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BAYESIAN SEQUENTIAL RELIABILITY FOR WEIBULL AND RELATED DISTRIBUTIONS*

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Abstract. Assume that the probability density function for the lifetime of a newly designed product has the form: $[H'(t)/Q(\theta)]\exp\{-H(t)/Q(\theta)\}$. The Exponential $\mathcal{E}(\theta)$, Rayleigh, Weibull $\mathcal{W}(\theta,\beta)$ and Pareto pdf's are special cases. $Q(\theta)$ will be assumed to have an inverse Gamma prior. Assume that m independent products are to be tested with replacement. A Bayesian Sequential Reliability Demonstration Testing plan is used to eigher accept the product and start formal production, or reject the product for reengineering. The test criterion is the intersection of two goals, a minimal goal to begin production and a mature product goal. The exact values of various risks and the distribution of total number of failures are evaluated. Based on a result about a Poisson process, the expected stopping time for the exponential failure time is also found. Included in these risks and expected stopping times are frequentist versions, thereof, so that the results also provide frequentist answers for a class of interesting stopping rules.

Key words and phrases: Bayesian sequential test, reliability demonstration test, exponential distribution, Weibull distribution, expected stopping time, producer's risk, consumer's risk.

1. Introduction

Reliability Demonstration Testing (RDT) is often used for the purpose of verifying whether a specified reliability has been achieved in a newly designed product. Based on a demonstration test, a decision is made to either accept the design and start formal production, or reject the design and send the product back for reengineering.

A serious problem with RDT is that a reliability test can be very expensive in terms of money and time, especially in the case of products that require very

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high reliability and have a long lifetime. A common solution is to take into consideration prior information, typically from engineering knowledge or knowledge of previous similar products, and to test in a sequential fashion. Bayesian Sequential Reliability Demonstration Testing (BSRDT) can, in many cases, significantly reduce the amount of testing required.

The following design questions are of interest. How many units need to be tested to reach a decision? How much time is required to make a decision? What are expected losses? There are a variety of Bayesian non-sequential reliability demonstration tests for answering these questions. For example, see Esterling (1975), Goel (1975), Schafer (1975), Schemee (1975), Goel and Coppola (1979), and Bivens *et al.* (1987).

Schafer and Singpurwalla (1970) proposed the following test procedure. One unit at a time is tested, where the lifetimes of units are independently identically exponentially distributed with mean θ. The unknown θ is assumed to have an inverse Gamma prior distribution. Choose a minimum acceptable value, say θ_1, and let $P_n = P(\theta \geq \theta_1 \mid \text{data})$. The test is terminated when $P_n \geq 1 - \alpha_2$ (or $\leq \alpha_1$), and a decision to accept (or reject) the product is made. Schafer and Singpurwalla (1970) were primarily concerned with the acceptance probability of this procedure, and developed approximations for it. Some related computations and approximations for other risks were done in Schafer and Sheffield (1971) and Mann *et al.* (1974). The approximations are only for the case where there is no indifference region (see Subsection 2.3), and while they are often reasonably accurate, they can be inaccurate. The extreme difficulty of all computations of this type is discussed in Martz and Waller (1982); one of the major motivations for this work is to show how such computations can be done explicitly, in closed form, and for a general class of distributions.

The stopping rule of Schafer and Singpurwalla (1970) is discrete, in the sense that one can only stop the test when a failure occurs. This can be inefficient when observations are very expensive and/or have long lifetimes. Barnett (1972) proposed a continuous BSRDT plan for the exponential failure rate problem. By his method, one can stop the test at any time that enough information has accumulated. Again, however, closed form answers were not obtained.

Related work can be found in Epstein and Sobel (1953), Ray (1965), Harris and Singpurwalla (1968, 1969), Soland (1969), MacFarland (1971), Goel and Coppola (1979), Martz and Waller (1979), Chandra and Singpurwalla (1981), Montagne and Singpurwalla (1985) and Lindley and Singpurwalla (1991*a*, 1991*b*).

In this paper, a general BSRDT plan, stimulated by the work of Barnett (1972), is considered for a general class of life distributions. The Weibull and Pareto distributions are special cases. The testing plan continues until the posterior loss is decisive according to a desired criterion, at which time testing terminates and a decision made concerning the quality of the product. For this plan, the exact values of various risks and the distribution of the total number of failures are evaluated. Also, bounds on the expected testing time are given and, for the special case of an exponential failure time, the expected testing time is computed explicitly. Included in these risks and expected stopping times are frequentist versions, thereof, so that the results also provide frequentist answers for a class of interesting stopping rules.

In Section 2 of the paper, the basic model and the BSRDT plan are introduced. The total number of units tested, the testing time, and several other important features are introduced and evaluated in Section 3. Most proofs are given in Appendix.

2. Structure of the problem

2.1 *The model*

The basic model is as follows. Suppose that units are independently tested on m machines. Whenever a unit fails, it is replaced by a new unit and testing is continued until enough information has been obtained. This model includes the case in which m machines are tested themselves and, upon failure, a machine is repaired or rebuilt (immediately) so that the repaired machine is as good as new. The inter-failure times for machine i will be denoted by $t_{i1}, t_{i2}, \ldots$, for $i = 1, 2, \ldots, m$. Given θ, the life times t_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$, are iid. random variables with reliability function

$$R_\theta(t) = P(t_{11} > t \mid \theta) = \exp\{-H(t)/Q(\theta)\}, \qquad t > 0, \tag{2.1}$$

i.e., t_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$ have the probability density function

$$f(t \mid \theta) = \frac{H'(t)}{Q(\theta)} \exp\left\{-\frac{H(t)}{Q(\theta)}\right\}, \qquad t > 0. \tag{2.2}$$

Here $H(\cdot)$ is a known increasing function satisfying $H(0^+) = 0$ and $\lim_{t\to\infty} H(t) = \infty$, $Q(\cdot)$ is a known and strictly increasing function, and θ is the unknown characteristic life.

The density of (2.2) is a special form of the exponential family and encompasses many common reliability distributions. The case when $Q(\theta) = \theta$ is known as the *proportional hazard family* and has been considered by Jewell (1977) for non-sequential tests. Jewell (1977) also discussed the problem of identifying the prototype function $H(x)$. Here are some other examples. The Weibull density, $\mathcal{W}(\theta, \beta)$, given by $f(t \mid \theta) = (\beta t^{\beta-1}/\theta^\beta) \exp\{-(t/\theta)^\beta\}$, $t > 0$, arises from $Q(x) \equiv H(x) = x^\beta$ $(x > 0)$ in (2.2), for some known positive constant β. Known β can arise when the product is in a class of similar well-studied products, whose distributions have been seen to arise from Weibull's with similar shape. Of course, the exponential and Raleigh distributions are two important cases, obtained when $\beta = 1$ and 2, respectively. Soland (1968) gives a justification for this situation. Assume that, for given θ, X_1, X_2 are independent random variables and X_i has reliability function $\exp\{-t^{\beta_i}/\theta\}$, where β_1 and β_2 are known positive constants. Then the p.d.f. $\min\{X_1, X_2\}$ is also a special case of (2.2) with $H(t) = t^{\beta_1} + t^{\beta_2}$ and $Q(\theta) = \theta$. If we let $H(t) = \ln(t + 1)$ $(t > 0)$ in (2.2), the lifetimes have a Pareto distribution with p.d.f. $f(t \mid \theta) = 1/[Q(\theta)(t + 1)^{1/Q(\theta)+1}]$, $t > 0$.

From Billingsley ((1986), (21.9) on p. 282), we know that

$$E(t_{ij} \mid \theta) = \int_0^\infty P(t_{ij} > s \mid \theta) ds = \int_0^\infty R_\theta(s) ds.$$

Since $Q(\cdot)$ is strictly increasing, we get $Q(\theta_1) < Q(\theta_2)$, if $\theta_1 < \theta_2$. Thus $R_{\theta_1}(t) < R_{\theta_2}(t)$, which implies that $E(t_{ij} \mid \theta_1) < E(t_{ij} \mid \theta_2)$, for $\theta_1 < \theta_2$. In other words, larger θ provide larger expected lifetime.

2.2 *The prior and posterior*

Prior information about the unknown parameter θ is assumed available in the form of a prior density function $\pi(\cdot)$. Schafer (1969) and Schafer and Sheffield (1971) observed that the inverse Gamma prior distributions are often reasonable for exponential failure problems. Here, the prior p.d.f. of θ for the family (2.2) will be assumed to belong to the conjugate family

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \frac{Q'(\theta)}{Q^{a+1}(\theta)} \exp\left\{-\frac{b}{Q(\theta)}\right\}, \quad \text{for} \quad \theta > 0. \tag{2.3}$$

Note that then $Q(\theta)$ has an inverse gamma distribution $\mathcal{IG}(a, b)$. Methods of choosing a and b will be discussed in Appendix A.1.

Define $N_i(t) = \max\{j : t_{i1} + \cdots + t_{ij} \le t\}$, the number of failures in machine i at time t, and

$$N(t) = N_1(t) + \cdots + N_m(t), \tag{2.4}$$

the total number of failures at time t. Let

$$V = V(t) = \sum_{i=1}^{m} \sum_{j=1}^{N_i(t)} H(t_{ij}) + \sum_{i=1}^{m} H\left(t - \sum_{j=1}^{N_i(t)} t_{ij}\right). \tag{2.5}$$

Here we define $\sum_{j=1}^{0} \cdot = 0$. Note that V is often called the transformed total time on test or the rescaled total time on test. See Barlow and Campo (1975) and Klefsjö (1991). Since $H(\cdot)$ is continuous and strictly increasing, so is $V(t)$, $t > 0$. Then the likelihood function of θ is proportional to $Q(\theta)^{-N(t)} \exp\{-V(t)/Q(\theta)\}$. It follows directly that if we stop the test at time t, the posterior density of θ is

$$\pi(\theta \mid \text{data}) = \frac{(V(t)+b)^{N(t)+a}}{\Gamma(N(t)+a)} \frac{Q'(\theta)}{Q^{N(t)+a+1}(\theta)} \exp\left\{-\frac{V(t)+b}{Q(\theta)}\right\}, \quad \text{for} \quad \theta > 0, \tag{2.6}$$

i.e., $Q(\theta)$ has, *a posteriori*, an inverse gamma distribution, $\mathcal{IG}(N(t)+a, V(t)+b)$. In particular, it follows that the posterior α-th quantile is

$$q^*(\alpha) = Q^{-1}\left(\frac{2(V(t)+b)}{\chi^2_{2(N(t)+a)}(1-\alpha)}\right), \quad \text{for} \quad 0 < \alpha < 1, \tag{2.7}$$

where $Q^{-1}(\cdot)$ is the inverse function of $Q(\cdot)$ and $\chi^2_j(1-\alpha)$ is the $(1-\alpha)$-th quantile of the χ^2 distribution with j degrees of freedom.

2.3 *The BSRDT plan*

There are a variety of possible goals for sequential experimentation. The following BSRDT plan is the intersection of two goals.

1. Let θ_1 be the goal to begin production, in the sense that the experiment will stop and production begin if there is $100(1-\alpha_1)\%$ "confidence" that $\theta > \theta_1$.

2. Let θ_2 be the mature product goal, in the sense that the experiment will stop and the product will be rejected (sent back for reengineering) if there is $100(1-\alpha_2)\%$ "confidence" that $\theta < \theta_2$.

Here α_1 and α_2 are two usually somewhat small numbers, and $\theta_1 < \theta_2$ are two prespecified values. The region $\theta_1 < \theta < \theta_2$ is often called the indifference region.

The BSRDT plan also arises in formal decision models. Suppose that a product with small $\theta < (\theta_1)$ should be rejected and with large $\theta > (\theta_2)$ should be accepted. Let $l(\theta)$ be the loss for making a wrong decision, where $l(\theta)$ is nonincreasing and nondecreasing in $(0, \theta_1]$ and $[\theta_2, \infty)$, respectively, and $l(\theta) = 0$ for $\theta \in (\theta_1, \theta_2)$. The test will stop and production begin if the posterior loss of accepting the product $(\int_0^{\theta_1} l(\theta)\pi(\theta \mid \text{data})d\theta)$ is small enough, and the test will stop and the product be rejected if the posterior loss of rejecting the product $(\int_{\theta_2}^{\infty} l(\theta)\pi(\theta \mid \text{data})d\theta)$ is small enough. The BSRDT plan arises if $l(\theta)$ is constant on both $(0, \theta_1]$ and $[\theta_2, \infty)$.

It is easy to see that the testing plan is equivalent to

$$\begin{cases} \text{Stop and accept the product,} & \text{if } q^*(\alpha_1) > \theta_1; \\ \text{Stop and reject the product,} & \text{if } q^*(1-\alpha_2) \le \theta_2; \\ \text{Continue testing,} & \text{otherwise,} \end{cases}$$

where $q^*(\alpha)$ is the α-th posterior quantile. From (2.7), this is equivalent to

$$\begin{cases} \text{Stop and accept the product,} \\ \qquad \text{if } V(t) + b > (1/2)Q(\theta_1)\chi^2_{2(N(t)+a)}(1-\alpha_1), \\ \text{Stop and reject the product,} \\ \qquad \text{if } V(t) + b \le (1/2)Q(\theta_2)\chi^2_{2(N(t)+a)}(\alpha_2). \end{cases} \tag{2.8}$$

As with certain classical sequential tests these procedures are "semicontinuous" (see Epstein and Sobel (1955)): one can "accept" when the continuous time of accumulated nonfailure is large enough, but can "reject" only on the (discrete) occurrence of a failure. It is possible to graph the stopping boundaries of this test by defining

$$c_i = \frac{1}{2}Q(\theta_2)\chi^2_{2(a+i)}(\alpha_2) \quad \text{and} \quad d_i = \frac{1}{2}Q(\theta_1)\chi^2_{2(a+i)}(1-\alpha_1), \qquad i \ge 0. \tag{2.9}$$

Since both α_1 and α_2 are small, it can be assumed that $\alpha_2 < 1 - \alpha_1$. Then it can be seen that $c_i \ge d_{i-1}$ when i is large enough, as long as $\theta_1 < \theta_2$. Therefore there is an i_0 such that

$$i_0 = \min\{i = 1, 2, \ldots : c_i \ge d_{i-1}\}. \tag{2.10}$$

Note that if $i_0 > 1$, then $c_1 < d_0$, which implies that $c_0 < d_0$, but for $i_0 = 1$ we will need to assume that $c_0 < d_0$. If $b \le c_0$, we should reject the product without

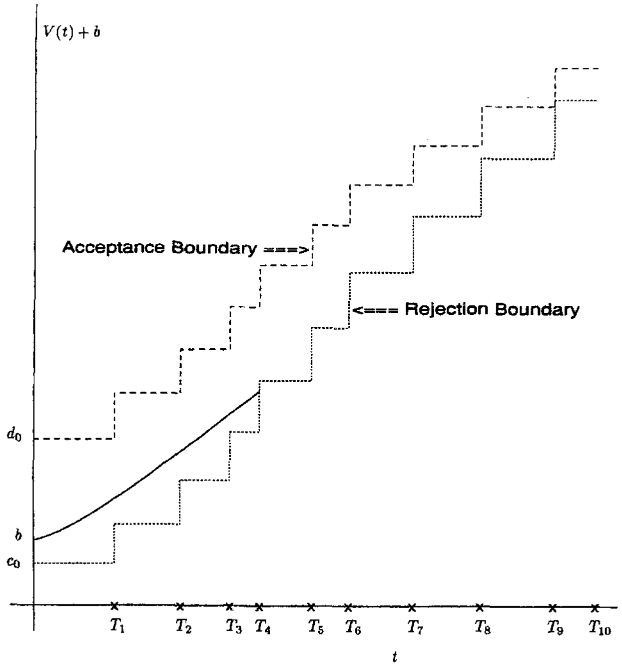


Fig. 1. An example of stopping boundaries.

test, and if $b \geq d_0$, we should accept the product without test. Obviously, these two cases are unlikely to occur in practice, so we consider only the case where $c_0 < b < d_0$.

Let $T_1 \leq T_2 \leq \cdots \leq T_n$ be the first n ordered failure times for all the m machines. For instance, $T_1 = \min\{t_{11}, t_{21}, \ldots, t_{m1}\}$, and if the first failure occurs on machine i_1, then $T_2 = \min\{t_{i_1 1} + t_{i_1 2}, t_{i1}, i \neq i_1\}$. Let T be the stopping time or the testing time, that is

$$T = \min\left\{t > 0 : V(t) + b > \frac{1}{2}Q(\theta_1)\chi^2_{2(N(t)+a)}(1-\alpha_1) \text{ or } \leq \frac{1}{2}Q(\theta_2)\chi^2_{2(N(t)+a)}(\alpha_2)\right\}.$$

Note that T is different from the well-known total time on test. A graph of $V(t)+b$ with respect to t is shown in Fig. 1 using simulated data from a $\mathcal{W}(6800, 1.35)$ distribution. For the graph, $a = 2.5$, $b = 255{,}000$, $\alpha_1 = 0.10$, $\alpha_2 = 0.10$, $\theta_1 = 6{,}400$ and $\theta_2 = 8{,}650$, which implies $c_9 < d_8$ and $i_0 = 8$. Note that both the acceptance boundary and the rejection boundary have random jump points, but have fixed height at each point. The boundaries in Fig. 1 are different than the boundaries in traditional exponential sequential life testing procedures, in that they are data-dependent, whereas in other sequential procedures, both Bayesian and frequentist, they are predetermined. Also, the boundaries here have the form of a step function, whereas $V(t) + b$ is smooth, while the reverse is true for the other procedures. In

Fig. 1, at time T, $V(T)+b$ hits the rejection boundary. In general, at the stopping time T, $V(T)+b$ will hit either a horizontal segment on the acceptance boundary or a vertical segment on the rejection boundary. If $\theta_1 < \theta_2$, $i_0 < \infty$ and the sampling region is closed. If $\theta_1 < \theta_2$, $i_0 = \infty$ and the sampling region is open. Although we have derived this class of stopping boundaries from a Bayesian perspective, they can be used simply as specified boundaries in a frequentist analysis. Indeed frequentist measures of performance will also be considered.

3. Features of the BSRDT plan

3.1 *Design criteria and risks*

Let $\mathcal{A}$ and $\mathcal{R}$ denote the action (or, by an abuse of notation, the region) of accepting the product and the action of rejecting the product, respectively. Several risk criteria, defined in Chapter 10 of Martz and Waller (1982), can also be applied here to measure the goodness of the BSRDT plan. The following names of these risks are borrowed from related conventions in quality control and are widely used in nonsequential context.

1. *Classical producer's risk*, $\gamma = P(\mathcal{R} \mid \theta_2)$, and *classical consumer's risk*, $\delta = P(\mathcal{A} \mid \theta_1)$. Here γ is the probability that a product at the mature product goal will fail the BSRDT and δ is the probability that a product at the goal to begin production will pass the BSRDT. Note that these are frequentist risks. It can be seen that $P(\mathcal{A} \mid \theta)$ is monotonically increasing in θ. Thus $P(\mathcal{R} \mid \theta) < \gamma$ for $\theta > \theta_2$, and $P(\mathcal{A} \mid \theta) < \delta$ for $\theta < \theta_1$.

2. *Average producer's risk*, $\tilde{\gamma} = P(\mathcal{R} \mid \theta \geq \theta_2)$, and *average consumer's risk*, $\tilde{\delta} = P(\mathcal{A} \mid \theta \leq \theta_1)$. Here $\tilde{\gamma}$ is the probability of rejecting a good product and $\tilde{\delta}$ is the probability of accepting a bad product. Note that computation of these risks involves the prior.

3. *Posterior producer's risk*, $\gamma^* = P(\theta \geq \theta_2 \mid \mathcal{R})$, and *posterior consumer's risk*, $\delta^* = P(\theta \leq \theta_1 \mid \mathcal{A})$. Here γ^* is the posterior probability that a rejected product is good, and δ^* is the posterior that an accepted product is bad.

4. *Rejection probability*, $P(\mathcal{R}) = \int_\Theta P(\mathcal{R} \mid \theta)\pi(\theta)d\theta$, and *acceptance probability*, $P(\mathcal{A}) = 1 - P(\mathcal{R})$. Here $P(\mathcal{A})$ is the unconditional probability of the product passing the BSRDT.

Of course, one can not find an "optimal" BSRDT in the sense that all above risks are very small. However, it is possible to choose several risks. For the fixed sample size problem, many papers are available concerning how to choose the criteria. For example, Balaban (1975) favors the mixed classical/Bayesian pair (γ, δ^*) to determine a Bayesian reliability demonstration test. Also see Easterling (1970), Schafer and Sheffield (1971), Schick and Drnas (1972), Goel and Joglekar (1976). We make no effort to compare the criteria here; our results are of use for computation with any of them.

Let $N = N(T)$ be the number of failed units at the time when testing stops. Let N_{TU} denote the total sample size or the total number of testing units put on test. Finding the expected stopping time, $E(T) = \int E(T \mid \theta)\pi(\theta)d\theta$, and the expected sample size, $E(N_{TU}) = \int E(N_{TU} \mid \theta)\pi(\theta)d\theta$, is important for design. The following relationship between the total sample size and the number of failures follows immediately from the definition of the stopping rule, and allows us to consider $E(N)$ instead of $E(N_{TU})$.

THEOREM 3.1. *$N_{TU} = N + m - I(\mathcal{R})$ and $E(N_{TU}) = E(N) + m - P(\mathcal{R})$, where $I(\cdot)$ is the indicator function.*

In Subsection 3.3, we will find expressions for all the above risks for our decision rules. In Subsection 3.4, we will find the distribution and expected value of the number of failures. Expressions for $E(T)$ are developed in Subsection 3.5. Some examples are given in Subsection 3.6.

3.2 *Technical preliminaries*

For $n = 1, \ldots, i_0$, let

$$G_n = \{(y_1, \ldots, y_n) : y_j > 0, c_j - b < y_1 + \cdots + y_j \le d_{j-1} - b, j = 1, \ldots, n\}, \tag{3.1}$$

let $\|G_0\| = 1$ and let $\|G_n\|$ denote the volume or Lebesgue measure of G_n $(n \ge 1)$. It will be seen that all the risk expressions involve $\|G_n\|$ $(n < i_0)$. For all theorems involving $\|G_n\|$, it will be assumed that

$$c_j < d_{j-1}, \qquad \text{for} \quad j = 1, 2, \ldots, n, \tag{3.2}$$

where both $\{c_j\}_{j\ge0}$ and $\{d_j\}_{j\ge0}$ are increasing sequences of positive numbers. If (3.2) is violated, i.e., if there is j $(\le n)$, such that $c_j \ge d_{j-1}$, then G_n is an empty set, and hence $\|G_n\| = 0$. An analytic formula for $\|G_n\|$ is given in Lemma 3.2. The following notation will be needed.

For $j = 1, 2, \ldots,$ $i = 0, 1, 2, \ldots,$ and $y \ge 0$, define

$$a_{ij}(y) = \begin{cases} (c_{i+j} \wedge d_{j-1}) \vee y - (c_j \vee y), & \text{if } i \ge 1, \\ d_{j-1} - c_j \vee y, & \text{if } i = 0, \end{cases} \tag{3.3}$$

where $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$. Then (3.2) implies that

$$a_{ij}(c_j) = \begin{cases} c_{i+j} \wedge d_{j-1} - c_j, & \text{if } i \ge 1, \\ d_{j-1} - c_j, & \text{if } i = 0. \end{cases} \tag{3.4}$$

For $n \ge 2$, define two sets of partitions of n by

$$\Psi_n = \left\{ (i_1, \ldots, i_k) : \sum_{j=1}^{k} i_j = n, k \ge 1, i_j \ge 1 \right\} \tag{3.5}$$

and

$$\Psi_n^* = \{(i_1, \ldots, i_k) : (i_1, \ldots, i_k) \in \Psi_n, i_k \ge 2\}. \tag{3.6}$$

For example, the first few Ψ_n and Ψ_n^* are as follows:

$$\begin{cases} \Psi_1 = \{(1)\}, \quad \Psi_2 = \{(1,1),(2)\}, \quad \Psi_3 = \{(1,1,1),(1,2),(2,1),(3)\}, \\ \Psi_4 = \{(1,1,1,1),(1,1,2),(1,2,1),(2,1,1),(3,1),(1,3),(2,2),(4)\}, \\ \Psi_1^* = \emptyset, \quad \Psi_2^* = \{(2)\}, \quad \Psi_3^* = \{(1,2),(3)\}, \\ \Psi_4^* = \{(1,1,2),(1,3),(2,2),(4)\}. \end{cases} \tag{3.7}$$

For $1 \le r \le n$, $(i_1, \ldots, i_k) \in \Psi_n$, define

$$\rho_1(b; n; r; i_1, \ldots, i_k) \equiv \rho_1(b; n; r; i_1, \ldots, i_k; c_j, d_{j-1}, 1 \le j \le n) = \begin{cases} \dfrac{a^r_{n-r,r}(b)}{r!}, & \text{if } k = 1, \\ \dfrac{a^{i_1}_{n-r,r}(c_r)}{\prod_{j=1}^k i_j!} \left\{ \prod_{j=2}^{k-1} a^*_{r,j}(c^*_{r,j}) \right\} a^*_{r,k}(b), & \text{if } k \ge 2, \end{cases} \tag{3.8}$$

and for $2 \le l \le r \le n$, $(i_1, \ldots, i_k) \in \Psi^*_l$,

$$\begin{aligned} &\rho_2(b; n; r; l; i_1, \ldots, i_k) \\ &\quad \equiv \rho_2(b; n; r; l; i_1, \ldots, i_k; c_j, d_{j-1}, 1 \le j \le n) \\ &\quad = \begin{cases} \dfrac{a^l_{n-r,r}(d_{r-l})}{l!} \|G_{r-l}\|, & \text{if } k = 1, \\ \dfrac{a^{i_1}_{n-r,r}(c_r)}{\prod_{j=1}^k i_j!} \left\{ \prod_{j=2}^{k-1} a^*_{r,j}(c^*_{r,j}) \right\} a^*_{r,k}(d_{r-l}) \|G_{r-l}\|, & \text{if } k \ge 2, \end{cases} \end{aligned} \tag{3.9}$$

where $i_0 = 0$, $i_{(j)} = i_0 + i_1 + \cdots + i_j$, $a^*_{r,j}(\cdot) = a^{i_j}_{i_{j-1}, r-i_{(j)}}(\cdot)$, $a_{ij}(\cdot)$ is given by (3.3), $c^*_{n,j} = c_{n-i_{(j)}}$, and $\prod_{j=2}^1 \cdot = 1$.

The following lemma will be used in the proof of Lemma 3.2. It also provides the number of terms in the formula for $\|G_n\|$.

LEMMA 3.1. *The numbers of elements in Ψ_n and Ψ^*_n are*

$$\#\{\Psi_n\} = 2^{n-1} \quad \textit{and} \quad \#\{\Psi^*_n\} = 2^{n-2}, \tag{3.10}$$

respectively. Furthermore, we have the following recursive formulas:

$$\Psi_{n+1} = \bigcup_{(i_1, \ldots, i_k) \in \Psi_n} \{(i_1, \ldots, i_k + 1), (i_1, \ldots, i_k, 1)\}, \tag{3.11}$$

$$\Psi^*_{n+1} = \bigcup_{(i_1, \ldots, i_k) \in \Psi_n} \{(i_1, \ldots, i_k + 1)\}. \tag{3.12}$$

PROOF. See Appendix A.2. □

LEMMA 3.2. *The volume of G_n $(n \ge 1)$ is*

$$\begin{aligned} \|G_n\| = &\sum_{(i_1, \ldots, i_k) \in \Psi_n} \rho_1(b; n; n; i_1, \ldots, i_k) \\ &- \sum_{l=2}^n \sum_{(i_1, \ldots, i_k) \in \Psi^*_l} \rho_2(b; n; n; l; i_1, \ldots, i_k), \end{aligned} \tag{3.13}$$

where $\rho_1(b; n; n; i_1, \ldots, i_k)$ *and* $\rho_2(b; n; n; l; i_1, \ldots, i_k)$ *are given by* (3.8) *and* (3.9), *respectively.*

PROOF. See Appendix A.2. □

In equation (3.9), $\|G_1\|, \ldots,$ and $\|G_{n-2}\|$ appear. Therefore, (3.13) is essentially providing an iterative algorithm for their computation. Using (3.10), the total number of terms in (3.13) is $2^{n-1} + \sum_{l=2}^{n} 2^{l-2} = 2^n - 1$.

3.3 *Evaluation of risks*

Define $I_\Gamma(x; y) = \int_0^y t^{x-1} e^{-t} dt$, for $x, y > 0$.

LEMMA 3.3. *For any fixed* $\mu > 0$, *we have*

$$P(\theta \geq \mu) = I_\Gamma\left(a, \frac{b}{Q(\mu)}\right) \Big/ \Gamma(a), \tag{3.14}$$

$$P(\mathcal{A} \mid \theta) = \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{Q^n(\theta)} \exp\left\{-\frac{d_n - b}{Q(\theta)}\right\}, \qquad \theta > 0, \tag{3.15}$$

and

$$P((\theta \geq \mu) \cap \mathcal{A}) = \sum_{n=0}^{i_0-1} \frac{b^a \|G_n\|}{d_n^{a+n} \Gamma(a)} I_\Gamma\left(a + n, \frac{d_n}{Q(\mu)}\right). \tag{3.16}$$

PROOF. See Appendix A.2. □

The following theorem concerning the rejection and acceptance probabilities follows from Lemma 3.3 and the fact that $P(\mathcal{A}) = \lim_{\mu \to 0} P((\theta \geq \mu) \cap \mathcal{A})$.

THEOREM 3.2. *The rejection probability is*

$$P(\mathcal{R}) = 1 - P(\mathcal{A}) = 1 - \sum_{n=0}^{i_0-1} \frac{b^a \Gamma(a+n)}{d_n^{a+n} \Gamma(a)} \|G_n\|. \tag{3.17}$$

THEOREM 3.3. *The classical risks are*

$$\gamma = P(\mathcal{R} \mid \theta_2) = 1 - \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{Q^n(\theta_2)} \exp\left\{-\frac{d_n - b}{Q(\theta_2)}\right\}, \tag{3.18}$$

$$\delta = P(\mathcal{A} \mid \theta_1) = \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{Q^n(\theta_1)} \exp\left\{-\frac{d_n - b}{Q(\theta_1)}\right\}. \tag{3.19}$$

PROOF. The results follow immediately from (3.15). □

THEOREM 3.4. *The average risks are*

$$(3.20)\quad \tilde{\gamma} = P(\mathcal{R} \mid \theta \geq \theta_2) = 1 - \sum_{n=0}^{i_0-1} \frac{b^a \|G_n\|}{d_n^{a+n}} \frac{I_\Gamma(a+n, d_n/Q(\theta_2))}{I_\Gamma(a, b/Q(\theta_2))},$$

$$(3.21)\quad \tilde{\delta} = P(\mathcal{A} \mid \theta \leq \theta_1) = \sum_{n=0}^{i_0-1} \frac{b^a \|G_n\|}{d_n^{a+n}} \frac{\Gamma(a+n) - I_\Gamma(a+n, d_n/Q(\theta_1))}{\Gamma(a) - I_\Gamma(a, b/Q(\theta_1))}.$$

PROOF. Note that

$$(3.22)\qquad \tilde{\gamma} = P(\mathcal{R} \mid \theta \geq \theta_2) = 1 - P((\theta \geq \theta_2) \cap \mathcal{A})/P(\theta \geq \theta_2).$$

So (3.20) follows by substituting (3.14) and (3.16) into (3.22). Similarly,

$$\tilde{\delta} = P(\mathcal{A} \mid \theta \leq \theta_1) = \frac{P(\mathcal{A}) - P((\theta \geq \theta_1) \cap \mathcal{A})}{1 - P(\theta \geq \theta_1)}.$$

Combining this, (3.14), (3.16) and (3.17), we get (3.21). □

THEOREM 3.5. *The posterior risks are*

$$(3.23)\quad \gamma^* = P(\theta \geq \theta_2 \mid \mathcal{R}) = \frac{I_\Gamma\left(a, \frac{b}{Q(\theta_2)}\right) - \sum_{n=0}^{i_0-1} b^a \|G_n\| I_\Gamma\left(a+n, \frac{d_n}{Q(\theta_2)}\right) \Big/ d_n^{a+n}}{\Gamma(a) - \sum_{n=0}^{i_0-1} b^a \|G_n\| \Gamma(a+n)/d_n^{a+n}},$$

$$(3.24)\quad \delta^* = P(\theta \leq \theta_1 \mid \mathcal{A}) = \alpha_1.$$

PROOF. Note that

$$(3.25)\qquad \gamma^* = P(\theta \geq \theta_2 \mid \mathcal{R}) = \frac{P(\theta \geq \theta_2) - P((\theta \geq \theta_2) \cap \mathcal{A})}{P(\mathcal{R})}.$$

Substituting (3.14), (3.16) and (3.17) into (3.25), one gets (3.23). Similarly,

$$(3.26)\quad \delta^* = P(\theta \leq \theta_1 \mid \mathcal{A}) = 1 - \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{d_n^{a+n}} I_\Gamma\left(a+n, \frac{d_n}{Q(\theta_1)}\right) \Big/ \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{d_n^{a+n}} \Gamma(a+n).$$

Since $d_n/Q(\theta_1) = (1/2)\chi^2_{2(a+n)}(1-\alpha_1)$, $I_\Gamma(a+n, d_n/Q(\theta_1))/\Gamma(a+n) = P(\chi^2_{2(a+n)} \leq \chi^2_{2(a+n)}(1-\alpha_1)) = 1 - \alpha_1$. Therefore, $\delta^* = \alpha_1$. This completes the proof of Theorem 3.5. □

Intuitively, it is easy to understand why $P(\theta \leq \theta_1 \mid \mathcal{A}) = \alpha_1$. Indeed, $\mathcal{A} = \{\text{data} : P(\theta \leq \theta_1 \mid \text{data}) = \alpha_1\}$, and a standard measure-theoretic argument immediately gives the conclusion. Similarly, since $\mathcal{R} = \{\text{data} : P(\theta \geq \theta_2 \mid \text{data}) \leq \alpha_2$, with a positive probability that strict inequality holds$\}$, we conclude that

Fact 3.1. $P(\theta \geq \theta_2 \mid \mathcal{R}) < \alpha_2$.

From (3.24) and Fact 3.1, one is able to control the two posterior risks by choosing α_1 and α_2.

3.4 *Distribution and expected value of the number of failures*

Recall that N is the number of failures at the time we stop the test and make a decision. It is easy to see that $0 \leq N \leq i_0$. The distribution and the expected value of N are as follows.

THEOREM 3.6. *Let $P_\theta(\cdot) = P(\cdot \mid \theta)$. The cumulative distribution of N for given θ is*

$$P_\theta(N \leq n) = \begin{cases} \exp\left\{-\dfrac{d_0 - b}{Q(\theta)}\right\}, & \text{if } n = 0, \\ 1 - J_{\theta,n} + \displaystyle\sum_{i=n-1}^{n} \frac{\|G_i\|}{Q^i(\theta)} \exp\left\{-\frac{d_i - b}{Q(\theta)}\right\}, & \text{if } 1 \leq n < i_0; \end{cases} \tag{3.27}$$

the expected number of failed units for given θ is

$$E(N \mid \theta) = \sum_{n=0}^{i_0-1} J_{\theta,n} - 2\sum_{n=0}^{i_0-2} \frac{\|G_n\|}{Q^n(\theta)} \exp\left\{-\frac{d_n - b}{Q(\theta)}\right\} - \frac{\|G_{i_0-1}\|}{Q^{i_0-1}(\theta)} \exp\left\{-\frac{d_{i_0-1} - b}{Q(\theta)}\right\}; \tag{3.28}$$

the marginal cumulative distribution of N is

$$P(N \leq n) = \begin{cases} \left(\dfrac{b}{d_0}\right)^a, & \text{if } n = 0, \\ 1 - J_n + \displaystyle\sum_{i=n-1}^{n} \|G_i\| \frac{b^a \Gamma(a+i)}{d_i^{a+i}\Gamma(a)}, & \text{if } 1 \leq n < i_0; \end{cases} \tag{3.29}$$

and the expected number of failed units is

$$E(N) = \sum_{n=0}^{i_0-1} J_n - 2\sum_{n=0}^{i_0-2} \|G_n\| \frac{b^a\Gamma(a+n)}{d_n^{a+n}\Gamma(a)} - \|G_{i_0-1}\| \frac{b^a\Gamma(a+i_0-1)}{d_{i_0-1}^{a+i_0-1}\Gamma(a)}; \tag{3.30}$$

where $\|G_0\| = 1$, $\|G_n\|$ is the volume of G_n, $\sum_0^{-1} \cdot = 0$, $J_{\theta,0} = J_0 = 1$,

$$J_{\theta,n} = \begin{cases} \exp\left\{-\dfrac{c_1 \vee b - b}{Q(\theta)}\right\}, & \text{if } n = 1, \\ \dfrac{1}{Q(\theta)^{n-1}} \displaystyle\int \cdots \int_{G_{n-1}} \exp\left\{-\frac{(c_n - b) \vee s_{n-1}}{Q(\theta)}\right\} dy_{n-1} \cdots dy_1, & \text{if } n \geq 2, \end{cases} \tag{3.31}$$

and

$$J_n = \begin{cases} \dfrac{b^a}{(c_1 \vee b)^a}, & \textit{if } n = 1, \\ \dfrac{b^a\Gamma(a+n-1)}{\Gamma(a)} \displaystyle\int \cdots \int_{G_{n-1}} \dfrac{dy_{n-1}\cdots dy_1}{[c_n \vee (s_{n-1}+b)]^{a+n-1}}, & \textit{if } n \geq 2. \end{cases} \tag{3.32}$$

Here $s_0 = b$, $s_{n-1} = y_1 + \cdots + y_{n-1}$.

PROOF. See Appendix A.2. □

The formulas for computing $J_{\theta,n}$ and J_n are given by the following lemma.

LEMMA 3.4. *For* $2 \leq n \leq i_0 - 1$, *we have*

$$J_{\theta,n} = \exp\left\{-\frac{c_n \vee b - b}{Q(\theta)}\right\} - \sum_{i=0}^{n-2} \frac{\exp\left\{-\dfrac{c_n \vee d_i - b}{Q(\theta)}\right\}}{Q^i(\theta)} \|G_i\| + \exp\left\{-\frac{c_n - b}{Q(\theta)}\right\} \sum_{r=1}^{n-1} \frac{\xi_{n,r}}{Q^r(\theta)}, \tag{3.33}$$

and

$$J_n = \frac{b^a}{(c_n \vee b)^a} - \sum_{i=0}^{n-2} \frac{b^a\Gamma(a+i)\|G_i\|}{\Gamma(a)(c_n \vee d_i)^{a+i}} + \sum_{r=1}^{n-1} \frac{b^a\Gamma(a+r)}{c_n^{a+r}\Gamma(a)} \xi_{n,r}, \tag{3.34}$$

where

$$\xi_{n,r} = \sum_{(i_1,\ldots,i_k)\in\Psi_r} \rho_1(b;n;r;i_1,\ldots,i_k) - \sum_{l=2}^{r} \sum_{(i_1,\ldots,i_k)\in\Psi_l^*} \rho_2(b;n;r;l;i_1,\ldots,i_k). \tag{3.35}$$

Here $\rho_1(b;n;r;i_1,\ldots,i_k)$ *and* $\rho_2(b;n;r;l;i_1,\ldots,i_k)$ *are defined by* (3.8) *and* (3.9), *respectively.*

PROOF. See Appendix A.2. □

The above computation is quite complicated, but there is a simple inequality for J_n:

$$\frac{b^a\|G_n\|\Gamma(a+n-1)}{(c_n \vee d_{n-2})^{a+n-1}\Gamma(a)} \leq J_n \leq \frac{b^a\|G_n\|\Gamma(a+n-1)}{c_n^{a+n-1}\Gamma(a)}.$$

If $d_{n-2} \leq c_n$, then $J_n = b^a\|G_n\|\Gamma(a+n-1)/\{c_n^{a+n-1}\Gamma(a)\}$. Since $d_{i_0-1} \leq c_{i_0}$, we have $d_{n-2} \leq c_n$ for somewhat large n.

3.5 *Expected stopping time* $E(T)$

The expected stopping time in general cannot be evaluated in closed form. In this section, an upper bound and a lower bound for $E(T \mid \theta)$ are found. By using a result about Poisson processes, a closed form is given for exponential failure time.

3.5.1 *Bounds for* $E(T)$

Note that

$$E(N_j(T) \mid \theta)E(t_{11} \mid \theta) \le E(T \mid \theta) \le E(N_j(T) + 1 \mid \theta)E(t_{11} \mid \theta),$$

where $N_j(T)$ is the number of failure units for machine j at time T. Therefore,

$$\frac{1}{m}E(N \mid \theta)E(t_{11} \mid \theta) \le E(T \mid \theta) \le \left(\frac{1}{m}E(N \mid \theta) + 1\right) E(t_{11} \mid \theta),$$

where N is the total number of failure units. Note that $E(N \mid \theta)$ can be evaluated by (3.28), and

$$E(t_{11} \mid \theta) = \int_0^\infty tf(t \mid \theta)dt = \int_0^\infty \frac{tH'(t)}{Q(\theta)} \exp\left\{-\frac{H(t)}{Q(\theta)}\right\} dt.$$

Remark 3.1. For the Weibull distribution, $\mathcal{W}(\theta, \beta)$,

$$E(t_{11} \mid \theta) = \frac{\beta}{\theta^\beta}\int_0^\infty t^\beta \exp\{-t^\beta/\theta^\beta\}dt = \Gamma(1 + 1/\beta)\theta,$$
$$0 \le E(T \mid \theta) - T_\theta^* \le E(t_{11} \mid \theta),$$
$$0 \le E(T) - T^* \le E(t_{11}) \equiv \Gamma(1 + 1/\beta)\left[\frac{b^a\Gamma(a - 1/\beta)}{\Gamma(a)}\right]^{1/\beta},$$

where

$$(3.36) \qquad T_\theta^* = \frac{\Gamma\left(1 + \frac{1}{\beta}\right)}{m}\left\{\sum_{n=0}^{i_0-1} \theta J_{\theta,n} - 2\sum_{n=0}^{i_0-2} \frac{\|G_n\| \exp\left\{-\frac{d_n - b}{\theta^\beta}\right\}}{\theta^{n\beta-1}} - \frac{\|G_{i_0-1}\|}{\theta^{(i_0-1)\beta-1}} \exp\left\{-\frac{d_{i_0-1} - b}{\theta^\beta}\right\}\right\},$$

$$(3.37) \qquad T^* = \frac{\Gamma\left(1 + \frac{1}{\beta}\right)}{m}\left\{\sum_{n=0}^{i_0-1} J_n^* - 2\sum_{n=0}^{i_0-2} \|G_n\| \frac{b^a\Gamma\left(a + n - \frac{1}{\beta}\right)}{d_n^{a+n-1/\beta}\Gamma(a)} - \|G_{i_0-1}\|\frac{b^a\Gamma\left(a + i_0 - 1 - \frac{1}{\beta}\right)}{d_{i_0-1}^{a+i_0-1-1/\beta}\Gamma(a)}\right\},$$

$\|G_0\| = 1$, $\sum_0^{-1} \cdot = 0$, $J_{\theta,n}$ is given by (3.31), and

$$J_n^* = \begin{cases} \dfrac{b^{1/\beta}\Gamma(a-1/\beta)}{\Gamma(a)}, & \text{if } n=0,\\ \dfrac{b^a\Gamma(a-1/\beta)}{\Gamma(a)(c_1\vee b)^{a-1/\beta}}, & \text{if } n=1,\\ \dfrac{b^a}{\Gamma(a)}\left\{\dfrac{\Gamma(a-1/\beta)}{(c_n\vee b)^{a-1/\beta}} - \displaystyle\sum_{i=0}^{n-2}\dfrac{\|G_i\|\Gamma(a+i-1/\beta)}{(c_n\vee d_i)^{a+i-1/\beta}} \right. & \\ \qquad \left. + \displaystyle\sum_{r=1}^{n-1}\dfrac{\xi_{n,r}\Gamma(a+r-1/\beta)}{c_n^{a+r-1/\beta}}\right\}, & \text{if } n\geq 2. \end{cases}$$

3.5.2 *$E(T)$ for exponential failure time*

Suppose that $M(t)$ ($t \geq 0$) is a Poisson process with intensity $1/\theta$. Thus $M(t)$ has a Poisson distribution with mean t/θ. For two increasing sequences of constants $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ satisfying $a_0 < 0 < b_0$, define

$$T = \inf\{t > 0 : t > b_{M(t)} \text{ or } t \leq a_{M(t)}\}.$$

LEMMA 3.5. *If $\exists i_0$ such that $b_{i_0-1} \leq a_{i_0}$, then*

$$E\{M(T)\} = E(T)/\theta. \tag{3.38}$$

PROOF. See Appendix A.2. □

Note that, for an arbitrary stopping time T, equation (3.38) may be invalid. For example, if $T = \inf\{t > 0 : t > M(t) + 1\}$, then it is easy to show that $T = M(T) + 1$ and $\tilde{T} = M(\tilde{T}) + 1$ for $\tilde{T} = \min(T, 3)$. Thus $E\{M(T)\} \neq E(T)/\theta$ and $E\{M(\tilde{T})\} \neq E(\tilde{T})/\theta$.

THEOREM 3.7. *For the BSRDT plan, if the product lifetime is $\mathcal{E}(\theta)$ distributed, then $E(T \mid \theta) = T_\theta^*$ and*

$$E(T) = \frac{1}{m}\left\{\sum_{n=0}^{i_0-1} J_n^* - 2\sum_{n=0}^{i_0-2}\|G_n\|\frac{b^a\Gamma(a+n-1)}{d_n^{a+n-1}\Gamma(a)} - \|G_{i_0-1}\|\frac{b^a\Gamma(a+i_0-1-1)}{d_{i_0-1}^{a+i_0-1-1}\Gamma(a)}\right\},$$

where T_θ^* is given by (3.36) with $\beta = 1$ and

$$J_n^* = \begin{cases} \dfrac{b}{a-1}, & \text{if } n=0,\\ \dfrac{b^a}{(a-1)(c_1\vee b)^{a-1}}, & \text{if } n=1,\\ \dfrac{b^a}{\Gamma(a)}\left\{\dfrac{\Gamma(a-1)}{(c_n\vee b)^{a-1}} - \displaystyle\sum_{i=0}^{n-2}\dfrac{\|G_i\|\Gamma(a+i-1)}{(c_n\vee d_i)^{a+i-1}} + \displaystyle\sum_{r=1}^{n-1}\dfrac{\xi_{n,r}\Gamma(a+r-1)}{c_n^{a+r-1}}\right\}, & \text{if } n\geq 2. \end{cases}$$

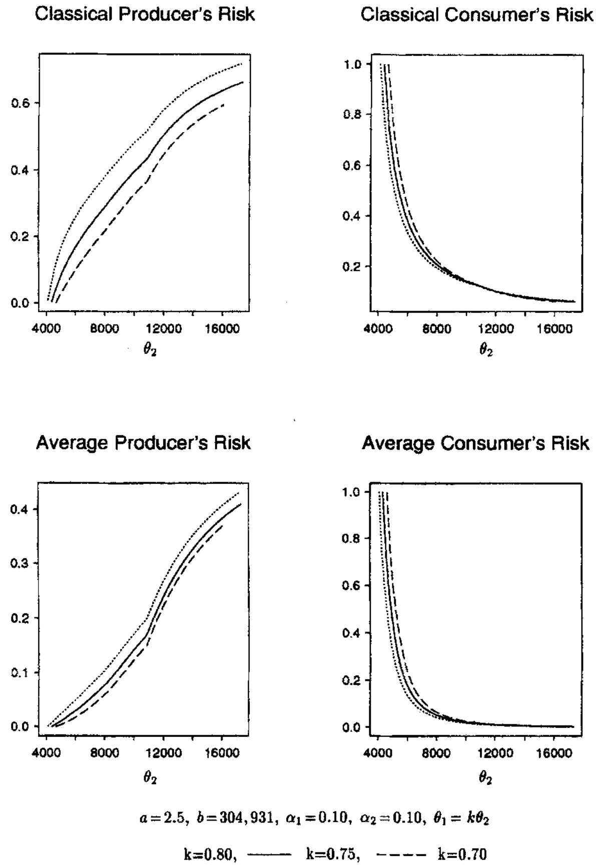


Fig. 2. Classical and average risks.

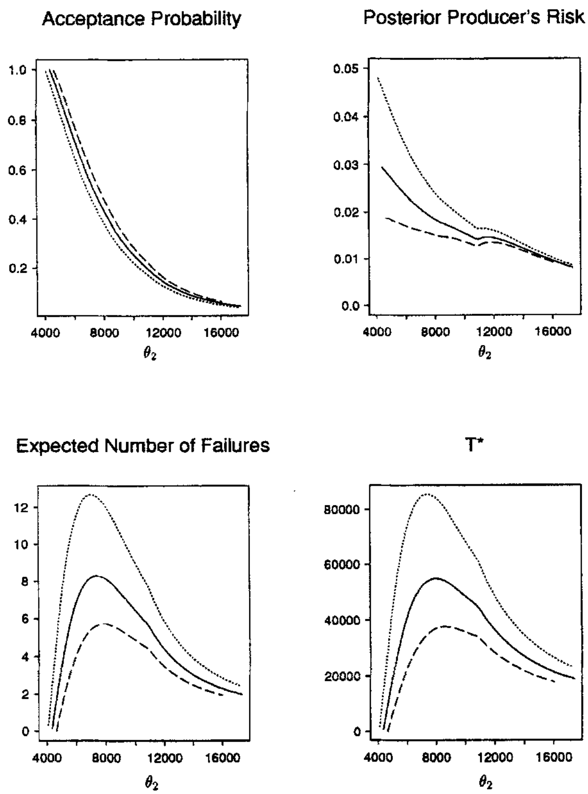
PROOF. The first result follows from Lemma 3.5 immediately. The rest follows from the expression of T_θ^* and the assumption $\theta \sim \mathcal{IG}(a, b)$. □

3.6 *Numerical examples*

A Fortran program has been developed to compute the cumulative distribution of N, $E(N)$, T^*, and all the risks. For fixed θ_1, θ_2, α_1 and α_2, if $i_0 = 10$, the computing time is about two seconds to compute all the risks and stopping quantities on a Sun 3/60 workstation. It takes about 15 seconds if $i_0 = 15$ and 30 minutes for $i_0 = 30$.

Example. Suppose that the p.d.f. of failure time is $\mathcal{W}(\theta, \beta)$, a guess for the mean of θ is $\theta_0 = 8{,}000$ (hours), and a guess for the standard deviation of θ is $\sigma_0 = 6{,}000$ (hours). Assume that $\beta = 1.35$. Then we can compute that the parameters a and b in the prior distribution are 2.5 and 304,931, respectively.

Choose $\alpha_1 = 0.10$ and $\alpha_2 = 0.10$, and $\theta_1 = k\theta_2$ for various k mentioned below. We choose θ_1 to be proportional to θ_2 for convenience in graphing the results. Various risks, T^* (with $m = 1$), $E(N)$, and the cumulative distributions of N with



$a=2.5,\ b=304,931,\ \alpha_1=0.10,\ \alpha_2=0.10,\ \theta_1=k\theta_2$

$\cdots\cdots$ k=0.80, ——— k=0.75, - - - - k=0.70

Fig. 3. $P(\mathcal{A})$, $P(\theta \geq \theta_2 \mid \mathcal{R})$, $E(N)$ and T^*

respect to θ_2 are shown in Figs. 2–4 for $k = 0.70$ (with a dashed curve), $k = 0.75$ (with a solid curve) and $k = 0.80$ (with a dotted curve).

The classical consumer's risk, average consumer's risk and the acceptance probability have a similar shape for these three situations. Almost all the curves bend at the value $\theta_2 = [2b/\chi^2_{2(a+1)}(1-\alpha_1)]^{1/\beta}$. The curve of the posterior consumer's risk, $P(\theta \leq \theta_1 \mid \mathcal{A})$, has not been drawn since it is the constant α_1. The posterior producer's risk $P(\theta \geq \theta_2 \mid \mathcal{R})$ is not monotone with respect to θ_2 and is always less than $\alpha_2 = 0.10$, as shown in Fact 3.1. Also, it reaches its local minimum at $\theta_2 = \{2b/\chi^2_{2(a+i)}(\alpha_2)\}^{1/\beta}$, $i = 0, 1, \ldots$. Here T^* is the lower bound for $E(T)$, and the upper bound is $T^* + E(t_{11})$, where $E(t_{11}) = 7{,}335.91$. As intuition would suggest, a larger k induces a larger rejection region, and hence increases all the producer's risks, the expected number of failures and the expected testing time, but decreases all the consumer's risks.

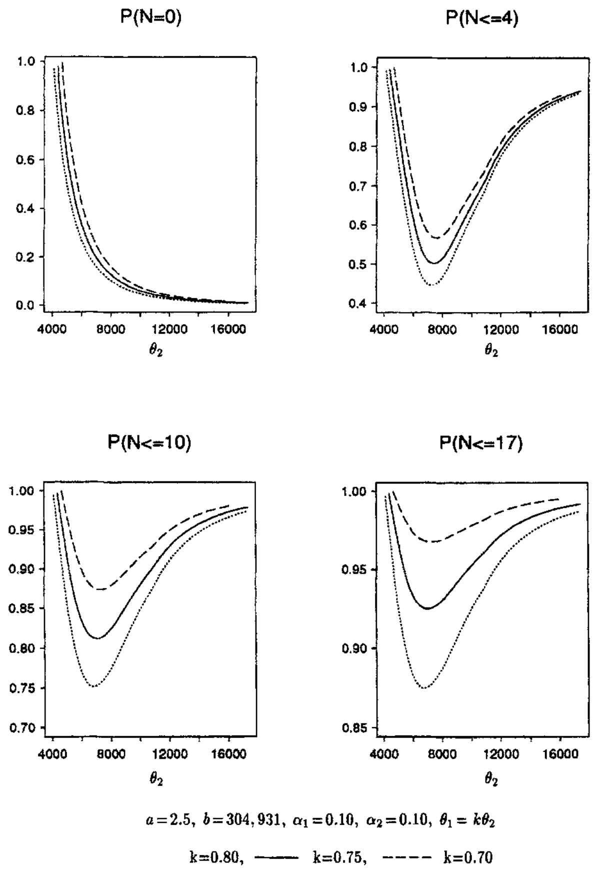


Fig. 4. $P(N \leq n)$, $n = 0$, 4, 10, 17.

Appendix

A.1 *Selecting a prior from the conjugate family*

Option 1: Ask for Q_0 (the best guess for the mean of $Q(\theta)$) and σ_{Q_0} (the standard deviation for the guess). Then a and b can be determined by the equations

$$\begin{cases} Q_0 = b\Gamma(a-1)/\Gamma(a), \\ \sigma_{Q_0}^2 + Q_0^2 = b^2\Gamma(a-2)/\Gamma(a), \end{cases} \quad \Rightarrow \quad \begin{cases} a = 2 + Q_0^2/\sigma_{Q_0}^2, \\ b = Q_0(a-1). \end{cases}$$

Option 2: For the Weibull case with known β, $Q(\theta) = \theta^\beta$. Ask for θ_0 (the best guess for the mean of θ), and σ_0 (the standard deviation for the guess). Then a and b can be determined by the equations

$$\begin{cases} \theta_0 = \dfrac{b^{1/\beta}\Gamma\left(a - \dfrac{1}{\beta}\right)}{\Gamma(a)}, \\ \\ \theta_0^2 + \sigma_0^2 = \dfrac{b^{2/\beta}\Gamma\left(a - \dfrac{2}{\beta}\right)}{\Gamma(a)}, \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\Gamma(a-2/\beta)\Gamma(a)}{\Gamma^2(a-1/\beta)} = 1 + \dfrac{\sigma_0^2}{\theta_0^2}, \\ b = \left[\dfrac{\theta_0\Gamma(a)}{\Gamma(a-1/\beta)}\right]^\beta. \end{cases}$$

Note that an approximate value of a can be determined by iteratively solving

$$a = 0.5 + \left[\ln\left(1 + \frac{\sigma_0^2}{\theta_0^2}\right) + \frac{2}{\beta}\ln\left(1 - \frac{1}{a\beta - 1}\right)\right] \Big/ \ln\left(1 - \frac{1}{(a\beta-1)^2}\right),$$

starting with the initial estimate

$$\hat{a} = \frac{1}{2\beta}\left\{3 + \exp\left[\frac{2}{\beta}\ln\left(1 + \frac{\sigma_0^2}{\theta_0^2}\right)\right]\right\}.$$

Actually, $\hat{a}$ is often a very reasonable approximation to a itself.

Option 3: For the Weibull p.d.f. with known β, ask for the 0.5 quantile, $q(0.5)$, and the 0.75 quantile, $q(0.75)$, of the predictive distribution. Then it can be shown that the matching choices of a and b are

$$a = \ln 2 / \ln\{[q(0.75)/q(0.5)]^\beta - 1\},$$
$$b = [q(0.5)]^\beta / (2^{1/a} - 1),$$

assuming that $q(0.75) > 2^{1/\beta} q(0.5)$. If this is not satisfied, the given functional form for the prior may not be suitable.

For example, assume that the p.d.f. of lifetime is $\mathcal{W}(\theta, \beta)$ with $\beta = 1.35$. If we have from engineering knowledge and/or knowledge of previous similar products that the best guess for the mean of θ is $\theta_0 = 8{,}000$ (hours) and the standard deviation for the guess is $\sigma_0 = 6{,}000$ (hours), then $a = 2.5$ and $b = 304{,}931$. If we know that $q(0.5) = 5{,}000$ (hours) and $q(0.75) = 9{,}147$ (hours), then $a = 3.0$ and $b = 379{,}112$.

A.2 *Proofs of theorems and lemmas*

Note that $\mathcal{A}$ and $\mathcal{R}$ can be written as disjoint unions:

(A.1) $$\mathcal{A} = \bigcup_{n=0}^{i_0 - 1} \mathcal{A} \cap (N = n), \qquad \mathcal{R} = \bigcup_{n=1}^{i_0} \mathcal{R} \cap (N = n),$$

where

(A.2) $\mathcal{A} \cap (N = n) = \{c_j < W_j + b \le d_{j-1}, j = 1, \ldots, n, W_{n+1} + b > d_n\}$,

(A.3) $\mathcal{R} \cap (N = n) = \{c_j < W_j + b \le d_{j-1}, j = 1, \ldots, n-1, W_n + b \le c_n\}$.

Here $W_n = V(T_n)$, and $V(\cdot)$ is defined by (2.5) and T_n is the n-th failure time. It is easy to see that evaluation of $E(N)$ and all the risks depends on the joint distribution of $(W_1, W_2, \ldots, W_{i_0})$, which is given by Lemma A.1 in the following subsection. We will use the notation, $Y_1 = W_1$, $Y_j = W_j - W_{j-1}$ $(j \geq 2)$. Also, let $P_\theta(\cdot)$ denote the conditional probability $P(\cdot \mid \theta)$ for fixed θ.

A.2.1 *A crucial lemma*

The following lemma gives the distribution of $Y_1, \ldots, Y_n$ and is of independent interest.

LEMMA A.1. *For given θ and any n, $Y_1, Y_2, \ldots, Y_n$ are i.i.d. exponential random variables with common mean $Q(\theta)$.*

In order to prove the lemma, a known result about point processes will be stated here. Let $M(t)$ be a point process which is right continuous and piecewise constant with jump 1, and satisfy $M(0) = 0$. Assume that $\lambda(t)$ is an intensity for this process, i.e., for fixed $t > 0$, $\lambda(t)$ is measurable with respect to the σ-field $\mathcal{F}_{t^-} \equiv \sigma\{M(s) : 0 \leq s < t\}$, and satisfies

$$\lim_{\Delta t \to 0} P(M(t + \Delta t) - M(t) \geq 1 \mid \mathcal{F}_{t^-}) = \lambda(t), \qquad \text{a.s.}$$

The following theorem follows from Theorem T16 of Bremaud ((1981), p. 41).

THEOREM A.1. *Define $\tau(t) = \inf\{s \geq 0 : \int_0^s \lambda(r)dr = t\}$ $(t \geq 0)$. Then, $M(\tau(t))$ $(t \geq 0)$ is a standard Poisson process with the intensity* 1.

PROOF OF LEMMA A.1. Since θ is considered to be fixed, without loss of generality, assume that $Q(\theta) = 1$. It is easy to see that $N(t)$, the total number of failures at time t, is a point process which is right continuous and piecewise constant with jump 1, and satisfies $N(0) = 0$.

It follows from the assumptions on $H(\cdot)$ that $V(t)$, defined by (2.5), is a strictly increasing and continuous function of t and satisfies that $V(0) = 0$ and $\lim_{t\to\infty} V(t) = \infty$. Note that units are tested independently on all machines, it is easy to see that $(d/dt)V(t)$ is an intensity for the process $N(t)$. Therefore $\tau(t) = V^{-1}(t)$, where $V^{-1}(\cdot)$ is the inverse function of $V(\cdot)$. From Theorem A.1, $N(V^{-1}(t))$ $(t \geq 0)$ is a standard Poisson process with the intensity 1. Thus $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $\mathcal{E}(1)$ r.v.s. This completes the proof of Lemma A.1. □

A.2.2 *Proofs of technical preliminaries*

PROOF OF LEMMA 3.1. First, for fixed $2 \leq k \leq n$, we will show that

$$\#\left\{(i_1, i_2, \ldots, i_k) : i_j \geq 1, \sum_{j=1}^{k} i_j = n\right\} = \binom{n-1}{k-1}, \tag{A.4}$$

$$\#\left\{(i_1, i_2, \ldots, i_k) : i_j \geq 1, i_k \geq 2, \sum_{j=1}^{k} i_j = n\right\} = \binom{n-2}{k-1}. \tag{A.5}$$

Induction on k will be used to prove (A.4). If $k = 2$ and $n \geq k$,

$$\#\{(i,j) : i,j \geq 1, i+j = n\} = \#\{(1,n-1),(2,n-2),\dots,(n-1,1)\}$$
$$= n-1 = \binom{n-1}{k-1}.$$

So (A.4) holds for $k = 2$. Suppose, next, that (A.4) is true for all $k \leq n$; then, for $k+1 \leq n$,

$$\#\left\{(i_1,\dots,i_k,i_{k+1}) : i_j \geq 1, \sum_{j=1}^{k+1} i_j = n\right\}$$
$$= \sum_{l=1}^{n-k} \#\left\{(i_1,\dots,i_k,l) : i_j \geq 1, \sum_{j=1}^{k} i_j = n-l\right\},$$

which equals, by the induction assumption, $\sum_{l=1}^{n-k}\binom{n-l-1}{k-1} = \binom{n-1}{k} = \binom{n-1}{(k+1)-1}$. This means that (A.4) is also true for $k+1 \leq n$. The proof of (A.5) is similar. Since

$$\#\{\Psi_n\} = \#\{(n)\} + \sum_{k=2}^{n} \#\{(i_1,\dots,i_k) : i_j \geq 1, i_1 + \cdots + i_k = n\},$$

equation (A.4) yields $\#\{\Psi_n\} = 1 + \sum_{k=2}^{n}\binom{n-1}{k-1} = 2^{n-1}$. Similarly, $\#\{\Psi_n^*\} = 2^{n-2}$, using (A.5). So (3.10) holds. Note that the right hand side of (3.11) is a subset of Ψ_{n+1}, and the number of elements in the right hand side of (3.11) is $2(2^{n-1}) = 2^{(n+1)-1}$, by (3.10). Thus (3.11) is true. Similarly we can prove (3.12). This completes the proof. □

In order to prove Lemma 3.2, an integral formula is needed.

LEMMA A.2. *For any integers $i \geq 0$ and $j \geq 2$, and any real number $x > 0$,*

$$\text{(A.6)} \qquad \int_{c_1 \vee b}^{d_0} a_{ij}^x(y)dy = a_{ij}^x(c_j)a_{j-1,1}(b) + \frac{1}{x+1}a_{ij}^{x+1}(b) - \frac{1}{x+1}a_{ij}^{x+1}(d_0),$$

where $a_{ij}(\cdot)$ is defined by (3.3).

PROOF OF LEMMA A.2. We first consider the case $i = 0$. Denote

$$A_0 \equiv \int_{c_1 \vee b}^{d_0} a_{0j}^x(y)dy = \int_{c_1 \vee b}^{d_0} (d_{j-1} - c_j \vee y)^x dy.$$

Since $j \geq 2$ and $c_0 < b < d_0$, there are 3 different situations to consider: $c_0 < b \leq c_j < d_0$, $c_0 < c_j < b < d_0$, and $c_0 < b < d_0 \leq c_j$. By a simple calculation, A_0 equals

$$\begin{cases} (d_{j-1}-c_j)^x(c_j-c_1\vee b)+\frac{1}{x+1}(d_{j-1}-c_j)^{x+1}-\frac{1}{x+1}(d_{j-1}-d_0)^{x+1}, & \text{if } c_0<b\le c_j<d_0 \\ \frac{1}{x+1}(d_{j-1}-b)^{x+1}-\frac{1}{x+1}(d_{j-1}-d_0)^{x+1}, & \text{if } c_0<c_j<b<d_0 \\ (d_{j-1}-c_j)^x(d_0-c_1\vee b), & \text{if } c_0<b<d_0\le c_j \end{cases}$$

$$= (d_{j-1}-c_j)^x\{(c_j\wedge d_0)\vee b - c_1\vee b\} + \frac{(d_{j-1}-c_j\vee b)^{x+1}}{x+1} - \frac{(d_{j-1}-c_j\vee d_0)^{x+1}}{x+1}$$

$$= a_{0j}^x(c_j)a_{j-1,1}(b)+\frac{1}{x+1}a_{0j}^{x+1}(b)-\frac{1}{x+1}a_{0j}^{x+1}(d_0),$$

by the definition of (3.3). So (A.6) holds for $i=0$. Similarly, we get for $i\ge 1$ that

$$\int_{c_1\vee b}^{d_0} a_{ij}^x(y)dy = \int_{c_1\vee b}^{d_0}\{(c_{i+j}\wedge d_{j-1})\vee y - c_j\vee y\}^x dy$$

$$= \begin{cases} (c_{i+j}-c_j)^x(c_j-c_1\vee b)+\frac{1}{x+1}(c_{i+j}-c_j)^{x+1}, & \text{if } c_0<b\le c_j<c_{i+j}<d_0, \\ \frac{1}{x+1}(c_{i+j}-b)^{x+1}, & \text{if } c_0<c_j<b\le c_{i+j}<d_0, \\ 0, & \text{if } c_0<c_j<c_{i+j}<b<d_0, \\ (c_{i+j}\wedge d_{j-1}-c_j)^x(c_j-c_1\vee b) \\ \quad +\frac{1}{x+1}(c_{i+j}\wedge d_{j-1}-c_j)^{x+1}-\frac{1}{x+1}(c_{i+j}\wedge d_{j-1}-d_0)^{x+1}, & \text{if } c_0<b\le c_j<d_0\le c_{i+j}, \\ \frac{1}{x+1}(c_{i+j}\wedge d_{j-1}-b)^{x+1}-\frac{1}{x+1}(c_{i+j}\wedge d_{j-1}-d_0)^{x+1}, & \text{if } c_0<c_j<b<d_0\le c_{i+j}, \\ (c_{i+j}\wedge d_{j-1}-c_j)^x(d_0-c_1\vee b), & \text{if } c_0<b<d_0\le c_j, \end{cases}$$

$$= (c_{i+j}\wedge d_{j-1}-c_j)^x((c_j\wedge d_0)\vee b - c_1\vee b) + \frac{1}{x+1}((c_{i+j}\wedge d_{j-1})\vee b - c_j\vee b)^{x+1} - \frac{1}{x+1}((c_{i+j}\wedge d_{j-1})\vee d_0 - c_j\vee d_0)^{x+1},$$

which also equals the right hand side of (A.6). Thus Lemma A.2 is proved. □

PROOF OF LEMMA 3.2. We will also use induction to prove Lemma 3.2. Note that

$$\|G_n\| = \int_{0\vee(c_1-b)}^{d_0-b}\int_{0\vee(c_2-(b+y_1))}^{d_1-(b+y_1)}\cdots\int_{0\vee(c_n-(b+y_1+\cdots+y_{n-1}))}^{d_{n-1}-(b+y_1+\cdots+y_{n-1})} dy_n\cdots dy_2dy_1.$$

If we write $\|G_n\| = \|G_n\|(b;c_1,\ldots,c_n;d_0,\ldots,d_{n-1})$, then

$$\text{(A.7)}\qquad \|G_{n+1}\| = \int_{0\vee(c_1-b)}^{d_0-b}\|G_n\|(b+y_1;c_2,\ldots,c_{n+1};d_1,\ldots,d_n)dy_1 = \int_{c_1\vee b}^{d_0}\|G_n\|(y;c_2,\ldots,c_{n+1};d_1,\ldots,d_n)dy.$$

Now, for $n = 1$, $\|G_1\| = a_{01}(b)$, so (3.13) holds. For $n = 2$, it follows from (A.7) that

$$\|G_2\| = \int_{c_1 \vee b}^{d_0} \|G_1\|(y; c_2; d_1)dy = \int_{c_1 \vee b}^{d_0} a_{02}(y)dy$$
$$= a_{02}(c_2)a_{1,1}(b) + \frac{1}{2}a_{02}^2(b) - \frac{1}{2}a_{02}^2(d_0).$$

The last equality follows from (A.6). Thus (3.13) is true for $n = 2$. Suppose (3.13) is true for n. From (A.7), we then know that, for $n + 1$,

$$\text{(A.8)} \quad \|G_{n+1}\| = \sum_{(i_1,\ldots,i_k)\in\Psi_n} \int_{c_1 \vee b}^{d_0} \rho_1(y; n; n; i_1, \ldots, i_k; c_j, d_{j-1}, 2 \le j \le n+1)dy$$
$$- \sum_{l=2}^{n} \sum_{(i_1,\ldots,i_k)\in\Psi_l^*} \int_{c_1 \vee b}^{d_0} \rho_2(y; n; n; l; i_1, \ldots, i_k; c_j, d_{j-1}, 2 \le j \le n+1)dy.$$

By the definition of ρ_1 and the fact that $n + 1 - i_{k+1}^+ = i_k + 1$,

$$\text{(A.9)} \quad \int_{c_1 \vee b}^{d_0} \rho_1(y; n; n; i_1, \ldots, i_k; c_j, d_{j-1}, 2 \le j \le n+1)dy$$
$$= \frac{1}{\prod_{h=1}^k i_h!} \left\{ \prod_{j=1}^{k-1} a_{n+1,j}^*(c_{n+1,j}^*) \right\} \int_{c_1 \vee b}^{d_0} a_{n+1,k}^*(y)dy$$
$$= \frac{1}{\prod_{h=1}^k i_h!} \prod_{j=1}^{k-1} a_{n+1,j}^*(c_{n+1,j}^*)$$
$$\cdot \{a_{n+1,k}^*(c_{n+1,k}^*)a_{n+1,k+1}^*(b) + a_{n+1,k}^*(b) - a_{n+1,k}^*(d_0)\}$$
$$= \rho_1(b; n+1; n+1; i_1, \ldots, i_k, 1) + \rho_1(b; n+1; n+1; i_1, \ldots, i_k + 1)$$
$$- \rho_2(b; n+1; n+1; n+1; i_1, \ldots, i_k + 1),$$

where the second equality follows from Lemma A.2. In addition,

$$\text{(A.10)} \quad \int_{c_1 \vee b}^{d_0} \rho_2(y; n; n; l; i_1, \ldots, i_k; c_j, d_{j-1}, 2 \le j \le n+1)dy$$
$$= \frac{1}{\prod_{h=1}^k i_h!} \left\{ \prod_{j=1}^{k-1} a_{n+1,j}^*(c_{n+1,k}^*) \right\} a_{n+1,k}^*(d_{n+1-l})$$
$$= \rho_2(b; n+1; n+1; l; i_1, \ldots, i_k).$$

Substituting (A.9) and (A.10) into (A.8), $\|G_{n+1}\|$ equals

$$\sum_{(i_1,\ldots,i_k)\in\Psi_n} \{\rho_1(b; n+1; n+1; i_1, \ldots, i_k, 1) + \rho_1(b; n+1; n+1; i_1, \ldots, i_k + 1)\}$$

$$
\begin{aligned}
&- \sum_{(i_1,\ldots,i_k)\in\Psi_n} \rho_2(b;n+1;n+1;n+1;i_1,\ldots,i_k+1) \\
&- \sum_{l=2}^{n} \sum_{(i_1,\ldots,i_k)\in\Psi_l^*} \rho_2(b;n+1;n+1;l;i_1,\ldots,i_k) \\
= &\sum_{(i_1,\ldots,i_k)\in\Psi_{n+1}} \rho_1(b;n+1;n+1;i_1,\ldots,i_k) \\
&- \sum_{l=2}^{n+1} \sum_{(i_1,\ldots,i_k)\in\Psi_l^*} \rho_2(b;n+1;n+1;l;i_1,\ldots,i_k).
\end{aligned}
$$

The last equality follows from (3.11) and (3.12). So (3.13) is true for $n+1$. This completes the induction argument. □

A.2.3 *Proof of lemma about risks*

PROOF OF LEMMA 3.3. Since $Q(\theta) \sim \mathcal{IG}(a,b)$, i.e., $1/Q(\theta) \sim \Gamma(a,b)$, we have $b/Q(\theta) \sim \Gamma(a,1)$. Thus

$$
P(\theta \geq \mu) = \frac{1}{\Gamma(a)} \int_0^{b/Q(\mu)} x^{a-1} e^{-x} dx = I_\Gamma \left(\frac{b}{Q(\mu)}; a \right) \Big/ \Gamma(a).
$$

So (3.14) is true. Since

$$
P_\theta(\mathcal{A} \cap (N=0)) = P_\theta(mT_1 > d_0 - b) = \exp\left\{ -\frac{d_0 - b}{\theta} \right\}, \tag{A.11}
$$

(3.15) follows from (A.1) and (A.11), for $i_0 = 1$. If $i_0 \geq 2$, it follows from (A.2) that, for $1 \leq n < i_0$, $P_\theta(\mathcal{A} \cap (N=n))$ equals

$$
\begin{aligned}
\text{(A.12)}\quad & P_\theta \left\{ 0 \vee \left(c_j - b - \sum_{k=1}^{j-1} Y_k \right) < Y_j \leq d_{j-1} - b - \sum_{k=1}^{j-1} Y_k, 1 \leq j \leq n, \right. \\
& \qquad\qquad\qquad\qquad \left. \sum_{k=1}^{n+1} Y_k > d_n - b \right\} \\
&= \int \cdots \int_{G_n} P_\theta \left(Y_{n+1} > d_n - b - \sum_{k=1}^{n} y_k \right) \\
&\qquad \cdot \frac{1}{\theta^n} \exp\left\{ -\frac{y_1 + \cdots + y_n}{\theta} \right\} dy_n \cdots dy_1 \\
&= \frac{\|G_n\|}{\theta^n} \exp\left\{ -\frac{d_n - b}{\theta} \right\}.
\end{aligned}
$$

So (3.15) holds for $i_0 \geq 2$. Finally,

$$
P((\theta \geq \mu) \cap \mathcal{A}) = \int_\mu^\infty P(\mathcal{A} \mid \theta) \pi(\theta) d\theta \tag{A.13}
$$

and

$$\int_{\mu}^{\infty} \frac{e^{-(d_n-b)/Q(\theta)}}{Q^n(\theta)} \pi(\theta) d\theta = \frac{b^a}{\Gamma(a)} \int_{Q(\mu)}^{\infty} \frac{e^{-d_n/s}}{s^{a+n}} ds = \frac{b^a}{d_n^{a+n}\Gamma(a)} I_\Gamma\left(a+n, \frac{d_n}{Q(\mu)}\right). \tag{A.14}$$

Equation (3.16) follows from combining (3.15), (A.13) and (A.14). □

A.2.4 *Proofs about* $E(N)$

PROOF OF THEOREM 3.6. First we have

$$P_\theta(n) = P_\theta(\mathcal{A} \cap (N=n)) + P_\theta(\mathcal{R} \cap (N=n)). \tag{A.15}$$

For $i_0 = 1$, equation (3.27) follows from combining (A.1), (A.2) and (A.11). If $i_0 \geq 2$,

$$P_\theta(\mathcal{R} \cap (N=1)) = P_\theta(Y_1 + b \leq c_1) = 1 - \exp\left\{-\frac{0 \vee (c_1 - b)}{Q(\theta)}\right\}.$$

Combining this and (A.12) establishes (3.27) for $i_0 = 2$. If $i_0 \geq 3$, $P_\theta(\mathcal{R} \cap (N=n))$ equals

$$\begin{aligned}
&P_\theta\left(0 \vee \left(c_j - b - \sum_{k=1}^{j-1} Y_k\right) < Y_j \leq d_{j-1} - b - \sum_{k=1}^{j-1} Y_k, 1 \leq j \leq n-1,\right.\\
&\qquad\qquad\left.\sum_{k=1}^{n} Y_k \leq c_n - b\right)\\
&= \int \cdots \int_{G_{n-1}} \{1 - P_\theta[Y_n > 0 \vee (c_n - b - s_{n-1})]\}\\
&\qquad \cdot \frac{1}{Q(\theta)^{n-1}} \exp\left\{-\frac{s_{n-1}}{Q(\theta)}\right\} dy_{n-1} \cdots dy_1\\
&= \frac{1}{Q(\theta)^{n-2}} \int \cdots \int_{G_{n-2}} \exp\left\{-\frac{(c_{n-1} - b) \vee s_{n-2}}{Q(\theta)}\right\} dy_{n-2} \cdots dy_1\\
&\quad - \frac{1}{Q(\theta)^{n-1}} \int \cdots \int_{G_{n-1}} \exp\left\{-\frac{(c_n - b) \vee s_{n-1}}{Q(\theta)}\right\} dy_{n-1} \cdots dy_1\\
&\quad - \frac{\|G_{n-2}\|}{Q(\theta)^{n-2}} \exp\left\{-\frac{d_{n-2} - b}{Q(\theta)}\right\}\\
&= J_{\theta,n-1} - J_{\theta,n} - \frac{\|G_{n-2}\|}{Q(\theta)^{n-2}} \exp\left\{-\frac{d_{n-2} - b}{Q(\theta)}\right\},
\end{aligned}$$

for $2 \leq n < i_0$, where $s_n = y_1 + \cdots + y_n$. Equations (3.27) and (3.28) follow by algebra and from (A.12). Equations (3.29) and (3.30) follow immediately from the assumption on the prior. □

PROOF OF LEMMA 3.4. It is enough to prove (3.33). The proof is similar to the proof of Lemma 3.2 and is hence only outlined.

1. By an argument similar to that leading to (A.6), the following integral formula can be established:

$$\text{(A.16)}\quad \int_{c_1\vee b}^{d_0} \exp\left\{-\frac{c_j \vee y}{\lambda}\right\} dy = a_{j-1,1}(b)\exp\left\{-\frac{c_j}{\lambda}\right\} + \lambda \exp\left\{-\frac{c_j\vee b}{\lambda}\right\} - \lambda \exp\left\{-\frac{c_j \vee d_0}{\lambda}\right\},$$

for $j = 2, 3, \ldots,$ and $\lambda > 0$, where $a_{j-1,1}(\cdot)$ is given by (3.3).

2. Define

$$\begin{aligned}\tilde{J}_{\theta,n} &= e^{-b/Q(\theta)} J_{\theta,n}\\ &= \frac{1}{Q(\theta)^{n-1}} \int_{c_1\vee b}^{d_0} \cdots \int_{c_{n-1}\vee s_{n-2}}^{d_{n-2}} \exp\left\{-\frac{c_n \vee s_{n-1}}{Q(\theta)}\right\} ds_{n-1}\cdots ds_1.\end{aligned}$$

Then prove (3.33) by induction, Lemma 3.6 and equation (A.16).

A.2.5 *Proof of the lemma about a Poisson process*

PROOF OF LEMMA 3.5. Replace $c_j - b$ and $d_{j-1} - b$ in (3.1) by a_j and b_{j-1}, respectively. From Theorem 3.6,

$$\text{(A.17)}\quad E(M(T)) = \sum_{n=0}^{i_0-1} J_{\theta,n} - 2\sum_{n=0}^{i_0-2} \frac{\|G_n\|}{\theta^n}\exp\left\{-\frac{b_n}{\theta}\right\} - \frac{\|G_{i_0-1}\|}{\theta^{i_0-1}}\exp\left\{-\frac{b_{i_0-1}}{\theta}\right\}.$$

Let $Y_1, Y_2, \ldots,$ be iid exponential random variables with common mean $1/\theta$. Then

$$\text{(A.18)}\quad T = \sum_{n=1}^{i_0}\{H_{1n} + H_{2n}\},$$

where

$$\begin{aligned}
H_{11} &= b_0 I(Y_1 > b_0),\\
H_{1n} &= b_{n-1} I\{((Y_1, \ldots, Y_{n-1}) \in G_{n-1}) \cap (S_n > b_{n-1})\}, \qquad 2 < n \le i_0,\\
H_{21} &= Y_1 I(Y_1 \le a_1),\\
H_{2n} &= S_n I\{((Y_1, \ldots, Y_{n-1}) \in G_{n-1}) \cap (S_n \le a_n)\}, \qquad 2 \le n < i_0,\\
H_{2i_0} &= S_{i_0-1} I\{((Y_1, \ldots, Y_{i_0-1}) \in G_{i_0-1}) \cap (S_n \le b_{i_0-1})\}.
\end{aligned}$$

Note that, for $1 \le n \le i_0$,

$$EH_{1n} = \frac{b_{n-1}\|G_n\|}{\theta^{n-1}}\exp\left\{-\frac{b_{n-1}}{\theta}\right\}.$$

Define $H_{31} = H_{11}$ and

$$H_{3n} = S_n I\{((Y_1, \dots, Y_{n-1}) \in G_{n-1}) \cap (S_n \le b_n)\}, \qquad \text{for} \quad 2 \le n \le i_0.$$

Then

$$EH_{3n} = EH_{3,n-1} - EH_{2,n-1} - EH_{1,n} + \theta J_{\theta,n} - \sum_{i=n-1}^{n} \frac{\|G_i\|}{\theta^i} \exp\left\{-\frac{b_n}{\theta}\right\},$$

which implies that

$$E(H_{3n} - H_{3,n-1}) + EH_{2,n-1} + EH_{1,n} = \theta J_{\theta,n} - \sum_{i=n-1}^{n} \frac{\|G_i\|}{\theta^i} \exp\left\{\frac{b_n}{\theta}\right\}.$$

Therefore

$$\begin{aligned} ET &= \sum_{n=1}^{i_0} EH_{1,n} + \sum_{n=1}^{i_0} EH_{2,n} \\ &= \sum_{n=1}^{i_0} EH_{1,n} + \sum_{n=1}^{i_0} E(H_{3,n} - H_{3,n-1}) + EH_{3,1} + \sum_{i=2}^{i_0} EH_{2,n-1} \\ &= EH_{11} + EH_{31} + \sum_{i=2}^{i_0} \theta J_{\theta,n-1} - \theta \sum_{n=2}^{i_0} \sum_{i=n-2}^{n-1} \frac{\|G_i\|}{\theta^i} \exp\left\{-\frac{b_i}{\theta}\right\}. \end{aligned}$$

Since $E(H_{11} + H_{31}) = \theta(1 - \exp\{-b_n/\theta\})$,

$$ET = \theta \sum_{n=0}^{i_0-1} J_{\theta,n} - 2\theta \sum_{n=0}^{i_0-2} \frac{\|G_n\|}{\theta^n} \exp\left\{-\frac{b_n}{\theta}\right\} - \theta \frac{\|G_{i_0-1}\|}{\theta^{i_0-1}} \exp\left\{-\frac{b_{i_0-1}}{\theta}\right\}.$$

Thus (3.38) holds. □

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