Ann. Inst. Statist. Math.
Vol. 46, No. 2, 351–360 (1994)

CHARACTERIZATIONS OF THE POISSON PROCESS AS A RENEWAL PROCESS VIA TWO CONDITIONAL MOMENTS

SHUN-HWA LI, WEN-JANG HUANG* AND MONG-NA LO HUANG

Institute of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan, 80424, R.O.C.

(Received July, 27, 1992; revised August 2, 1993)

Abstract. Given two independent positive random variables, under some minor conditions, it is known that from $E(X^r \mid X+Y) = a(X+Y)^r$ and $E(X^s \mid X+Y) = b(X+Y)^s$, for certain pairs of r and s, where a and b are two constants, we can characterize X and Y to have gamma distributions. Inspired by this, in this article we will characterize the Poisson process among the class of renewal processes via two conditional moments. More precisely, let $\{A(t), t \geq 0\}$ be a renewal process, with $\{S_k, k \geq 1\}$ the sequence of arrival times, and F the common distribution function of the inter-arrival times. We prove that for some fixed n and k, $k \leq n$, if $E(S_k^r \mid A(t) = n) = at^r$ and $E(S_k^s \mid A(t) = n) = bt^s$, for certain pairs of r and s, where a and b are independent of t, then $\{A(t), t \geq 0\}$ has to be a Poisson process. We also give some corresponding results about characterizing F to be geometric when F is discrete.

Key words and phrases: Characterization, exponential distribution, gamma distribution, geometric distribution, Poisson process, renewal process.

1. Introduction

Since Lukacs (1955) characterized two independent non-degenerate positive random variables X and Y to be gamma distributed by the independence of X/Y and $X+Y$, many papers have been devoted to the problem of characterizing the gamma distributions. In particular Wesolowski (1989, 1990a) replaced the independence condition of X/Y and $X+Y$ by the following regression assumptions:

$$(1.1) \quad E(X \mid X+Y) = a(X+Y) \quad \text{and} \quad E(X^2 \mid X+Y) = b(X+Y)^2,$$

and

$$(1.2) \quad E(X \mid X+Y) = a(X+Y) \quad \text{and} \quad E(X^{-1} \mid X+Y) = b(X+Y)^{-1},$$

* Support for this research was provided in part by the National Science Council of the Republic of China, Grant No. NSC 81-0208-M110-06.

respectively, where a and b are constants. The following is another similar characterization.

THEOREM 1.1. *Let X and Y be two independent non-degenerate positive random variables such that $E(X^{-2}) < \infty$ and $E(Y^2) < \infty$. If the conditions*

$$E(X^{-1} \mid X+Y) = a(X+Y)^{-1} \quad \textit{and} \quad E(X^{-2} \mid X+Y) = b(X+Y)^{-2} \tag{1.3}$$

hold for some constants a and b, then both X and Y have gamma distributions with the same scale parameter.

PROOF. The assumptions imply

$$E(Y/X \mid X+Y) = a-1 \tag{1.4}$$

and

$$E((Y/X)^2 \mid X+Y) = b-2a+1. \tag{1.5}$$

From this it is easy to obtain $(b+1)/2 > a > 1$ and $b > a^2$. Now for $\theta > 0$, let $f(\theta) = E(X^{-2}e^{-\theta X})$ and $g(\theta) = E(e^{-\theta Y})$. Then (1.4) and (1.5) imply

$$f'(\theta)g'(\theta) = (a-1)f''(\theta)g(\theta) \tag{1.6}$$

and

$$f(\theta)g''(\theta) = (b-2a+1)f''(\theta)g(\theta), \tag{1.7}$$

respectively.

Solving (1.6) and (1.7), along the lines of the proof of Wesolowski (1990a), yields

$$f(\theta) = k_1(1+k_2\theta)^{-a(a-1)/(b-a^2)} \tag{1.8}$$

and

$$g(\theta) = (1+k_2\theta)^{-(b-a)(a-1)/(b-a^2)}, \tag{1.9}$$

where $k_1 = E(X^{-2})$, $k_2 = k_3(b-a^2)/(k_1a(a-1))$ and $k_3 = E(X^{-1})$. Consequently,

$$\begin{aligned} E(e^{-\theta X}) = f''(\theta) &= \frac{k_3^2(b-a^2)}{k_1a(a-1)}(1+k_2\theta)^{-(2b-a-a^2)/(b-a^2)} \\ &= (1+k_2\theta)^{-(2b-a-a^2)/(b-a^2)}. \end{aligned} \tag{1.10}$$

Here we have used the fact that being a Laplace transform, $f''(0) = 1$, hence $k_3^2(b-a^2)/(k_1a(a-1)) = 1$. Therefore we have shown that both X and Y have gamma distributions with the same scale parameter k_2^{-1} and shape parameters

$$\frac{2b-a-a^2}{b-a^2} \quad \text{and} \quad \frac{(b-a)(a-1)}{b-a^2},$$

respectively. This completes the proof. □

But for giving the other pairs of conditional moments, the computations become very complicated. So that except (1.1), (1.2) and (1.3), we still do not have other similar conditions to characterize gamma distributions.

The following is an application of the above results. Tollar (1988) has established that if $W > 0$ and $0 < V < 1$ are two independent non-degenerate random variables, then VW and $(1-V)W$ are independent if and only if W has a $\Gamma(\alpha+\beta, \lambda)$ distribution and V has a $\mathcal{B}e(\alpha, \beta)$ distribution, for some $\alpha, \beta, \lambda > 0$. Based on the previous constant regression characterizations of the gamma distributions, instead of assuming the independence of W and V, we have the following immediate generalization.

THEOREM 1.2. *Let $W > 0$ and $0 < V < 1$ be two non-degenerate random variables such that VW and $(1-V)W$ are independent. Then W has a $\Gamma(\alpha+\beta, \lambda)$ distribution and V has a $\mathcal{B}e(\alpha, \beta)$ distribution, for some $\alpha, \beta, \lambda > 0$, if and only if one of the following three holds*:

(i) $E(W^2) < \infty$, $E(V \mid W) = a_1$ *and* $E(V^2 \mid W) = b_1$,

(ii) $E(W) < \infty$, $E(W^{-1}) < \infty$, $E(V \mid W) = a_2$ *and* $E(V^{-1} \mid W) = b_2$,

(iii) $E(W^{-2}) < \infty$, $E(V^{-1} \mid W) = a_3$ *and* $E(V^{-2} \mid W) = b_3$,

where $a_j, b_j, j = 1, 2, 3$, *are constants.*

Other characterizations of distributions by constant regression or moment's approach can be found in papers such as Laha and Lukacs (1960), Hall and Simons (1969), Shanbhag (1971), Alzaid (1990), Wesolowski (1990*b*), Li *et al.* (1992), and Pusz and Wesolowski (1992). On the other hand, Çinlar and Jagers (1973), Huang *et al.* (1993) and many others have characterized Poisson process among the class of renewal processes through some conditional expectations. More precisely, let $\{A(t), t \geq 0\}$ be a renewal process with $\{X_k, k \geq 1\}$ the sequence of inter-arrival times, $\{S_k, k \geq 1\}$ the sequence of arrival times, where $S_k = \sum_{j=1}^{k} X_j$, and let F be the common distribution function of the inter-arrival times, δ_t the current life at t, and γ_t the residual life at t. Huang *et al.* (1993) proved that, for a single positive integer n, under mild conditions, as long as $E(G(\delta_t) \mid A(t) = n) = E(G(X_1) \mid A(t) = n)$, $\forall t \geq 0$, or $E(G(\gamma_t) \mid A(t) = n) = c$, $\forall t \geq 0$, where G is a monotone function and c is a constant, then $\{A(t), t \geq 0\}$ is a Poisson process.

An immediate consequence of the previous constant regression characterizations of gamma law enables us to characterize the inter-arrival times of the renewal process to be gamma distributed by using one of the following conditions:

(i) $E(S_k^2 \mid S_n) = aS_n^2$,

(ii) $E(S_k^{-1} \mid S_n) = bS_n^{-1}$,

where k and n, $1 \leq k \leq n-1$, are two fixed integers, a and b are some constants.

Inspired by the above results, in this article, we will characterize a renewal process $\{A(t), t \geq 0\}$ to be a Poisson process, by the identities such as

$$E(S_k^r \mid A(t) = n) = at^r \quad \text{and} \quad E(S_k^s \mid A(t) = n) = bt^s, \qquad \forall t \geq 0, \tag{1.11}$$

for some fixed $1 \leq k \leq n$ and $(r, s) = (1, 2)$, $(1, -1)$ or $(-1, -2)$, where a and b are constants. Actually when $(r, s) = (1, 2)$ the right side of the latter equation in (1.11) with more general form will yield a similar result. A parallel result for δ_t will also be presented. We also give some corresponding results about characterizing F to be geometrically distributed when F is assumed to be discrete.

2. Results for the continuous case

If the renewal process $\{A(t), t \geq 0\}$ is Poisson, then given that $A(t) = n$, $S_1 \leq S_2 \leq \cdots \leq S_n$ are distributed as order statistics of n i.i.d. random variables with the common uniform distribution on $[0, t]$. From this it can be shown that $E(S_k \mid A(t) = n) = kt/(n+1)$ and $E(S_k^2 \mid A(t) = n) = k(k+1)t^2/[(n+1)(n+2)]$. Conversely, we have the following result.

THEOREM 2.1. *Let the common inter-arrival time distribution function F of the renewal process $\{A(t), t \geq 0\}$ be continuous with $F(0) = 0$ and $E(X_1^2) < \infty$. Assume for some fixed integers k and n, $1 \leq k \leq n$,*

$$E(S_k \mid A(t) = n) = at \tag{2.1}$$

and

$$E(S_k^2 \mid A(t) = n) = bt^2 + ct, \tag{2.2}$$

hold for some constants a, b and c, for every $t > 0$, whenever $P(A(t) = n) > 0$. Then

(i) $a = k/(n+1)$, $b = k(k+1)/[(n+1)(n+2)]$ *and* $c = 0$;
(ii) $\{A(t), t \geq 0\}$ *is a Poisson process.*

PROOF. From (2.1) and (2.2) we obtain for every $t > 0$,

$$\int_0^t y(F_{n-k}(t-y) - F_{n+1-k}(t-y))dF_k(y) = at(F_n(t) - F_{n+1}(t)) \tag{2.3}$$

and

$$\begin{aligned} \int_0^t y^2(F_{n-k}(t-y) - F_{n+1-k}(t-y))dF_k(y) \\ = (bt^2 + ct)(F_n(t) - F_{n+1}(t)), \end{aligned} \tag{2.4}$$

where F_n is the n-fold convolution of F with itself. Taking the Laplace transforms of both sides of (2.3) and (2.4) with respect to θ, we obtain for every $\theta > 0$,

$$(\phi^k(\theta))' \frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} = a\left(\frac{\phi^n(\theta) - \phi^{n+1}(\theta)}{\theta}\right)' \tag{2.5}$$

and

$$\begin{aligned} (\phi^k(\theta))'' \frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} \\ = b\left(\frac{\phi^n(\theta) - \phi^{n+1}(\theta)}{\theta}\right)'' - c\left(\frac{\phi^n(\theta) - \phi^{n+1}(\theta)}{\theta}\right)', \end{aligned} \tag{2.6}$$

where

$$\phi(\theta) = E(e^{-\theta X_1}) = \int_0^\infty e^{-\theta x} dF(x). \tag{2.7}$$

From (2.5), it follows

$$(2.8)\quad (1-a)(\phi^k(\theta))'\frac{\phi^{n-k}(\theta)-\phi^{n+1-k}(\theta)}{\theta} = a\phi^k(\theta)\left(\frac{\phi^{n-k}(\theta)-\phi^{n+1-k}(\theta)}{\theta}\right)'.$$

Consequently $a \neq 0, 1$. Solving (2.8) we deduce that

$$(2.9)\qquad \frac{\phi^{n-k}(\theta)-\phi^{n+1-k}(\theta)}{\theta} = K\phi^{ka^{-1}(1-a)}(\theta),$$

where $K > 0$ is a constant. After dividing the both sides of (2.9) by $\phi^{n-k}(\theta)$ and using the fact that $K = \lim_{\theta\to 0}(1-\phi(\theta))/\theta = \mu_1 = E(X_1)$, we have

$$(2.10)\qquad \frac{1-\phi(\theta)}{\theta} = \mu_1\phi^{ka^{-1}-n}(\theta).$$

From (2.10) we have $E(X_1^2) = 2(ka^{-1}-n)\mu_1^2$. As $\mathrm{Var}(X_1) > 0$, therefore we obtain $ka^{-1}-n > 1/2$. Replacing $(1-\phi(\theta))/\theta$ in the left side of (2.5) by the right side of (2.10), it follows

$$(2.11)\qquad \left(\frac{\phi^n(\theta)-\phi^{n+1}(\theta)}{\theta}\right)' = \frac{k\mu_1}{a}\phi^{k(a^{-1}-1)}(\theta)\phi'(\theta).$$

Furthermore, by using (2.10) and (2.11) and after some simplifications, (2.6) becomes

$$(2.12)\qquad \left(1-\frac{b}{a}\right)\frac{\phi''(\theta)}{\phi(\theta)} + \left(k-1-\frac{b}{a}\left(\frac{k}{a}-1\right)\right)\left(\frac{\phi'(\theta)}{\phi(\theta)}\right)^2 + \frac{c}{a}\frac{\phi'(\theta)}{\phi(\theta)} = 0.$$

Substituting

$$(2.13)\qquad \xi(\theta) = \frac{\phi'(\theta)}{\phi(\theta)}$$

into (2.12), it yields

$$(2.14)\qquad \left(1-\frac{b}{a}\right)\xi'(\theta) + \left(1-\frac{b}{a^2}\right)k\xi^2(\theta) + \frac{c}{a}\xi(\theta) = 0.$$

From (2.14) we find that $a \neq b$, otherwise $\phi(\theta) \equiv 1$ or $\phi(\theta) = \exp\{-(c/a)((1-b/a^2)k)^{-1}\theta\}$, which contradicts the assumption that F is continuous. Similarly, $a^2 \neq b$, otherwise it would also lead to a contradiction. Now (2.14) can be rewritten as

$$(2.15)\qquad \xi'(\theta) + C_1\xi^2(\theta) + C_2\xi(\theta) = 0,$$

where

$$(2.16)\qquad C_1 = \left(1-\frac{b}{a^2}\right)k\Big/\left(1-\frac{b}{a}\right) \neq 0,$$

$$(2.17)\qquad C_2 = \frac{c}{a}\Big/\left(1-\frac{b}{a}\right).$$

In the following we will prove $C_2 = 0$ first. Assume $C_2 \neq 0$, then the solution of (2.15) is

$$\frac{\xi(\theta)}{\xi(\theta) + C_2/C_1} = K_1 e^{-C_2\theta}, \tag{2.18}$$

for some constant K_1. Differentiating both sides of (2.10), we obtain

$$\frac{-\phi'(\theta)\theta - (1-\phi(\theta))}{\theta^2} = (ka^{-1} - n)\frac{1-\phi(\theta)}{\theta}\frac{\phi'(\theta)}{\phi(\theta)}, \tag{2.19}$$

hence

$$\xi(\theta) = \frac{\phi'(\theta)}{\phi(\theta)} = \frac{-\phi'(\theta) - (1-\phi(\theta))/\theta}{(ka^{-1} - n)(1-\phi(\theta))}. \tag{2.20}$$

From the right side of (2.20), we find that as $\theta \to \infty$, $\xi(\theta) \to 0$. Using this fact in (2.18) it in turn implies $C_2 > 0$. On the other hand, from (2.18) we have

$$\phi(\theta) = \left(\frac{1 - K_1 e^{-C_2\theta}}{1 - K_1}\right)^{C_1^{-1}}, \tag{2.21}$$

which would not imply $\lim_{\theta\to\infty} \phi(\theta) = 0$. The contradiction implies $C_2 = 0$, hence $c = 0$. Therefore the solution of (2.15) is

$$\xi(\theta) = (C_1\theta - \mu_1^{-1})^{-1}, \tag{2.22}$$

since $\lim_{\theta\to 0}\xi(\theta) = -\mu_1$. Furthermore, as $\xi(\theta) \leq 0$, $\forall\theta > 0$, it yields $C_1 < 0$. Comparing (2.13) and (2.22), it follows

$$\phi(\theta) = (1 - C_1\mu_1\theta)^{C_1^{-1}}. \tag{2.23}$$

Substituting $\phi(\theta)$ in (2.23) into the right side of (2.10), we have

$$1 - \phi(\theta) = \mu_1\theta(1 - C_1\mu_1\theta)^{C_1^{-1}(ka^{-1}-n)}. \tag{2.24}$$

By letting $\theta \to \infty$ in the both sides of (2.24), we get (noting that $ka^{-1} - n > 0$ and $C_1 < 0$)

$$C_1^{-1}(ka^{-1} - n) = -1 \quad \text{and} \quad C_1 = -1. \tag{2.25}$$

Hence $a = k/(n+1)$ and

$$\phi(\theta) = (1 + \mu_1\theta)^{-1}. \tag{2.26}$$

Thus $b = k(k+1)/[(n+1)(n+2)]$, as can be seen by replacing $C_1 = -1$ and $a = k/(n+1)$ into (2.16). This completes the proof. □

Remark 1. As mentioned in Section 1, by choosing $G(x) = x$, for a fixed $n \geq 1$,

$$E(\delta_t \mid A(t) = n) = E(X_1 \mid A(t) = n), \qquad \forall t > 0, \tag{2.27}$$

implies $\{A(t), t \geq 0\}$ is a Poisson process. Consequently, let $1 \leq k \leq n$ be two fixed integers, if

$$E(S_k \mid A(t) = n) = \frac{k}{n+1} t, \tag{2.28}$$

then $\{A(t), t \geq 0\}$ is a Poisson process. Thus in order to force the process $\{A(t), t \geq 0\}$ to be Poisson, the condition (2.2) in Theorem 2.1 may be replaced by $a = k/(n+1)$. This also can be seen by noting that (2.1) is equivalent to (2.10) which in turn implies (2.26) immediately when $a = k/(n+1)$.

On the other hand, noting that $\phi'(\theta)|_{\theta=0} = -E(X_1)$ and $\phi''(\theta)|_{\theta=0} = E(X_1^2)$, from (2.10) we obtain

$$E(X_1^2) = 2E^2(X_1)(ka^{-1} - n). \tag{2.29}$$

Hence if

$$E(X_1^2) = 2E^2(X_1), \tag{2.30}$$

then $a = k/(n+1)$. Therefore condition (2.2) may be also replaced by (2.30).

Yet only the condition (2.1) does not suffice to characterize the Poisson process, since (2.10) is not enough to imply (2.26). The point is, by giving both (2.1) and (2.2), not only the process $\{A(t), t \geq 0\}$, but also the constants a, b and c can be determined. However, we do not even know whether given the condition (2.1) together with an $a \neq k/(n+1)$ can yield a solution which is a Laplace transform or not. For the case $a = k/(n+2)$, (2.10) becomes a quadratic equation, solving it we obtain $\phi(\theta) = 2(1 + \sqrt{1 + 4\mu_1\theta})^{-1}$ which is indeed a Laplace transform. Yet when $a = 2k/(2n+1)$, (2.10) reduces to a quadratic equation again, and it can be checked easily that neither of the two solutions of (2.10) is a Laplace transform in this case. For other a's, it is not easy to solve equation (2.10), so this part is still open.

The next two theorems are concerned with the situations of moments with negative orders. As the proofs follow standard technique of solving the differential equations as in the proof of Theorem 2.1, we only give outlines.

THEOREM 2.2. *Let F be continuous with $F(0) = 0$ and $E(X_1) < \infty$. Assume for some fixed integers k and n, $2 \leq k \leq n$,*

$$E(S_k \mid A(t) = n) = at \tag{2.31}$$

and

$$E(S_k^{-1} \mid A(t) = n) = bt^{-1}, \tag{2.32}$$

hold for some constants a and b, for every $t > 0$ whenever $P(A(t) = n) > 0$. Also assume $E(S_k^{-1}) < \infty$. Then

(i) *$a = k/(n+1)$ and $b = n/(k-1)$;*

(ii) *$\{A(t), t \geq 0\}$ is a Poisson process.*

PROOF. Again, first (2.31) and (2.32) can be converted into

$$p_k''(\theta)\frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} = a(q_n''(\theta) - q_{n+1}''(\theta)) \tag{2.33}$$

and

$$p_k(\theta)\frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} = b(q_n(\theta) - q_{n+1}(\theta)), \tag{2.34}$$

respectively, where $\phi(\theta)$ is as defined in (2.7), $p_k(\theta) = \int_0^\infty e^{-\theta y}y^{-1}dF_k(y)$, and $q_j(\theta) = \int_0^\infty e^{-\theta y}y^{-1}F_j(y)dy$, $j = n, n+1$. Solving the above differential equations we obtain the assertions. □

THEOREM 2.3. *Let F be continuous with $F(0) = 0$ and $E(X_1) < \infty$. Assume for some fixed integers k and n, $3 \leq k \leq n$,*

$$E(S_k^{-1} \mid A(t) = n) = at^{-1} \tag{2.35}$$

and

$$E(S_k^{-2} \mid A(t) = n) = bt^{-2}, \tag{2.36}$$

hold for some constants a and b, for every $t > 0$ whenever $P(A(t) = n) > 0$. Also assume $E(S_k^{-2}) < \infty$. Then

(i) *$a = n/(k-1)$ and $b = n(n-1)/[(k-1)(k-2)]$;*

(ii) *$\{A(t), t \geq 0\}$ is a Poisson process.*

PROOF. As before the proof follows by converting (2.35) and (2.36) into

$$\psi_k'(\theta)\frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} = a(\xi_n'(\theta) - \xi_{n+1}'(\theta)) \tag{2.37}$$

and

$$\psi_k(\theta)\frac{\phi^{n-k}(\theta) - \phi^{n+1-k}(\theta)}{\theta} = b(\xi_n(\theta) - \xi_{n+1}(\theta)), \tag{2.38}$$

respectively, where $\psi_k(\theta) = \int_0^\infty e^{-\theta y}y^{-2}dF_k(y)$, and $\xi_j(\theta) = \int_0^\infty e^{-\theta y}y^{-2}F_j(y)dy$, $j = n, n+1$. The remaining argument is similar to the one used in the proof of Theorem 2.1. □

Note that when $\{A(t), t \geq 0\}$ is a Poisson process both $E(S_1^{-1} \mid A(t) = n)$ and $E(S_2^{-2} \mid A(t) = n)$ are infinite, hence the conditions that $k \geq 2$ and $k \geq 3$ are needed in Theorems 2.2 and 2.3, respectively. For the current life δ_t, we have the following result in Theorem 2.4 under similar conditions as in Theorem 2.1. Again since when $\{A(t), t \geq 0\}$ is a Poisson process both $E(\delta_t^{-1} \mid A(t) = n)$

and $E(\delta_t^{-2} \mid A(t) = n)$ are infinite, we do not have results which are parallel to Theorems 2.2 or 2.3.

THEOREM 2.4. *Let F be continuous with $F(0) = 0$ and $E(X_1^2) < \infty$. Assume for some fixed integer n,*

$$E(\delta_t \mid A(t) = n) = at \tag{2.39}$$

and

$$E(\delta_t^2 \mid A(t) = n) = bt^2 + ct, \tag{2.40}$$

hold for some constants a, b and c, for every $t > 0$ whenever $P(A(t) = n) > 0$. Then

(i) $a = 1/(n+1)$, $b = 2/[(n+1)(n+2)]$ *and* $c = 0$;

(ii) $\{A(t), t \geq 0\}$ *is a Poisson process.*

PROOF. Since given $A(t) = n$, $\delta_t = t - S_n$, the assertions now follow from Theorem 2.1 immediately. □

3. Results for the discrete case

When the inter-arrival distribution of the renewal process $\{A(t), t \geq 0\}$ is discrete, we have the following result which can be compared with Theorem 2.1. Yet again the results corresponding to Theorems 2.2 and 2.3 do not hold any more in this case as will be seen from an example given later.

THEOREM 3.1. *Let F, $F(0) = 0$, be arithmetic with span 1 and $E(X_1^2) < \infty$. For some fixed integers k and n, $1 \leq k \leq n$, if*

$$E(S_k \mid A(t) = n) = a_1 t + a_0 \tag{3.1}$$

and

$$E(S_k^2 \mid A(t) = n) = b_2 t^2 + b_1 t + b_0, \tag{3.2}$$

hold for some constants a_r, b_s, $r = 0, 1$, $s = 0, 1, 2$, for every $t = n, n+1, \ldots,$ whenever $P(A(t) = n) > 0$, then

(i) $a_1 = a_0 = k/(n+1)$, $b_2 = k(k+1)/[(n+1)(n+2)]$, $b_1 = k(3k - n + 1)/[(n+1)(n+2)]$, $b_0 = k(2k - n)/[(n+1)(n+2)]$;

(ii) $\{A(t), t \geq 0\}$ *is a geometric renewal process.*

Again, by letting $f(\theta) = E(e^{-\theta X_1}) = \sum_{x=1}^{\infty} e^{-\theta x} P(X_1 = x)$, the proof of the above theorem is much in the spirt as that of Theorem 2.1 and is omitted. Also similar to the continuous case, condition (3.3) may also be replaced by $a_1 = a_0 = k/(n+1)$. This is exactly Theorem 2 of Huang *et al.* (1993). Next, just as in Theorem 2.4, it is easy to obtain a corresponding result for δ_t. We omit it here.

Finally, we give an example to illustrate that for the discrete case we do not have similar characterizations based on the conditional moments with negative orders.

Example 1. Let $\{A(t), t \geq 0\}$ be a geometric renewal process, that is, assume $P(X_1 = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$, where $0 < p < 1$. Then

$$E(S_2^{-1} \mid A(t) = 2) = 2t^{-1} - 2\sum_{y=2}^{t} y^{-1}/[t(t-1)], \tag{3.3}$$

which is not the form of $a_1 t^{-1} + a_0$ for any constants a_0 and a_1.

4. Discussion

In this paper we have characterized the Poisson process using some suitable conditional moments. By using this kind of conditions we could develope an asymptotic test from one of the characterizations, which does not need all the informations about the arrival times, therefore gives an alternative way of testing the Poisson process other than all those tests already exist. Further investigation about this kind of testing problem will be discussed elsewhere.

Acknowledgements

The authors would like to thank the referees for their many constructive and useful suggestions which enable them to improve the manuscript. They also thank the referees for bringing to their notice the papers by Laha and Lukacs (1960), Hall and Simons (1969), Shanbhag (1971) and Pusz and Wesolowski (1992).

References

Alzaid, A. A. (1990). A moment's approach to some characterization problems, *Ann. Inst. Statist. Math.*, **32**, 281–285.

Çinlar, E. and Jagers, P. (1973). Two mean values which characterize the Poisson process, *J. Appl. Probab.*, **10**, 678–681.

Hall, W. J. and Simons, G. (1969). On characterizations of the gamma distribution, *Sankhyā Ser. A*, **31**, 385–390.

Huang, W. J., Li, S. H. and Su, J. C. (1993). Some characterizations of the Poisson process and geometric renewal process, *J. Appl. Probab.*, **30**, 121–130.

Laha, R. G. and Lukacs, E. (1960). On a problem connected with quadratic regression, *Biometrika*, **47**, 335–343.

Li, S. H., Huang, W. J. and Huang, M. N. L. (1992). Characterizations of the Poisson process via two conditional moments, Tech. Report, #92-1, Institute of Applied Mathematics, National Sun Yat-sen University.

Lukacs, E. (1955). A characterization of the gamma distribution, *Ann. Math. Statist.*, **26**, 319–324.

Pusz, J. and Wesolowski, J. (1992). A non-Lukacsian regressional characterization of the gamma distribution, *Appl. Math. Lett.*, **5**(4), 81–84.

Shanbhag, D. N. (1971). An extension of Lukacs' result, *Proc. Camb. Phil. Soc.*, **69**, 301–303.

Tollar, E. S. (1988). A characterization of the gamma distribution from a random difference equation, *J. Appl. Probab.*, **25**, 142–149.

Wesolowski, J. (1989). A characterization of the gamma process by conditional moments, *Metrika*, **36**, 299–309.

Wesolowski, J. (1990*a*). A constant regression characterization of the gamma law, *Adv. in Appl. Probab.*, **22**, 488–490.

Wesolowski, J. (1990*b*). Characterizations of distributions by constant regression of quadratic statistics on a linear one, *Sankhyā Ser. A*, **52**, 383–386.