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DISTRIBUTIONS OF NUMBERS OF FAILURES AND SUCCESSES UNTIL THE FIRST CONSECUTIVE k SUCCESSES*

SIGEO AKI[1] AND KATUOMI HIRANO[2]

[1] *Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Machikaneyama-cho, Toyonaka, Osaka 560, Japan*
[2] *The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan*

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Abstract. Exact distributions of the numbers of failures, successes and successes with indices no less than l $(1 \leq l \leq k-1)$ until the first consecutive k successes are obtained for some $\{0,1\}$-valued random sequences such as a sequence of independent and identically distributed (iid) trials, a homogeneous Markov chain and a binary sequence of order k. The number of failures until the first consecutive k successes follows the geometric distribution with an appropriate parameter for each of the above three cases. When the $\{0,1\}$-sequence is an iid sequence or a Markov chain, the distribution of the number of successes with indices no less than l is shown to be a shifted geometric distribution of order $k-l$. When the $\{0,1\}$-sequence is a binary sequence of order k, the corresponding number follows a shifted version of an extended geometric distribution of order $k-l$.

Key words and phrases: Geometric distribution, discrete distributions, Markov chain, waiting time, geometric distribution of order k, iid sequence, binary sequence of order k, inverse sampling.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of $\{0,1\}$-valued random variables. We often call X_n the n-th trial and we say success and failure for the outcomes "1" and "0", respectively. The distribution of the number of trials until the first consecutive k successes is well known when the sequence is an iid sequence with success probability p (e.g. cf. Feller (1968)). The probability generating function (pgf) of it is given by

$$\frac{p^k(1-pt)t^k}{1-t+qp^kt^{k+1}},$$

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where $q = 1 - p$. Philippou *et al.* (1983) called it the geometric distribution of order k and derived its exact probability function. Since then exact distribution theory for so called discrete distributions of order k has been extensively developed and some of the results were applied to problems on the reliability of the consecutive-k-out-of-n:F system by many authors (cf. Aki *et al.* (1984), Aki (1985, 1992), Charalambides (1986), Hirano (1986), Philippou (1986, 1988), Ling (1988), Aki and Hirano (1989, 1993), Ebneshahrashoob and Sobel (1990), Sobel and Ebneshahrashoob (1992) and references therein). Every distribution of order 1 is of course the usual corresponding discrete distribution. This is an explicit relation between distributions of order k and that of order 1. However, as far as we know, no other relations among distributions of different orders have been studied.

In this paper, we find out a vector of random variables whose marginal distributions follow the geometric or the extended geometric (cf. Aki (1985) or Section 3) distributions of order l $(1 \leq l \leq k-1)$. This leads us a new type of geneses of the geometric distributions of order l. Surprisingly, this property only depends on the conditional probability $P(X_{n+1} = 1 \mid X_n = 1)$ and hence we can treat homogeneous $\{0,1\}$-valued Markov chain similarly as the iid sequence.

Throughout the paper let k and l be fixed positive integers such that $l \leq k-1$. We denote by $G_k(p)$ the geometric distribution of order k, by $G_k(p,a)$ the shifted geometric distribution of order k so that its support begins with a, and by $G_k(p,a;x)$ the probability of $G_k(p,a)$ at x $(x = a, a+1, a+2, \dots)$. By definition, $G_k(p) = G_k(p,k)$ holds for every $k > 1$. Here the geometric distribution, to be denoted by $G(p)$, is defined as the distribution of the number of failures preceding the first success. Note that $G(p) = G_1(p,0)$.

In Section 2, only an iid sequence is treated and our main idea is illustrated. Exact distributions of the numbers of occurrences of "0", "1" and "1" with indices no less than l until the first consecutive k successes are shown to be $G(p^k)$, $G_{k-1}(p,k)$ and $G_{k-l}(p,k-l+1)$, respectively. In Section 3, results given in Section 2 are extended for dependent sequences such as the homogeneous Markov chain and the binary sequences of order k. When the $\{0,1\}$-sequence follows the Markov chain, the distribution of the number of occurrences of "1" with indices no less than l until the first consecutive k successes becomes $G_{k-l}(p_{11}, k-l+1)$ where $p_{11} = P(X_{i+1} = 1 \mid X_i = 1)$ for $i = 0,1,2,\dots$. When the $\{0,1\}$-sequence is a binary sequence of order k with parameters $p_1, \dots, p_k$, the distribution of the number of occurrences of "1" with indices no less than l until the first consecutive k successes is shown to be the shifted extended geometric distribution of order $k-l$ with parameters $p_{l+1}, \dots, p_k$ whose support begins with $k-l+1$.

2. Independent trials

2.1 *Number of failures until the first consecutive k successes*

Let $X_1, X_2, \dots$ be a sequence of iid $\{0,1\}$-valued random variables with $P(X_1 = 1) = p$ (success probability). We denote by τ the waiting time (the number of trials) for the first consecutive k successes. Let ν be the number of occurrences of "0" (failure) among $X_1, X_2, \dots, X_\tau$. For nonnegative integers m

and N, let $d(m, N) \equiv P(\nu = m, \tau = N)$. Then we have

$$
(2.1) \qquad \begin{cases} d(m, N) = 0 \quad \text{if } 0 \leq N < k, \\ d(0, k) = p^k, \\ d(m, k) = 0 \quad \text{if } m \neq 0, \\ d(m, N) = 0 \quad \text{if } N > k \text{ and } m < 1, \\ d(m, N) = \sum_{l=0}^{k-1} p^l q d(m-1, N-l-1). \end{cases}
$$

The last recurrence relation in (2.1) follows immediately if we consider all possibilities of the first occurrence of "0".

Let $\phi(s, t)$ be the joint pgf of (ν, τ), i.e.,

$$
\phi(s, t) = \sum_{m=0}^{\infty} \sum_{N=0}^{\infty} d(m, N) s^m t^N.
$$

The relations (2.1) imply

PROPOSITION 2.1. *The pgf $\phi(s, t)$ is given by*

$$
\phi(s, t) = \frac{p^k t^k}{1 - \sum_{l=0}^{k-1} p^l q s t^{l+1}}.
$$

COROLLARY 2.1. *The marginal distribution of ν is the geometric distribution with parameter p^k, i.e., $\nu \sim G(p^k)$.*

We denote by E_1 and E_2 a "1"-run of length k and a "0"-run of length r, respectively. Let ν_0 be the number of occurrences of E_2 among $X_1, X_2, \ldots, X_\tau$. Let $p_2(u)$ be the probability of the event that at the u-th trial the sooner event between E_1 and E_2 occurs and the sooner event is E_2. We set

$$
\psi(t) = \sum_{u=r}^{\infty} p_2(u) t^u
$$

and

$$
d_1(m, N) = P(\nu_0 = m, \tau = N).
$$

Then, we can obtain

$$
(2.2) \qquad \begin{cases} d_1(m, N) = 0 \quad \text{if } 0 \leq N < k, \\ d_1(0, k) = p^k, \\ d_1(m, k) = 0 \quad \text{if } m \neq 0, \\ d_1(m, N) = \sum_{u=r}^{N-k} p_2(u) d_1(m-1, N-u). \end{cases}
$$

Let $\phi_1(s,t)$ be the joint pgf of (ν_0, τ), i.e.,

$$\phi_1(s,t) = \sum_{m=0}^{\infty}\sum_{N=0}^{\infty} d_1(m,N)s^m t^N.$$

From the relations (2.2) we have

PROPOSITION 2.2. *The pgf $\phi_1(s,t)$ is given by*

$$\phi_1(s,t) = \frac{\sum_{N=0}^{\infty} d_1(0,N)t^N}{1 - s\psi(t)}.$$

COROLLARY 2.2. *The marginal distribution of ν_0 is the geometric distribution with parameter $1 - \psi(1)$, i.e.,*

$$\phi_1(s,1) = \frac{1-\psi(1)}{1-s\psi(1)},$$

where

$$\psi(1) = \frac{q^{r-1}(1-p^k)}{1-(1-p^{k-1})(1-q^{r-1})}. \tag{2.3}$$

Remark. By definition, $\psi(1) = P(\{E_2 \text{ comes sooner}\})$. Hence, the formula (2.3) can be obtained anyway. But, one of the easiest way to find out it is to set $x = 0$, $y = 1$ and $t = 1$ in the generalized pgf (2.2) in Ebneshahrashoob and Sobel (1990).

We can give an intuitive explanation why ν_0 has the geometric distribution. We cut the $\{0,1\}$-sequence at the trial when the event E_1 or E_2 has just occurred. Then every divided part of the sequence has the distribution of the waiting time for the sooner event between E_1 and E_2. Hence, we have

$$P(\{\nu_0 = m\}) = \{P(\{E_2 \text{ comes sooner}\})\}^m P(\{E_1 \text{ comes sooner}\}).$$

2.2 *Number of successes until the first consecutive k successes*

In this subsection we consider the distribution of the number of successes until the first occurrence of consecutive k successes. We denote by ν_1 the number of occurrences of "1" among $X_1, X_2, \ldots, X_\tau$. Set $A(x) = P(\nu_1 = x)$.

PROPOSITION 2.3. *The probabilities $\{A(x)\}_{x=0}^{\infty}$ satisfy the relations*

$$\begin{cases} A(x) = 0 \quad \textit{if } 0 \le x < k, \\ A(k) = p^{k-1}, \\ A(x) = qA(x) + pqA(x-1) + \cdots + p^{k-1}qA(x-k+1) \quad \textit{if } x > k. \end{cases} \tag{2.4}$$

PROOF. Note that $\{\nu_1 = k\} = \cup_{j=0}^{\infty}\{\underbrace{0\cdots0}_{j}\underbrace{1\cdots1}_{k}\}$. Then,

$$A(k) = \sum_{j=0}^{\infty} q^j p^k = p^{k-1}.$$

By considering all possibilities of the first occurrence of "0", we easily obtain the above recurrence relations. □

Let $\phi_1(t)$ be the pgf of ν_1, i.e., $\phi_1(t) = \sum_{x=0}^{\infty} A(x)t^x$. From (2.4) we immediately obtain

PROPOSITION 2.4. *The pgf $\phi_1(t)$ is given by*

$$\phi_1(t) = \frac{p^{k-1}(1-pt)t^k}{1-t+qp^{k-1}t^k}.$$

Remark. Proposition 2.4 shows that ν_1 follows the shifted geometric distribution of order $k-1$ so that the support begins with k, i.e., $\nu_1 \sim G_{k-1}(p,k)$.

Now, in order to derive a general result, we define the following "words" or "sets of words" in $\{0,1\}$-sequences as follows:

Let S = "1", F_1 = "0" and $\mathcal{F}_1 \equiv \{F_1\}$. We define a set of words $\mathcal{F}_2 \equiv \cup_{j=1}^{\infty}\{\underbrace{0\cdots0}_{j}1\}$ $(= \cup_{j=1}^{\infty}\{\underbrace{F_1\cdots F_1}_{j}S\})$. We say "$F_2$ occurs" when one of words in $\mathcal{F}_2$ occurs in the $\{0,1\}$-sequence. It is easy to see that any $\{0,1\}$-sequence which ends with "$\underbrace{1\cdots1}_{k}$" can be uniquely regarded as $\{S,F_2\}$-sequence which ends with "$\underbrace{S\cdots S}_{k-1}$". Next, we consider a set of words $\mathcal{F}_3 \equiv \cup_{j=1}^{\infty}\{\underbrace{F_2\cdots F_2}_{j}S\}$ in $\{S,F_2\}$-sequences which ends with "$\underbrace{S\cdots S}_{k-1}$". We say "$F_3$ occurs" when one of words in $\mathcal{F}_3$ occurs in the $\{S,F_2\}$-sequence. Then, any $\{0,1\}$-sequence which ends with "$\underbrace{1\cdots1}_{k}$" can be uniquely regarded as $\{S,F_3\}$-sequence which ends with $\underbrace{S\cdots S}_{k-2}$. By repeating these considerations, we can define $\mathcal{F}_1,\ldots,\mathcal{F}_k$ and $F_1,\ldots,F_k$. Thus, for $i=1,2,\ldots,k$, $\{0,1\}$-random sequence which ends with "$\underbrace{1\cdots1}_{k}$" can be regarded as the new binary $\{S,F_i\}$-sequence which ends with "$\underbrace{S\cdots S}_{k-i+1}$".

PROPOSITION 2.5. *For $i=1,\ldots,k$, $P(F_i)=q$, which does not depend on i.*

PROOF. For $i=1$, $P(F_1) = P(\{0\}) = q$. Assume that $P(F_i) = q$ holds. Then we have $P(F_{i+1}) = \sum_{j=1}^{\infty}(P(F_i))^j P(S) = \sum_{j=1}^{\infty} q^j p = q$. This completes the proof by the induction. □

Now, we denote by W_l the waiting time for the first occurrence of "$\underbrace{1 \cdots 1}_{l-1}$". Also $W_1 \equiv 0$. In a $\{0,1\}$-sequence, fix a success "1". We say that the "1" is a "1" (success) with indices m when there exists a "1"-run of length $m-1$ just before the "1". As an example we give a $\{0,1\}$-sequence "$0, 1_1, 0, 1_1, 1_2, 0, 1_1, 1_2, 1_3, 1_4, 1_5, 0, 1_1$", where every subscript of "1" shows its indices. Fix an integer l . Let ν_l be the number of 1's with indices no less than l until the first consecutive k successes. Note that

$$\nu_l = \text{the number of occurrences of "1" with indices no less than } l \\ \text{between } W_l \text{ and } W_{k+1}.$$

Remark. The number of "1's" with indices no less than l coincides with the number of success runs of length l by the overlapping way of counting (cf. Ling (1988)).

THEOREM 2.1. *The distribution of ν_l is the shifted geometric distribution of order $k-l$, i.e., $\nu_l \sim G_{k-l}(p, k-l+1)$.*

PROOF. Note that

$$P(\nu_l = x) = \sum_w P(W_l = w) P(\nu_l = x \mid W_l = w).$$

For the $\{0,1\}$-sequence, regard the part just after W_l as the binary $\{S, F_l\}$-sequence. Then, the number of occurrences of S until the first occurrence of a S-run of length $(k-l+1)$ is equal to ν_l. From Proposition 2.4, the distribution is the shifted geometric distribution of order $k-l$ and does not depend on the value w. Thus we have

$$P(\nu_l = x) = G_{k-l}(p, k-l+1; x).$$

This completes the proof. □

3. Dependent sequences

3.1 *Markov chain*

In this section we study the problems in the previous section based on two typical dependent sequences. One is a two-state homogeneous Markov chain and the other is a binary sequence of order k defined by Aki (1985).

First, we consider the problems based on the following Markov chain. Suppose that the sequence $X_0, X_1, X_2, \ldots$ is a $\{0,1\}$-valued Markov chain with $P(X_0 = 0) = p_0$, $P(X_0 = 1) = p_1$, and for $i = 0, 1, 2, \ldots$ $P(X_{i+1} = 0 \mid X_i = 0) = p_{00}$, $P(X_{i+1} = 1 \mid X_i = 0) = p_{01}$, $P(X_{i+1} = 0 \mid X_i = 1) = p_{10}$ and $P(X_{i+1} = 1 \mid X_i = 1) = p_{11}$. For dependent sequences, we use the same notations as the iid case. The distribution of the number of trials until the first consecutive k successes in the Markov chain was obtained by Aki and Hirano (1993).

THEOREM 3.1. *Let ν be the number of failures until the first consecutive k successes in the Markov chain. Then the conditional distribution of ν given that $X_0 = 0$ is the geometric distribution $G(p_{01}p_{11}^{k-1})$. The conditional distribution of ν given that $X_0 = 1$ is the mixture distribution $p_{11}^k \cdot \delta_0 + (1 - p_{11}^k) \cdot G_1(p_{01}p_{11}^{k-1}, 1)$, where δ_0 means the unit mass at the origin.*

PROOF. First, we consider the case that $X_0 = 0$. Define a word and a set of words by $S =$ "$\underbrace{1 \cdots 1}_{k}$" and $\mathcal{F} = \cup_{j=0}^{k-1}\{$"$\underbrace{1 \cdots 1}_{j}0$"$\}$. We say "$F$ occurs" when a word in $\mathcal{F}$ occurs. If we regard the $\{0,1\}$-sequence until the first "$\underbrace{1 \cdots 1}_{k}$" as the new binary $\{S, F\}$-sequence, then the new sequence becomes a Markov chain. Since we consider the $\{0,1\}$-sequence only up to the first "$\underbrace{1 \cdots 1}_{k}$", the all possible sequences we can observe are $\{\underbrace{FF \cdots F}_{j} S\}_{j=0}^{\infty}$. Then, from the Markov property of the sequences and the assumption that $X_0 = 0$, we have $P(\underbrace{FF \cdots F}_{j} S) = \{P(F \mid F)\}^j P(S \mid F)$. This probability law is the same as the iid trials with success probability $p_{01}p_{11}^{k-1}$. In the iid sequence, the number of occurrences of F until the first S is equal to ν, since every word in $\mathcal{F}$ has just one "0". Thus the conditional distribution given that $X_0 = 0$ is the geometric distribution $G(p_{01}p_{11}^{k-1})$. Next, we consider the case that $X_0 = 1$. If a "0" occurs before the first "$\underbrace{1 \cdots 1}_{k}$", then the distribution of the sequence just after the first "0" is the same as the above case. The probability that "0" does not occur until the first "$\underbrace{1 \cdots 1}_{k}$" is p_{11}^k. Hence, we have the desired result. □

THEOREM 3.2. *Let ν_1 be the number of successes until the first consecutive k successes in the Markov chain. Then ν_1 follows the shifted geometric distribution of order $k-1$, i.e., $\nu_1 \sim G_{k-1}(p_{11}, k)$.*

PROOF. Since the case that $k = 1$ is trivial, we assume that $k > 1$. Let W_2 be the waiting time for the first "1" and let W_{k+1} be the waiting time for the first "$\underbrace{1 \cdots 1}_{k}$". Note that $W_2 < W_{k+1}$ with probability one. Define a word and a set of words by $S =$ "1" and $\mathcal{F}_2 = \cup_{j=1}^{\infty}\{$"$\underbrace{0 \cdots 0}_{j}1$"$\}$. We say that "$F_2$ occurs" when a word in $\mathcal{F}_2$ occurs. We regard the $\{0,1\}$-sequence just after W_2 as the new binary $\{S, F_2\}$-sequence. Noting that

$$P(F_2) = \sum_{j=0}^{\infty} p_{10}p_{00}^j p_{01} = p_{10} \quad \text{and} \quad P(S) = p_{11},$$

we see that the new $\{S, F_2\}$-sequence is the iid trials with success probablity p_{11}. The event that "$\underbrace{1 \cdots 1}_{k}$" occurs for the first time for the $\{0,1\}$-sequence just after

W_2 is equivalent to the event that "$\underbrace{S\cdots S}_{k-1}$" occurs for the first time in the new $\{S, F_2\}$-sequence. Since $W_2 < W_{k+1}$ holds with probability one, we see that $\nu_1 = 1+$(the number of trials until the first consecutive $k-1$ successes in iid trials with success probability p_{11}). This completes the proof. □

Remark. The iid sequence treated in Section 2 is the special case of the Markov chain ($p_1 = p_{11} = p_{01} = p$). So, corresponding results in Section 2 can be regarded as corollaries of those in this section.

From the proof of Theorem 3.2, the $\{0,1\}$-sequence just after the first "1" can be regarded as Bernoulli trials with success probability p_{11}. Then Theorem 2.1 immediately implies

THEOREM 3.3. *For $l = 1, \ldots, k-1$, let ν_l be the number of successes with indices no less than l in the Markov chain. Then the distribution of ν_l is the shifted geometric distribution of order $k-l$, i.e., $\nu_l \sim G_{k-l}(p_{11}, k-l+1)$.*

3.2 *Binary sequence of order k*

A binary sequence of order k is defined and studied by Aki (1985, 1992). This is an extension of iid trials and determined by k parameters $p_1, \ldots, p_k$. The definition is as follows:

DEFINITION. A sequence $\{X_i\}_{i=0}^{\infty}$ of $\{0,1\}$-valued random variables is said to be a binary sequence of order k if there exist a positive integer k and k real numbers $0 < p_1, p_2, \ldots, p_k < 1$ such that

(1) $X_0 = 0$ almost surely, and

(2) $P(X_n = 1 \mid X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = p_j$ is satisfied for any positive integer n, where $j = r - [(r-1)/k] \cdot k$, and r is the smallest positive integer which satisfies $x_{n-r} = 0$.

The distribution of the number of trials until the first occurrence of consecutive k successes is the extended geometric distribution of order k, whose pgf is given by

$$\frac{p_1 \cdots p_k t^k}{1 - \sum_{i=1}^{k} p_1 \cdots p_{i-1} q_i t^i},$$

where $q_i = 1 - p_i$ (cf. Aki (1985)). We denote by $EG_k(p_1, \ldots, p_k, a)$ the shifted extended geometric distribution of order k so that its suport begins with a.

THEOREM 3.4. *Let ν_0 be the number of failures until the first occurrence of consecutive k successes in the binary sequence of order k. Then, ν_0 follows the geometric distribution $G(p_1 \cdots p_k)$.*

PROOF. If we define S and F samely as in the proof of Theorem 3.1, then the $\{S, F\}$-sequence obtained from the $\{0,1\}$-sequence becomes iid trials with success probability $p_1 \cdots p_k$. Since every F includes just one "0", the number ν_0 coincides with the number of occurrences of F until the first S. Hence the distribution of ν_0 is the geometric distribution, $G(p_1 \cdots p_k)$. $\square$

THEOREM 3.5. *Let ν_1 be the number of successes until the first consecutive k successes in the binary sequence of order k. Then ν_1 follows the shifted extended geometric distribution of order $k-1$, i.e., $\nu_1 \sim EG_{k-1}(p_2, \dots, p_k, k)$.*

PROOF. Since the case that $k = 1$ is trivial, we assume that $k > 1$. Let W_2 be the waiting time for the first "1" and let W_{k+1} be the waiting time for the first "$\underbrace{1 \cdots 1}_{k}$". Note that $W_2 < W_{k+1}$ with probability one. Define a word and a set of words by $S =$ "1" and $\mathcal{F}_2 = \cup_{j=1}^{\infty}\{$"$\underbrace{0 \cdots 0}_{j}1$"$\}$. We say that "$F_2$ occurs" when a word in $\mathcal{F}_2$ occurs. We regard the $\{0,1\}$-sequence just after W_2 as the new binary $\{S, F_2\}$-sequence. Then the $\{S, F_2\}$-sequence becomes the binary sequence of order $k-1$ with parameters $p_2, \dots, p_k$. Since both S and every word $F_2 \in \mathcal{F}_2$ include just one "1", we see that $\nu_1 = 1 +$ (the number of trials until the first occurrence of a S-run of length $k-1$ in the binary sequence of order $k-1$). This completes the proof. $\square$

THEOREM 3.6. *For $l = 1, \dots, k-1$, let ν_l be the number of successes with indices no less than l in the binary sequence of order k. Then the distribution of ν_l is the shifted extended geometric distribution of order $k-l$, i.e., $\nu_l \sim EG_{k-l}(p_{l+1}, \dots, p_k, k-l+1)$.*

PROOF. When $l = 1$, the statement is reduced to Theorem 3.5. So, we fix an integer l $(2 \le l \le k)$. Define $W_1, \dots, W_k$ and W_{k+1} similarly as in Section 2. Then, we can see that the number ν_l is equal to the number of occurrences of "1" with indices no less than l between W_l and W_{k+1}. If we regard the $\{0,1\}$-sequence just after W_l as $\{S, F_l\}$-sequence, then the sequence becomes the binary sequence of order $(k-l+1)$ with parameters $p_l, \dots, p_k$. Noting that the number of occurrences of "1" with indices no less than l between W_l and W_{k+1} coincides with the number of occurrences of S in the $\{S, F_l\}$-sequence until the first occurrence of a S-run of length $(k-l+1)$. Then, Theorem 3.5 implies that the distribution of ν_l is $EG_{k-l}(p_{l+1}, \dots, p_k, k-l+1)$. This completes the proof. $\square$

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