldots, y_p)$ the Taylor series of $|I+CY|^{-\alpha} = |\sum_{n=0}^{\infty} \binom{-\alpha}{n}(CY)^n|$ is absolutely convergent for all values $|y_i| \le 1$ if $\|C\| < 1$ (cf. 2.1.17, 2.1.18 in Royen (1991b)).

We shall need the following function for $\alpha, x > 0$, $y \in \mathbb{R}$:

$$G_\alpha(x,y) = e^{-y} \int_0^x {}_0F_1(\alpha; \xi y) g_\alpha(\xi) d\xi \tag{2.3}$$

$$= e^{-y} \sum_{n=0}^{\infty} G_{\alpha+n}(x) y^n/n! \tag{2.4}$$

$$= \sum_{n=0}^{\infty} G_{\alpha+n}^{(n)}(x)(-y)^n/n! \tag{2.5}$$

$$= e^{-y} \sum_{n=0}^{\infty} {}_0F_1(\alpha+1+n; xy) g_{\alpha+1+n}(x) \tag{2.6}$$

$$= \begin{cases} e^{-x-y} \sum_{n=0}^{\infty} (\sqrt{x/y})^{\alpha+n} I_{\alpha+n}(2\sqrt{xy}), & y > 0 \\ e^{-x-y} \sum_{n=0}^{\infty} (\sqrt{-x/y})^{\alpha+n} J_{\alpha+n}(2\sqrt{-xy}), & y < 0. \end{cases} \tag{2.7}$$

The last series are suitable for $|y| > x$ (cf. Section 3 in Royen (1991a)). Here

$$G_{\alpha+n}^{(n)}(x) = \frac{d^n}{dx^n} G_{\alpha+n}(x) = g_{\alpha+1+n-1}^{(n-1)}(x) = \frac{L_{n-1}^{(\alpha)}(x)}{\binom{\alpha+n-1}{n-1}} g_{\alpha+1}(x) \tag{2.8}$$

with the generalized Laguerre polynomials $L_n^{(\alpha)}$ and

$$\begin{aligned} &|G_{\alpha+n}^{(n)}(x)| \le 2^\alpha g_{\alpha+1}(x/2) \quad (n \ge 1) && \text{(A.S.22.14.13)}, \\ &G_{\alpha+n}^{(n)}(x) = \underset{n\to\infty}{\mathrm{O}}(n^{-\alpha/2-1/4}) && \text{(A.S.13.5.14)}. \end{aligned} \tag{2.9}$$

The derivative $g_\alpha(x,y) = \frac{\partial}{\partial x} G_\alpha(x,y)$ has the L.t.

$$\hat{g}_\alpha(t,y) = z^\alpha e^{-yu} = e^{-y} z^\alpha e^{yz} \qquad (z = (1+t)^{-1}, \quad u = 1 - z = tz). \tag{2.10}$$

Also

$$\begin{aligned} g_\alpha(x,y) &= e^{-y} g_\alpha(x) {}_0F_1(\alpha; xy) \\ &= \begin{cases} e^{-x-y}(\sqrt{x/y})^{\alpha-1} I_{\alpha-1}(2\sqrt{xy}) & y > 0 \\ e^{-x-y}(\sqrt{-x/y})^{\alpha-1} J_{\alpha-1}(2\sqrt{-xy}) & y < 0 \end{cases} \end{aligned} \tag{2.11}$$

and for $y < 0$ we have the bounds (A.S.9.1.60/62)

$$|J_{\alpha-1}(2\sqrt{-xy})| \le 1 \quad (\alpha \ge 1), \qquad |{}_0F_1(\alpha; y)| \le 1 \quad (\alpha \ge 1/2) \tag{2.12}$$

and consequently

$$|e^y G_\alpha(x,y)| \le G_\alpha(x). \tag{2.13}$$

Since $|I + RT|^{-\alpha}$ is not the L.t. of a d.f. for all combinations of $R > 0$, $\alpha > 0$, some remarks on this point are needed. According to Griffiths (1984) the L.t. $|I_{p\times p} + RT|^{-1}$ $(p \ge 3)$ is infinitely divisible (i.d.) iff the elements r^{ij} of R^{-1} satisfy the condition

$$(-1)^k \cdot r^{i_1 i_2} \cdot r^{i_2 i_3} \cdots r^{i_k i_1} \ge 0 \tag{2.14}$$

for all subsets $\{i_1, \dots, i_k\} \subseteq \{1, \dots, p\}$ $(3 \le k \le p)$. Thus $|I + RT|^{-1}$ is i.d. for any non-singular correlation matrix $R_{p\times p}$ $(p \ge 3)$ with R^{-1} of a tree type because of (1.3). However, as a consequence of the following lemma, $|I + RT|^{-1}$ is not i.d. if R itself has a tree type. In this case a d.f. arises at least for integer values of 2α or $\alpha > (p-1)/2$. It should be noticed, that the multivariate gamma distribution is a marginal distribution of a $W_p(R, 2\alpha)$-Wishart distribution with a non-singular density for any not necessarily integer values $2\alpha > p - 1$ (cf. formula 2.2.6 in Siotani *et al.* (1985)).

LEMMA 2.1. *Let $R_{p\times p}$ be a non-singular correlation matrix containing at least one irreducible $k \times k$-principal matrix with $k \ge 3$ and $T_{p\times p}$ a diagonal matrix of variables. Then $|I + RT|^{-1}$ and $|I + R^{-1}T|^{-1}$ are not both infinitely divisible.*

PROOF. Without loss of generality let R be irreducible and $|I + R^{-1}T|^{-1}$ i.d. According to Theorem 1 of Bapat (1989) there exist numbers $s_i = \pm 1$ $(i = 1, \dots, p)$ with

$$s_i r_{ij} s_j \le 0 \qquad \text{for all} \quad i \ne j. \tag{2.15}$$

If in addition $|I + RT|^{-1}$ would be i.d. then there would also exist numbers $s_i^* = \pm 1$ with

$$s_i^* r_{ij} s_j^* \ge 0 \qquad \text{for all} \quad i \ne j \tag{2.16}$$

according to Corollary 1 of the cited theorem.

If there is a cyclic product $r_{i_1i_2}\cdots r_{i_ki_1}\neq 0$ with an odd value of k then (2.16) contradicts (2.15) and $|I+RT|^{-1}$ cannot be i.d. If there is no such product, then R, after suitable renumbering, must contain a 3×3-principal matrix R_3 with the correlations $r_{12}\neq 0$, $r_{23}\neq 0$ and $r_{13}=0$. The product $r_{12}^2r_{23}^2$ of the three off-diagonal cofactors of R_3 is positive. It follows from (2.14) that the marginal L.t. $|I_3+R_3T_3|^{-1}$ is not i.d. Thus $|I+RT|^{-1}$ is not i.d. again. □

3. The multivariate gamma distribution with R or R^{-1} of a tree type

Let $R_{p\times p}=(r_{ij})$ or its standardized inverse $Q=(q_{ij})$ be a correlation matrix $C=(c_{ij})$ of a tree type. In any spanning tree $\mathcal{G}(C)$ the degree d_i of i is the number of edges $[i,j]$ of $\mathcal{G}$. We define

$$K=\{k\mid d_k=1\},\qquad I=\{i\mid d_i>1\}=\{1,\dots,p\}\backslash K \tag{3.1}$$

and for any $i\in I$ the possibly empty set

$$K_i=\{k\in K\mid c_{ik}\neq 0\}. \tag{3.2}$$

Furthermore we set

$$I_1=\{i\in I\mid K_i\neq\emptyset\},\qquad I_2=I\backslash I_1,\qquad \mathcal{I}=\{(i,j)\mid i,j\in I, i<j, c_{ij}\neq 0\}. \tag{3.3}$$

The notation $\sum_{(n)}$ means a summation over all partitions $n=\sum_{(i,j)\in\mathcal{I}}n_{ij}$ or $n=\sum_{1\le i<j\le p,c_{ij}\neq 0}n_{ij}$ with non-negative integers n_{ij} and $N_i:=\sum_{j=1,c_{ij}\neq 0}^{p}n_{ij}$, $n_i:=\sum_{j\in I,c_{ij}\neq 0}n_{ij}$ ($n_{ji}:=n_{ij}$, $n_{ii}:=0$). Let I be of size m. Then the following theorem holds:

THEOREM 3.1. (a) *If Q is of a tree type then the p-variate gamma d.f. with parameter α and the accompanying correlation matrix R is given by*

$$\begin{aligned}
&F(x_1,\dots,x_p;\alpha,R) \\
&\quad=|Q|^\alpha\int_{\mathcal{X}}\prod_{k\in K}\exp(q_{ki}^2\xi_i)G_\alpha(r^{kk}x_k;q_{ki}^2\xi_i) \\
&\qquad\times\prod_{(i,j)\in\mathcal{I}}{}_0F_1(\alpha;q_{ij}^2\xi_i\xi_j)\cdot\prod_{i\in I}g_\alpha(\xi_i)d\xi_i\qquad\left(\mathcal{X}=\underset{i\in I}{\times}(0,r^{ii}x_i)\right)
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
&\quad=\frac{|Q|^\alpha}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\sum_{(n)}\prod_{(i,j)\in\mathcal{I}}\frac{q_{ij}^{2n_{ij}}}{\Gamma(\alpha+n_{ij})n_{ij}!}\cdot\prod_{i\in I}\Gamma(\alpha+n_i) \\
&\qquad\times\prod_{i\in I_2}G_{\alpha+n_i}(r^{ii}x_i) \\
&\qquad\times\prod_{i\in I_1}\int_0^{r^{ii}x_i}\prod_{k\in K_i}\exp(q_{ki}^2\xi_i)G_\alpha(r^{kk}x_k;q_{ki}^2\xi_i)g_{\alpha+n_i}(\xi_i)d\xi_i
\end{aligned}\tag{3.5}$$

$$(3.6)\qquad = \frac{|Q|^{\alpha}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{\substack{1 \le i < j \le p \\ q_{ij} \neq 0}} \frac{q_{ij}^{2n_{ij}}}{\Gamma(\alpha + n_{ij}) n_{ij}!} \times \prod_{i=1}^{p} \Gamma(\alpha + N_i) G_{\alpha + N_i}(r^{ii} x_i).$$

(b) *If R is of a tree type and 2α is a positive integer or $\alpha > (p-1)/2$ (cf. remarks following* (2.14)) *then*

$$(3.7)\quad F(x_1, \dots, x_p; \alpha, R) = \int_{\mathbb{R}_+^m} \prod_{k \in K} G_\alpha(x_k; -r_{ki}^2 y_i) \cdot \prod_{(i,j) \in \mathcal{I}} {}_0F_1(\alpha; r_{ij}^2 y_i y_j) \times \prod_{i \in I} (\sqrt{x_i / y_i})^{\alpha} J_\alpha(2\sqrt{x_i y_i}) g_\alpha(y_i) dy_i$$

$$(3.8)\qquad = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{(i,j) \in \mathcal{I}} \frac{r_{ij}^{2n_{ij}}}{\Gamma(\alpha + n_{ij}) n_{ij}!} \cdot \prod_{i \in I} \Gamma(\alpha + n_i) \times \prod_{i \in I_2} G_{\alpha + n_i}^{(n_i)}(x_i) \times \prod_{i \in I_1} \int_0^{\infty} \prod_{k \in K_i} G_\alpha(x_k; -r_{ki}^2 y_i) (\sqrt{x_i / y_i})^{\alpha} J_\alpha(2\sqrt{x_i y_i}) g_{\alpha + n_i}(y_i) dy_i$$

$$(3.9)\qquad = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{\substack{i<j \\ r_{ij} \neq 0}} \frac{r_{ij}^{2n_{ij}}}{\Gamma(\alpha + n_{ij}) n_{ij}!} \cdot \prod_{i=1}^{p} \Gamma(\alpha + N_i) G_{\alpha + N_i}^{(N_i)}(x_i).$$

Remark 1. The proof of the density formula (2.1) in Blumenson and Miller (1963) can be extended to all correlation matrices R with R^{-1} of any tree type. However, to obtain directly a proof also holding for not integer values of 2α, a power series expansion of the L.t. (2.2) with simple coefficients is derived and inverted. Then the similarity of (2.1), (2.2) is used if R itself has a tree type.

Remark 2. Integral representations were included because of their remarkable reduction of the dimension. The dimension of the integrals is here the number m of the "inner" vertices $i \in I$. If e.g. $\mathcal{G}(R_{6\times 6})$ is a spanning tree with the degrees $(1, 1, 3, 3, 1, 1)$ then (3.7) provides a bivariate integral. For integral representations of the multivariate χ^2-d.f. with the lowest possible dimension see also Royen (1993). Besides the expansion (3.8), derived from (3.7), might converge more rapidly than (3.9) as suggested by some numerical examples with $p = 4$ and identical correlations $r_{i,i+1}$.

PROOF OF THEOREM 3.1. Let $\mathcal{G}(Q)$ be a spanning tree. At first $\|\dot{Q}\| < 1$ is shown. After a suitable index permutation $d_p = 1$ and $q_{p-1,p} \neq 0$ can be assumed. The polynomial $h_p(x) = |xI + \dot{Q}| = x h_{p-1} - q_{p-1,p}^2 h_{p-2}$ is inductively recognized as an even or odd function depending on the parity of p. From $Q = I + \dot{Q} > 0$ it follows that $h_p(x) > 0$ for all $x \geq 1$ and consequently $\|\dot{Q}\| < 1$.

Now the expansion

$$(3.10) \qquad |I + \dot{Q}Z|^{-\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{\substack{i<j \\ q_{ij} \neq 0}} \frac{q_{ij}^{2n_{ij}}}{n_{ij}! \Gamma(\alpha + n_{ij})} \cdot \prod_{i=1}^{p} \Gamma(\alpha + N_i) z_i^{N_i}$$

is derived inductively. The series is convergent for all values $0 \leq z_i \leq 1$ because of $\|\dot{Q}\| < 1$. It is sufficient to verify (3.10) with $Z = I$. Replacing the q_{ij} by $\sqrt{z}_i q_{ij} \sqrt{z}_j$ the general expansion follows.

In particular we have $|I_p + \dot{Q}| = 1 - \sum_{i=1}^{p-1} q_{ip}^2$ for $q_{ip} \neq 0$ $(i = 1, \ldots, p-1)$ and (3.10) is easily established with $N_i = n_{ip}$ $(i = 1, \ldots, p-1)$, $N_p = \sum_{i=1}^{p-1} n_{ip}$. Besides, after a suitable index permutation, (3.10) is seen to hold for all $Q_{3 \times 3}$ with exactly one $q_{ij} = 0$.

Now let be $p \geq 4$. There exists a vertex $k \in I_1$ and after a suitable index permutation we can assume $2 \leq k \leq p-2$, $K_k = \{1, \ldots, k-1\}$, i.e. $q_{ik} \neq 0$, $q_{ij} = 0$ $(i < k,\ j \neq k,\ j \neq i)$ and $q_{k,k+1} \neq 0$, $q_{kj} = 0$ $(j > k+1)$. With $d = |(q_{ij})_{i,j \leq k}| = 1 - \sum_{i=1}^{k-1} q_{ik}^2$, $d_m = |(q_{ij})_{i,j \geq k+m}|$ $(m = 0, 1, 2)$ we find

$$(3.11) \qquad d_0 = d_1 - d_2 q_{k,k+1}^2, \qquad |Q| = d d_0^* = d(d_1 - d_2 d^{-1} q_{k,k+1}^2).$$

Applying (3.10) to d_0 we get with $q_{k,k+1}^2$ replaced by $q_{k,k+1}^2/d$, $N_{ik} = \sum_{j=k, j\neq i, q_{ij}\neq 0}^{p} n_{ij}$, $N_{kk} = n_{k,k+1}$ and the expansion of $d^{-\alpha - n_{k,k+1}}$ the series

$$\begin{aligned} |Q|^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \sum_{\substack{n_{ij} \geq 0 \\ k \leq i < j \leq p, q_{ij} \neq 0}} \prod_{i<j} \frac{q_{ij}^{2n_{ij}}}{n_{ij}! \Gamma(\alpha + n_{ij})} \left(\prod_{i=k}^{p} \Gamma(\alpha + N_{ik}) \right) d^{-\alpha - n_{k,k+1}} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{\substack{n_{ij} \geq 0 \\ 1 \leq i < j \leq p, q_{ij} \neq 0}} \prod_{i<j} \frac{q_{ij}^{2n_{ij}}}{n_{ij}!} \cdot \frac{\prod_{i=k}^{p} \Gamma(\alpha + N_i)}{\prod_{\substack{k \leq i < j \leq p \\ q_{ij} \neq 0}} \Gamma(\alpha + n_{ij})}, \end{aligned}$$

which yields (3.10) since $N_i = n_{ik}$ for $i < k$.

From (2.2) and (3.10) we obtain by inversion for all positive α the density

$$\begin{aligned} (3.12) \quad f(x_1, \ldots, x_p; \alpha, R) &= |Q|^{\alpha} \prod_{i=1}^{p} r^{ii} g_{\alpha}(r^{ii} x_i) \cdot \prod_{1 \leq i < j \leq p} {}_0F_1(\alpha; r^{ij2} x_i x_j) \\ &= \frac{|Q|^{\alpha}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{i<j} \frac{q_{ij}^{2n_{ij}}}{\Gamma(\alpha + n_{ij}) n_{ij}!} \\ &\quad \times \prod_{i=1}^{p} \Gamma(\alpha + N_i) r^{ii} g_{\alpha + N_i}(r^{ii} x_i). \end{aligned}$$

The series (3.6) follows by integration, (3.4) is verified by differentiation and (3.5) follows from the series expansion of the factors ${}_0F_1$ in the integrand of (3.4).

Now let $\mathcal{G}(R)$ be a spanning tree. Using (3.10) the L.t. (2.1) is seen to have for all $t_i \geq 0$ the absolutely convergent series expansion

$$\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{\substack{i<j \\ r_{ij}\neq 0}} \frac{r_{ij}^{2n_{ij}}}{\Gamma(\alpha+n_{ij})n_{ij}!} \cdot \prod_{i=1}^{p} \Gamma(\alpha+N_i) z_i^{\alpha} u_i^{N_i}. \tag{3.13}$$

From this and (2.9), (3.9) follows by termwise inversion and integration.

To prove (3.7), (3.8) a second form of the L.t. of (3.12) is derived, which implies a corresponding form of (3.13). From (3.5) we obtain with (2.10) and $S_i := \sum_{k\in K_i} q_{ik}^2 z_k$ the L.t.

$$\begin{aligned} &\frac{|Q|^{\alpha}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{(n)} \prod_{(i,j)\in\mathcal{I}} \frac{q_{ij}^{2n_{ij}}}{\Gamma(\alpha+n_{ij})n_{ij}!} \\ &\times \prod_{i\in I} \left(\prod_{k\in K_i} z_k^{\alpha} \right) \Gamma(\alpha+n_i) \\ &\times \int_0^{\infty} \exp(r^{ii} S_i x_i - t_i x_i) r^{ii} g_{\alpha+n_i}(r^{ii} x_i) dx_i. \end{aligned} \tag{3.14}$$

With $1+t_i/r^{ii} = z_i^{-1}$ and the substitution $y_i = (z_i^{-1} - S_i) r^{ii} x_i$ the last two factors of (3.14) become

$$\prod_{i=1}^{p} z_i^{\alpha} \cdot \prod_{i\in I} \left(\int_0^{\infty} y_i^{\alpha-1+n_i} \exp(-y_i) dy_i \right) z_i^{n_i} (1 - z_i S_i)^{-\alpha-n_i}. \tag{3.15}$$

Since $Q > 0$ this also holds for the principal minor array of Q with the indices $k \in K_i$. Therefore $1 - \sum_{k\in K_i} q_{ik}^2$ and $1 - z_i S_i$ are positive.

With (3.14), (3.15) we find for all values $0 < z_i \leq 1$ the L.t.

$$\begin{aligned} &|Q|^{\alpha} \prod_{i=1}^{p} z_i^{\alpha} \cdot \prod_{i\in I} (1 - z_i S_i)^{-\alpha} \\ &\times \int_{\mathbb{R}_+^m} \prod_{(i,j)\in\mathcal{I}} {}_0F_1\left(\alpha; q_{ij}^2 \frac{y_i z_i}{1 - z_i S_i} \frac{y_j z_j}{1 - z_j S_j}\right) \cdot \prod_{i\in I} g_{\alpha}(y_i) dy_i. \end{aligned} \tag{3.16}$$

If $\mathcal{G}(R)$ is a spanning tree, then with $z_i = (1+t_i)^{-1}$, $u_i = t_i z_i$ and $S_i := \sum_{k\in K_i} r_{ik}^2 u_k$ the corresponding L.t. is

$$\begin{aligned} &\prod_{i=1}^{p} z_i^{\alpha} \cdot \prod_{i\in I} (1 - u_i S_i)^{-\alpha} \\ &\times \int_{\mathbb{R}_+^m} \prod_{(i,j)\in\mathcal{I}} {}_0F_1\left(\alpha; r_{ij}^2 \frac{u_i y_i}{1 - u_i S_i} \frac{u_j y_j}{1 - u_j S_j}\right) \cdot \prod_{i\in I} g_{\alpha}(y_i) dy_i. \end{aligned} \tag{3.17}$$

The value of this expression is $|R|^{-\alpha}$ if all the u_i and z_i are replaced by 1 because of (3.16) with $z_i \equiv 1$.

Now it is shown that the L.t. of (3.7) coincides with (3.17). Because of (2.11), (2.12) the function

$$f(x_1,\dots,x_p) = \int_{\mathbb{R}_+^m} \prod_{k\in K} g_\alpha(x_k; -r_{ki}^2 y_i) \times \prod_{i\in I}(x_i^{\alpha-1}/\Gamma(\alpha))_0F_1(\alpha; -x_i y_i) \times \prod_{(i,j)\in\mathcal{I}} {}_0F_1(\alpha; r_{ij}^2 y_i y_j)\cdot \prod_{i\in I} g_\alpha(y_i)dy_i \tag{3.18}$$

is well defined. If the integrand is replaced by its absolute value then the L.t. of this function is bounded by

$$|R|^{-\alpha}\prod_{k\in K}(1+t_k)^{-\alpha}\cdot\prod_{i\in I} t_i^{-\alpha}.$$

Now, changing the order of integration, the L.t. of (3.18) with $t_i > 0$ $(i \in I)$ is

$$\int_{\mathbb{R}_+^m}\prod_{i\in I}\prod_{k\in K_i} z_k^\alpha \exp(r_{ik}^2 u_k y_i)\cdot\prod_{i\in I}\exp(-y_i/t_i)t_i^{-\alpha} \times \prod_{(i,j)\in\mathcal{I}} {}_0F_1(\alpha; r_{ij}^2 y_i y_j)\cdot\prod_{i\in I} g_\alpha(y_i)dy_i.$$

With $t_i z_i = u_i$ and $u_i^{-1} = 1 + t_i^{-1}$ this becomes

$$\prod_{i=1}^{p} z_i^\alpha \int_{\mathbb{R}_+^m}\prod_{(i,j)\in\mathcal{I}} {}_0F_1(\alpha; r_{ij}^2 y_i y_j)\cdot\prod_{i\in I}\exp(-(1-u_iS_i)y_i/u_i)u_i^{-\alpha}y_i^{\alpha-1}/\Gamma(\alpha)dy_i,$$

which is seen to coincide with (3.17) after the substitution $(1-u_iS_i)y_i/u_i \to y_i$. Thus (3.7) is obtained from (3.18) by integration and (3.8) by the power series expansion of the factors ${}_0F_1(\alpha; r_{ij}^2 y_i y_j)$. □

The simplest case in (3.7) arises if $r_{ij} = 0$ for $i, j < p$ $(i \neq j)$. Then

$$F(x_1,\dots,x_p;\alpha,R) = \int_0^\infty \prod_{k=1}^{p-1} G_\alpha(x_k; -r_{kp}^2 y)\left(\sqrt{x_p/y}\right)^\alpha J_\alpha(2\sqrt{x_p y})g_\alpha(y)dy. \tag{3.19}$$

The correlation matrix R belongs to this class iff the elements q_{ij} of the standardized inverse satisfy the relations $q_{ij} = q_{ip}q_{jp}$ $(i, j < p,\ i \neq j)$, i.e. Q is the limit case of "one-factorial" correlation matrices (q_{ij}) with $q_{ij} = a_i a_j$ $(i \neq j)$, $a_i^2 < 1$ $(i = 1,\dots,p)$, $a_p^2 \to 1$.

The gamma d.f. with a 4×4-tridiagonal R is expressed by a double integral in (3.7). The expansion (3.8) yields

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} r_{23}^n L_n(x_1, x_2; \alpha, r_{12}^2) L_n(x_4, x_3; \alpha, r_{34}^2) \tag{3.20}$$

with

$$\begin{aligned}
L_n(x_1, x_2; \alpha, r^2) &= \int_0^\infty G_\alpha(x_1; -r^2 y)(\sqrt{x_2/y})^\alpha J_\alpha(2\sqrt{x_2 y}) g_{\alpha+n}(y) dy \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+n)k!} r^{2k} G_{\alpha+k}^{(k)}(x_1) G_{\alpha+n+k}^{(n+k)}(x_2) \\
&= (1-r^2)^{-n} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+n)k!} (-r^2/(1-r^2))^k \\
&\quad \times G_{\alpha+k}(x_1) G_{\alpha+n+k}^{(n+k)}(x_2/(1-r^2)).
\end{aligned} \tag{3.21}$$

The last identities are verified again by Laplace transformation using the transformation $\tilde{z}_2 = (1 + (1-r^2)t_2)^{-1} = z_2/(1 - r^2 u_2)$. The integral is evaluated with Kummer's transformation (A.S.13.1.27) and (A.S.22.5.54).

In particular with $\alpha = 1/2$ the p-variate "tetrachoric" expansion for probabilities of rectangular regions $\times_{i=1}^p [-\sqrt{2x_i}, \sqrt{2x_i}]$ under a $N(\mathbf{0}, R)$-distribution is obtained, if the Laguerre polynomials $L_n^{(1/2)}$ are expressed by Hermite polynomials. The condition $\|\dot{R}\| < 1$ however, sufficient for absolute convergence, also holds for the classical tetrachoric expansion on any unbounded regions (cf. Section 1 and remarks following Theorem 2.1 in Royen (1991*b*)).

The expansions in the above theorem may be numerically useful at least for small values of α and p. For larger values of α the central limit theorem and the multivariate Edgeworth expansion can be applied to the approximate computation of the multivariate gamma d.f. (cf. Khatri *et al.* (1977), Section 3).

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