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ON m-DEPENDENCE AND EDGEWORTH EXPANSIONS*

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Abstract. This paper contains two results. The first establishes, under mild assumptions, the validity of an Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with a kernel of degree two using observations from an m-dependent shift. The second result gives a necessary and sufficient condition for the distribution of a sum of m-dependent random variables to possess an Edgeworth expansion. This generalizes a result of Bickel and Robinson from the i.i.d. case to the m-dependent case.

Key words and phrases: Edgeworth expansions, U-statistics, m-dependence.

1. Introduction

This paper contains essentially two results on m-dependence and Edgeworth expansions. The first establishes sufficient conditions for the validity of a one term Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with a kernel h of degree two using observations from an m-dependent shift. More precisely, let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables and $f : R^{m+1} \to R$ be a measurable function. For $j \geq 1$, let

$$X_j = f(\xi_j, \ldots, \xi_{j+m}). \tag{1.1}$$

The sequence $X_1, X_2, \ldots$ is said to be an m-dependent shift and an immediate consequence is that $(X_1, \ldots, X_r)$ and $(X_s, X_{s+1}, \ldots)$ are stochastically independent whenever $s - r > m$. Next let $h : R^2 \to R$ be a measurable function symmetric in its two arguments. We shall assume throughout this paper that for some $p > 5/3$,

$$E|h(X_1, X_j)|^p < \infty, \qquad \forall 1 < j \leq m + 2. \tag{1.2}$$

Then $Eh(X_j, X_k)$ exists for all $j < k$ and we write

$$h_{j,k}(x, y) = h(x, y) - Eh(X_j, X_k), \qquad \forall x, y \in R.$$

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Now for $N \geq 2$, define a U-statistic with a kernel h of degree two as

$$U_N = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} h_{j,k}(X_j, X_k). \tag{1.3}$$

Theorem 2.1 establishes mild conditions such that

$$\sup_x |P(\hat{\sigma}_N^{-1} U_N \leq x) - F_N(x)| = o(N^{-1/2})$$

as $N \to \infty$, with F_N and $\hat{\sigma}_N$ as in (2.4) and (2.6) respectively. We think that a result such as the above would be useful in the investigation of the robustness of U-statistics when the independence of observations assumption is violated in the direction of an m-dependent shift (of which a moving average process of order m is a non-trivial special case).

There has been a great deal of research done on U-statistics based on independent and identically distributed observations. In this paragraph, we shall assume that the observations are independent and identically distributed. U-statistics were first discussed by Hoeffding (1948) who also showed the asymptotic normality of $\hat{\sigma}_N^{-1} U_N$ under very weak conditions. The rate of convergence to normality was investigated in increasing generality and precision by Grams and Serfling (1973), Bickel (1974), Chan and Wierman (1977), Callaert and Janssen (1978) and Helmers and van Zwet (1982). In particular, Helmers and van Zwet showed that if $p > 5/3$ and (1.2) and (2.3) hold, then

$$\sup_x |P(\hat{\sigma}_N^{-1} U_N \leq x) - \Phi(x)| = O(N^{-1/2}), \tag{1.4}$$

as $N \to \infty$ where Φ denotes the distribution function of a standard normal random variable. If furthermore we have $Eh^2(X_1, X_2) < \infty$, then $\hat{\sigma}_N$ can be replaced by the standard deviation of U_N in (1.4).

Berry-Esseen type bounds have been obtained by Yoshihara (1984) for U-statistics generated by absolutely regular processes, Rhee (1988) for U-statistics based on m-dependent observations and Zhao and Chen (1987) for finite population U-statistics.

Regarding the corresponding more involved problem of Edgeworth expansions, Callaert *et al.* (1980) and Bickel *et al.* (1986) established for a U-statistic with independent and identically distributed observations, the validity of a one (and two) term Edgeworth expansion with remainder $o(N^{-1/2})$ (and $o(N^{-1})$) respectively.

With dependent observations, the only result that we are aware of is by Kokic and Weber (1990) who established the validity of a one term Edgeworth expansion for U-statistics based on samples from finite populations. Recently Loh (1991) has obtained conditions for the vaildity of a one term Edgeworth expansion for U-statistics using weakly dependent observations. However the conditions given in that paper are stronger than those given here.

The second result of this paper gives a necessary and sufficient condition for the distribution of a sum of m-dependent random variables to possess an Edgeworth expansion. This generalizes a result of Bickel and Robinson (1982) from the independent and identically distributed case to the m-dependent case. Sufficient conditions for the validity of an Edgeworth expansion for a sum of m-dependent random variables have been obtained by Heinrich (1982, 1984 and 1985) but we are not aware of the existence of a necessary and sufficient condition except for the independent and identically distributed case. On the other hand as Bickel and Robinson observed, this result does not appear to give a practical criterion although it does pinpoint the relationship between the smoothness of the distribution function of the normalized sum of m-dependent random variables and its Edgeworth expansion.

The rest of the paper is organized as follows. Section 2 establishes the validity of an Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with a kernel of degree two using observations from an m-dependent shift. Theorems 2.1 and 2.2 contain the precise statements of these results. Section 3 gives a necessary and sufficient condition for the distribution of a sum of m-dependent random variables to have an Edgeworth expansion. This condition is stated in Theorem 3.1. Proofs of Theorems 2.1 and 3.1 are found in Sections 4 and 5 respectively. Finally the Appendix contains somewhat technical lemmas which are needed in previous sections.

2. U-statistics and Edgeworth expansions

The main objective of this section is to establish, under mild conditions, the validity of a one term Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with kernel h of degree two using observations from an m-dependent shift. Let X_j, $j \geq 1$, be as in (1.1) and h and U_N be as in (1.2) and (1.3) respectively.

For $N > 6m + 1$, we define

$$\begin{aligned} g(x) &= E[h_{j,k}(X_j, X_k) \mid X_j = x], \qquad \forall k - j > m, \\ \psi(x,y) &= h_{j,k}(x,y) - g(x) - g(y), \qquad \forall k - j > m, \\ \hat{U}_N &= (N - 6m - 1)\sum_{j=1}^{N} g(X_j), \\ \Delta_N &= \sum_{j=1}^{N-3m-1} \sum_{k=3m+j+1}^{N} \psi(X_j, X_k) + \sum_{j=1}^{N-1} \sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j, X_k) \\ &\quad + \sum_{j=1}^{3m}(3m - j + 1)g(X_j) + \sum_{j=N-3m+1}^{N} (3m + j - N)g(X_j). \end{aligned} \tag{2.1}$$

Straightforward calculations show that $U_N = \hat{U}_N + \Delta_N$. We suppose that

$$\sigma_g^2 = E\left[g^2(X_1) + 2\sum_{j=1}^{m} g(X_1)g(X_{j+1})\right] > 0 \tag{2.2}$$

and

$$E|g(X_1)|^3 < \infty. \tag{2.3}$$

Let $\hat{\sigma}_N^2$ denote the variance of $\hat{U}_N$. Then by the stationarity of the X_j's, we have

$$\begin{aligned}(2.4)\qquad \hat{\sigma}_N^2 &= (N-6m-1)^2 E\left[Ng^2(X_1) + 2\sum_{j=1}^{m}(N-j)g(X_1)g(X_{j+1})\right]\\ &= N^3\sigma_g^2 + O(N^2),\end{aligned}$$

as $N \to \infty$. Next let

$$\begin{aligned}(2.5)\qquad \kappa_3 = \sigma_g^{-3} E\Bigg\{ & g^3(X_1) + 3\sum_{j=1}^{m}[g^2(X_1)g(X_{j+1}) + g(X_1)g^2(X_{j+1})]\\ & + 6\sum_{j=2}^{m+1}\sum_{k=j+1}^{j+m} g(X_1)g(X_j)g(X_k)\\ & + 3\sum_{j=1}^{2m+1}\sum_{k=3m+2}^{5m+2} \psi(X_{m+1}, X_{4m+2})g(X_j)g(X_k)\Bigg\}.\end{aligned}$$

We observe that if $E|h(X_j, X_k)|^3 < \infty$ whenever $j < k$, then $\kappa_3 N^{-1/2}$ is an asymptotic approximation (with error $O(N^{-3/2})$) for the third cumulant of $\hat{\sigma}_N^{-1}U_N$. Define

$$(2.6)\qquad F_N(x) = \Phi(x) - \phi(x)\frac{\kappa_3}{6}N^{-1/2}(x^2-1),$$

where ϕ and Φ denote the standard normal density and distribution function respectively. Then we have

THEOREM 2.1. *Suppose* (1.2), (2.2), (2.3) *are satisfied and*

$$(2.7)\qquad \limsup_{|t|\to\infty} E|E[e^{it\sum_{j=1}^{m+1} g(X_j)} \mid \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]| < 1.$$

Then

$$\sup_x |P(\hat{\sigma}_N^{-1}U_N \le x) - F_N(x)| = o(N^{-1/2}),$$

as $N \to \infty$.

Theorem 2.1 though simple to state, has a somewhat tedious proof and hence we shall defer the proof to Section 4.

Remark. Götze and Hipp (1983) showed that (2.7) holds if ξ_1 has a probability density f_{ξ_1} with respect to Lebesgue measure and $gf : R^{m+1} \to R$ is continuously differentiable such that there exist $y_1, \ldots, y_{2m+1} \in R$ and an open subset $\Omega \supset \{y_1, \ldots, y_{2m+1}\}$ satisfying $f_{\xi_1} > 0$ on Ω and

$$\sum_{j=1}^{m+1} \frac{\partial}{\partial x_j} gf(x_1, \ldots, x_{1+m})|_{(x_1,\ldots,x_{1+m})=(y_j,\ldots,y_{j+m})} \neq 0.$$

Remark. If the observations are independent and identically distributed (that is $m = 0$), (2.7) reduces to the well known Cramér's condition.

In the case where $Eh^2(X_1, X_j) < \infty$ whenever $1 < j \leq m+2$, the variance σ_N^2 of U_N exists and $\sigma_N^2 = N^3\sigma_g^2 + O(N^2)$ as $N \to \infty$. Then we have

THEOREM 2.2. *Suppose that* (2.2), (2.3) *are satisfied,*

$$Eh^2(X_1, X_j) < \infty, \qquad \forall 1 < j \leq m+2$$

and

$$\limsup_{|t|\to\infty} E|E[e^{it\sum_{j=1}^{m+1} g(X_j)} \mid \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]| < 1.$$

Then

$$\sup_x |P(\sigma_N^{-1}U_N \leq x) - F_N(x)| = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. The proof of Theorem 2.2 is similar to that of Theorem 2.1 and hence is omitted. □

3. Smoothness and Edgeworth expansions

This section gives a necessary and sufficient condition for the distribution of a normalized sum of m-dependent random variables to possess an Edgeworth expansion. We begin by recalling the definition of m-dependence.

DEFINITION. A sequence $Y_1, Y_2, \ldots$ of random variables is m-dependent, where m is a nonnegative integer, if for any two subsets $A, B \subseteq \{1, 2, \ldots\}$ for which $\inf_{i\in A, j\in B} |i-j| > m$ holds, the sets of random variables $\{X_i : i \in A\}$ and $\{X_j : j \in B\}$ are independent.

From the above definition, we note that an independent sequence of random variables is 0-dependent and an m-dependent shift is also m-dependent. Let $Y_1, Y_2, \ldots$ be a sequence of m-dependent random variables with $EY_i = 0$, $i = 1, 2, \ldots$. We write

$$S_n = Y_1 + \cdots + Y_n, \qquad B_n^2 = ES_n^2,$$
$$M_{k,n} = \max_{1\leq j\leq n} E|Y_j|^k, \qquad \sigma_{k,n} = \max_{3\leq j\leq k+3} (nM_{j,n}/B_n^j)^{1/(j-2)}.$$

Let F_{S_n/B_n} denote the distribution function of S_n/B_n and $\Gamma_\nu(S_n)$ denote the ν-th order cumulant of S_n.

Next, for any $G : R \to R$ and $\sigma > 0$, we define the first difference operator Δ_σ by

$$\Delta_\sigma G(x) = G(x+\sigma) - G(x),$$

and the k-th difference operator Δ_σ^k as the k-th iterate of this. Thus

$$\Delta_\sigma^k G(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} G(x + j\sigma).$$

The interpolating polynomial to $G(y)$ of degree k at the points $x, x+\sigma, \ldots, x+k\sigma$ is

$$P_{k,\sigma}(y; x, G) = G(x) + \sum_{j=1}^{k} \sigma^{-j} (j!)^{-1} \Delta_\sigma^j G(x) \prod_{i=1}^{j} (y - x - (i-1)\sigma).$$

It is well known (see for example Bickel and Robinson (1982)) that if G has a bounded $(k+1)$-th derivative, then for all x and y,

$$|G(y) - P_{k,\sigma}(y; x, G)| \le C_0(|y - x|^{k+1} + \sigma^{k+1}) \sup_z |G^{(k+1)}(z)|, \tag{3.1}$$

where C_0 is a positive constant depending only on k and $G^{(k+1)}(z) = d^{k+1}G(z)/dz^{k+1}$. Also in the remainder of this section, the symbol C is used generically as a positive constant independent of n.

THEOREM 3.1. *Let $Y_1, Y_2, \ldots$ be a sequence of m-dependent random variables with $EY_j = 0$, $j = 1, 2, \ldots$. Suppose $\sigma_{k,n} \to 0$ as $n \to \infty$. Then the following statements are equivalent:*

(a) *F_{S_n/B_n} possesses an Edgeworth expansion to k terms. More precisely,*

$$\sup_x |F_{S_n/B_n}(x) - e_{k,n}(x)| \le C\sigma_{k,n}^{k+1},$$

where

$$e_{k,n}(x) = \Phi(x) - \sum_{\nu=1}^{k} \frac{1}{\sqrt{2\pi} B_n^\nu} e^{-x^2/2} \sum_{q=1}^{\nu} \frac{1}{q!} \times \sum_{\nu_1+\cdots+\nu_q=\nu, \nu_j \ge 1} H_{\nu+2q-1}(x) \prod_{i=1}^{q} \frac{\delta_{\nu_i+2,n}}{(\nu_i+2)!},$$

with

$$H_\nu(x) = \nu! \sum_{i=0}^{[\nu/2]} \frac{(-1)^i x^{\nu-2i}}{i!(\nu-2i)!2^i} \quad \textit{and} \quad \delta_{\nu,n} = \frac{\Gamma_\nu(S_n)}{B_n^2}.$$

(b) *For all x, y and n, there exists a constant C_1, independent of x, y and n, such that*

$$|F_{S_n/B_n}(y) - P_{k,\sigma_{k,n}}(y; x, F_{S_n/B_n})| \le C_1(|y - x|^{k+1} + \sigma_{k,n}^{k+1}).$$

Remark. H_ν, $\nu = 1, 2, \ldots$ are the Chebyshev-Hermite polynomials.

We now specialize Theorem 3.1 to the case of a stationary sequence of m-dependent random variables.

DEFINITION. A sequence $Y_1, Y_2, \ldots$ of random variables is said to be stationary if, for every pair t, j of natural numbers, the sequence $Y_{t+1}, \ldots, Y_{t+j}$ has the same distribution as $Y_1, \ldots, Y_j$.

COROLLARY 3.1. *Let $Y_1, Y_2, \ldots$ be a stationary sequence of m-dependent random variables with $EY_1 = 0$, $EY_1^2 = 1$ and $\lim B_n^2/n > 0$. If $E|Y_1|^{k+3} < \infty$, then the following statements are equivalent:*

(a) $\sup_x |F_{S_n/B_n}(x) - e_{k,n}(x)| \leq Cn^{-(k+1)/2}$.

(b) *For all x, y and n, there exists a constant C_1, independent of x, y and n, such that*

$$|F_{S_n/B_n}(y) - P_{k,1/\sqrt{n}}(y; x, F_{S_n/B_n})| \leq C_1(|y-x|^{k+1} + n^{-(k+1)/2}).$$

We shall defer the proofs of Theorem 3.1 and Corollary 3.1 to Section 5.

4. Proof of Theorem 2.1

PROOF OF THEOREM 2.1. Without loss of generality, we assume that $5/3 < p \leq 2$. To prove Theorem 2.1, we shall study the characteristic function (c.f.) of $\hat{\sigma}_N^{-1} U_N$. Let ϕ_N denote the c.f. of $\hat{\sigma}_N^{-1} U_N$, that is

$$\phi_N(t) = E \exp(it\hat{\sigma}_N^{-1} U_N),$$

and for κ_3, as in (2.5), let

$$\phi_N^*(t) = e^{-t^2/2}\left(1 - \frac{i\kappa_3}{6} N^{-1/2} t^3\right)$$

be the Fourier transform $\int \exp(itx) dF_N(x)$ of F_N in (2.6). By the smoothing lemma of Esseen (see for example, Feller (1971), p. 538), it suffices to show that

$$\int_{-N^{1/2}\log N}^{N^{1/2}\log N} \left| \frac{\phi_N(t) - \phi_N^*(t)}{t} \right| dt = o(N^{-1/2}), \tag{4.1}$$

as $N \to \infty$. However (4.1) is an immediate consequence of Propositions 4.1 and 4.2 whose statements and proofs are provided below. □

PROPOSITION 4.1. *Let $5/3 < p \leq 2$ and $0 < \varepsilon < (3p-5)/(2p)$. Then*

$$\int_{-N^\varepsilon}^{N^\varepsilon} \left| \frac{\phi_N(t) - \phi_N^*(t)}{t} \right| dt = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. It is well known that

$$\left| e^{ix} - \sum_{j=0}^{r} \frac{(ix)^j}{j!} \right| \le \min\left\{ \frac{2}{r!}|x|^{r+\theta}, \frac{|x|^{r+1}}{(r+1)!} \right\}, \qquad \forall \theta \in [0,1). \tag{4.2}$$

Hence

$$\begin{aligned} \phi_N(t) &= Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N}(1 + it\hat{\sigma}_N^{-1}\Delta_N) + O(E|t\hat{\sigma}_N^{-1}\Delta_N|^p) \\ &= Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N}(1 + it\hat{\sigma}_N^{-1}\Delta_N) + O(|t|^p N^{2-3p/2}). \end{aligned} \tag{4.3}$$

The last equality uses the fact that $E|\Delta_N|^p = O(N^2)$ (see for example Lemma 5-1 of Rhee (1988)). Define for $1 \le a < b \le N$,

$$S_{a,b}^{(\nu)} = (N - 6m - 1) \sum_{1 \le j \le N, |j-a| \wedge |j-b| > \nu m} g(X_j), \qquad \forall \nu \ge 1,$$

$$S_{a,b}^{(0)} = \hat{U}_N.$$

As $\hat{U}_N = S_{a,b}^{(0)}$, for all $a < b$, it follows from (4.3) and Lemma A.1 (see Appendix) that

$$\begin{aligned} &\phi_N(t) - e^{-t^2/2}\left(1 - \frac{i\hat{\kappa}_3}{6} N^{-1/2} t^3\right) \\ &\quad = Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\ &\qquad + O(|t|^p N^{2-3p/2}) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}], \end{aligned} \tag{4.4}$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$. It remains to approximate the term $Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N}$. Following a method of Tikhomirov (1980), we write

$$\begin{aligned} &\sum_{j=1}^{N-3m-1} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\ &\quad = \sum_{j=1}^{N-3m-1} \sum_{k=3m+j+1}^{N} E\Bigg\{ it\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}} \\ &\qquad + it\hat{\sigma}_N^{-1}\psi(X_j, X_k)\sum_{r=2}^{4}\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)} - S_{j,k}^{(l)})} - 1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}} \\ &\qquad + it\hat{\sigma}_N^{-1}\psi(X_j, X_k)\prod_{l=1}^{4}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)} - S_{j,k}^{(l)})} - 1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(4)}} \Bigg\} \\ &\quad = \sum_{j=1}^{N-3m-1} \sum_{k=3m+j+1}^{N} \sum_{r=2}^{4} it\hat{\sigma}_N^{-1} \\ &\qquad \times \left\{ E\psi(X_j, X_k)\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)} - S_{j,k}^{(l)})} - 1] \right\} \\ &\qquad \times [Ee^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}}] + O(|t|^6 N^{-2}), \end{aligned} \tag{4.5}$$

as $N \to \infty$ uniformly in t. The last equality uses Lemma A.3 and the independence of $S_{j,k}^{(r)}$ and $\psi(X_j, X_k)\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]$. Furthermore using Lemmas A.1, A.2 and A.3, we have

$$
\begin{aligned}
(4.6)\qquad & \sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N}\sum_{r=2}^{4} it\hat{\sigma}_N^{-1}\Bigg\{E\psi(X_j, X_k) \\
&\times \prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]\Bigg\}[Ee^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}}] \\
&= -\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} it^3 e^{-t^2/2}\sigma_g^{-3}N^{-5/2} \\
&\times \left[E\sum_{a=(j-m)\vee 1}^{j+m}\sum_{b=k-m}^{(k+m)\wedge N}\psi(X_j, X_k)g(X_a)g(X_b)\right] \\
&+ o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}]
\end{aligned}
$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$, where $\mathcal{P}(|t|)$ is a generic linear combination (not depending on N) of non-negative powers of $|t|$. Also for convenience of notation, $\mathcal{P}$ may represent different linear combinations at different occurrences. Thus it follows from (4.5) and (4.6) that

$$
\begin{aligned}
(4.7)\qquad & \sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\
&= -e^{-t^2/2}\frac{i}{6}N^{-1/2}t^3E \\
&\times \left[3\sigma_g^{-3}\sum_{j=1}^{2m+1}\sum_{k=3m+2}^{5m+2}\psi(X_{m+1}, X_{4m+2})g(X_j)g(X_k)\right] \\
&+ O(|t|^6N^{-2}) + o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}],
\end{aligned}
$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$. In a similar though less tedious way, we have

$$
\begin{aligned}
(4.8)\quad & Eit\hat{\sigma}_N^{-1}\sum_{j=1}^{N-1}\sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j, X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\
&= it\hat{\sigma}_N^{-1}\sum_{j=1}^{N-1}\sum_{k=j+1}^{(j+3m)\wedge N} E\{h_{j,k}(X_j, X_k)e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}} \\
&\quad + h_{j,k}(X_j, X_k)[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(0)}-S_{j,k}^{(1)})}-1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}}\} \\
&= O(|t|^2N^{-1})
\end{aligned}
$$

and

$$(4.9)\quad Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N} it\hat{\sigma}_N^{-1}\left[\sum_{j=1}^{3m}(3m-j+1)g(X_j) + \sum_{j=N-3m+1}^{N}(3m+j-N)g(X_j)\right] = O(|t|N^{-3/2}),$$

as $N \to \infty$ uniformly in $|t|$. Thus it follows from (2.1), (4.7), (4.8) and (4.9) that

$$\begin{aligned} &Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\ &\quad = -e^{-t^2/2}\frac{i}{6}N^{-1/2}t^3 E\left[3\sigma_g^{-3}\sum_{j=1}^{2m+1}\sum_{k=3m+2}^{5m+2}\psi(X_{m+1}, X_{4m+2})g(X_j)g(X_k)\right] \\ &\qquad + O(|t|N^{-3/2} + |t|^2N^{-1} + |t|^6N^{-2}) + o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^\varepsilon$. Hence we conclude from (4.4) that

$$\begin{aligned} \phi_N(t) - \phi_N^*(t) &= \phi_N(t) - e^{-t^2/2}\left(1 - \frac{i\kappa_3}{6}N^{-1/2}t^3\right) \\ &= O(|t|N^{-3/2} + |t|^2N^{-1} + |t|^6N^{-2} + |t|^pN^{2-3p/2}) \\ &\quad + o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^\varepsilon$ and hence

$$\int_{-N^\varepsilon}^{N^\varepsilon}\left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}),$$

as $N \to \infty$. This completes the proof of Proposition 4.1. □

Next we observe from (2.7) that there exists a constant $0 < \gamma < 1$ such that

$$(4.10)\qquad E|E[e^{it\sum_{j=1}^{m+1} g(X_j)} \mid \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]| \le 1 - \gamma,$$

for all $|t| \ge 1/(2\sigma_g)$. Also it follows from Lemma 3.2 of Götze and Hipp (1983) that there exists a constant $\mu > 0$ such that

$$(4.11)\qquad E|E[e^{it\sum_{j=1}^{m+1} g(X_j)} \mid \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]| \le e^{-\mu t^2},$$

for all $|t| \le 3/(2\sigma_g)$.

Proposition 4.2. *Let ε be as in Proposition 4.1. Then*

$$\int_{N^\varepsilon \le |t| \le N^{1/2}\log N}\left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. Let n be a positive integer such that for sufficiently large N

$$\left\lfloor \frac{n-2m}{2(m+1)} \right\rfloor - 3 = \left\lceil -\frac{\log N}{\log(1-\gamma)} \right\rceil$$

if $N^{1/2} \le |t| \le N^{1/2} \log N$, and $n = Kt^{-2} N \log N$ if $N^{\varepsilon} \le |t| \le N^{1/2}$ where K is some constant to be chosen later. Define

$$S(n) = (N - 6m - 1) \sum_{j=1}^{n} g(X_j)$$

and

$$\begin{aligned}\Delta_N(n) = &\sum_{j=1}^{n \wedge (N-3m-1)} \sum_{k=3m+j+1}^{N} \psi(X_j, X_k) \\ &+ \sum_{j=1}^{n \wedge (N-1)} \sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j, X_k) + \sum_{j=1}^{3m} (3m - j + 1) g(X_j).\end{aligned}$$

Then

$$|\phi_N(t)| = |Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]| + O(|t|^p n N^{1-3p/2}) \tag{4.12}$$

as $N \to \infty$ uniformly in t, since $E|\Delta_N(n)|^p = O(nN)$ (see Rhee (1988)).

We shall now approximate the first term of the r.h.s. of (4.12). For simplicity we let $\mathcal{A}_{j,k,n}$ denote the σ-field generated by the random variables ξ_l, $l \in [j, j+m] \cup [k, k+m] \cup [n+1, \infty)$. We observe from Lemma A.4 that K can be chosen such that

$$\begin{aligned}&|Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| \\ &\quad = |Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - S(n) - \Delta_N(n))} E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{j,k,n}]| \\ &\quad \le |t|\hat{\sigma}_N^{-1} E|\psi(X_j, X_k)| N^{-1},\end{aligned}$$

and hence

$$\left| \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))} \right| = O(|t| n N^{-3/2}), \tag{4.13}$$

as $N \to \infty$ uniformly over $N^{\varepsilon} \le |t| \le N^{1/2} \log N$.

In a similar way, we have

$$|Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(N^{-1}), \tag{4.14}$$

$$\left| \sum_{j=1}^{n} \sum_{k=j+1}^{j+3m} Eit\hat{\sigma}_N^{-1} h_{j,k}(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))} \right| = O(|t| n N^{-5/2}) \tag{4.15}$$

and

$$(4.16)\qquad \left|\sum_{j=1}^{3m} Eit\hat{\sigma}_N^{-1}(3m-j+1)g(X_j)e^{it\hat{\sigma}_N^{-1}(U_N-\Delta_N(n))}\right| = O(|t|N^{-5/2}),$$

as $N \to \infty$ uniformly over $N^\varepsilon \leq |t| \leq N^{1/2}\log N$. From (4.13), (4.15) and (4.16), we get

$$(4.17)\qquad |Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}(U_N-\Delta_N(n))}| = O(|t|nN^{-3/2}),$$

as $N \to \infty$ uniformly over $N^\varepsilon \leq |t| \leq N^{1/2}\log N$. Now it follows from (4.12), (4.14) and (4.17) that

$$|\phi_N(t)| = O(N^{-1} + |t|nN^{-3/2} + |t|^p nN^{1-3p/2}),$$

and from the definition of n, we have

$$(4.18)\qquad \int_{N^\varepsilon \leq |t| \leq N^{1/2}\log N} |\phi_N(t)/t|dt = o(N^{-1/2}),$$

as $N \to \infty$. □

5. Proof of Theorem 3.1

First we shall state a key result due to Heinrich ((1984), p. 14). We refer the reader to his paper for a sketch of the proof.

Lemma 5.2. *Let $Y_1, Y_2, \ldots$ be a sequence of m-dependent random variables with $EY_j = 0$ and $E|Y_j|^{k+3} < \infty$ whenever $j = 1, 2, \ldots$, for some $k \geq 0$. Then there exists positive constants B_1 and B_2, depending only on k and m, such that for all $|t| \leq B_1\sigma_{k,n}^{-1}$, we have*

$$|F^*_{S_n/B_n}(t) - e^*_{k,n}(t)| \leq B_2\sigma_{k,n}^{k+1}(|t|^{k+3} + |t|^{3(k+2)})\exp(-t^2/6),$$

where $F^*_{S_n/B_n}$ and $e^*_{k,n}$ denote the Fourier-Stieltjes transform of F_{S_n/B_n} and $e_{k,n}$ respectively.

Proof of Theorem 3.1. The proof closely parallels that given by Bickel and Robinson (1982) for the i.i.d. case. However we need to make the following changes in their proof to adapt it to the m-dependent case. First replace their equation (2.3) by that of Lemma 5.2. Also we observe from Heinrich (1985) that

$$|\delta_{\nu,n}| \leq CnM_{\nu,n}/B_n^2,$$

and hence

$$\left|\prod_{i=1}^q \delta_{\nu_i+2,n}/B_n^{\nu_i}\right| \leq C\prod_{i=1}^q nM_{\nu_i+2,n}/B_n^{\nu_i+2} \to 0$$

as $n \to \infty$ since $\sigma_{k,n} \to 0$. Thus we conclude that $\sup_x |e_{k,n}^{(k+1)}(x)| \le C$ and it follows from (3.1) that

$$|e_{k,n}(y) - P_{k,\sigma_{k,n}}(y; x, e_{k,n})| \le C_2(|y-x|^{k+1} + \sigma_{k,n}^{k+1}),$$

where C_2 is some positive constant independent of x, y and n. □

PROOF OF COROLLARY 3.1. Since $\lim B_n^2/n > 0$, it follows from the definition of $\sigma_{k,n}$ that $0 < \lim \sigma_{k,n}\sqrt{n} < \infty$. Now the proof proceeds as in Theorem 3.1 with $\sigma_{k,n}$ replaced by $n^{-1/2}$. □

Appendix

LEMMA A.1. *Suppose that* (2.2), (2.3) *are satisfied and r is a fixed nonnegative integer. Then*

$$Ee^{it\hat{\sigma}_N^{-1}S_{a,b}^{(r)}} = e^{-t^2/2}\left(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3\right) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}],$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $|t| \le N^{\varepsilon}$, where

$$\hat{\kappa}_3 = \sigma_g^{-3}E\left\{g^3(X_1) + 3\sum_{j=1}^{m}[g^2(X_1)g(X_{j+1}) + g(X_1)g^2(X_{j+1})] + 6\sum_{j=2}^{m+1}\sum_{k=j+1}^{j+m} g(X_1)g(X_j)g(X_k)\right\}.$$

PROOF. Let $\hat{\sigma}_{a,b}^{(r)}$ denote the standard deviation of $S_{a,b}^{(r)}$. We observe that the third cumulant of $(\hat{\sigma}_{a,b}^{(r)})^{-1}S_{a,b}^{(r)}$ is asymptotically $\hat{\kappa}_3 N^{-1/2}$ with error $O(N^{-3/2})$ uniformly over $1 \le a < b \le N$. Hence it follows from Heinrich ((1982), p. 513) that

$$Ee^{it(\hat{\sigma}_{a,b}^{(r)})^{-1}S_{a,b}^{(r)}} = e^{-t^2/2}\left(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3\right) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}],$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $|t| \le N^{\varepsilon+\delta}$, where δ is a small positive constant. We remark that Heinrich stated his result only for the case of a sum of 1-dependent random variables. However the extension to m-dependence is straightforward. Since $1 - (\hat{\sigma}_{a,b}^{(r)}/\hat{\sigma}_N)^2 = O(N^{-1})$ uniformly over $1 \le a < b \le N$, we have

$$\begin{aligned} Ee^{it\hat{\sigma}_N^{-1}S_{a,b}^{(r)}} &= Ee^{it(\hat{\sigma}_N^{-1}\hat{\sigma}_{a,b}^{(r)})(\hat{\sigma}_{a,b}^{(r)})^{-1}S_{a,b}^{(r)}} \\ &= e^{-t^2/2}\left(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3\right) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $|t| \le N^{\varepsilon}$. □

LEMMA A.2. *Let* $5/3 < p \leq 2$, $p^{-1} + q^{-1} = 1$ *and* $1 \leq a < b \leq N$ *with* $b - a > 3m$. *Then*

$$
\begin{aligned}
&E it\hat{\sigma}_N^{-1}\psi(X_a, X_b)\{\exp[it\hat{\sigma}_N^{-1}(S_{a,b}^{(0)} - S_{a,b}^{(1)})] - 1\} \\
&\quad = -it^3\sigma_g^{-3}N^{-5/2}E\sum_{j=(a-m)\vee 1}^{a+m}\sum_{k=b-m}^{(b+m)\wedge N}\psi(X_a, X_b)g(X_j)g(X_k) \\
&\qquad + O(|t|^3N^{-7/2} + |t|^{2+3/q}N^{-2-3/(2q)}),
\end{aligned}
$$

as $N \to \infty$ *uniformly in* a, b *and* t.

PROOF. We observe that

$$
S_{a,b}^{(0)} - S_{a,b}^{(1)} = (N - 6m - 1)\left[\sum_{j=(a-m)\vee 1}^{a+m} g(X_j) + \sum_{k=b-m}^{(b+m)\wedge N} g(X_k)\right].
$$

For $1 \leq c \leq N$, we define

$$
R_c = it\hat{\sigma}_N^{-1}(N - 6m - 1)\sum_{j=(c-m)\vee 1}^{(c+m)\wedge N} g(X_j). \tag{A.1}
$$

Then

$$
\begin{aligned}
&E it\hat{\sigma}_N^{-1}\psi(X_a, X_b)\{\exp[it\hat{\sigma}_N^{-1}(S_{a,b}^{(0)} - S_{a,b}^{(1)})] - 1\} \\
&\quad = E it\hat{\sigma}_N^{-1}\psi(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\
&\qquad + R_a(e^{R_b} - 1 - R_b) + R_b(e^{R_a} - 1 - R_a) + R_aR_b].
\end{aligned} \tag{A.2}
$$

The last equality uses the observation that

$$
E\psi(X_a, X_b) = E[\psi(X_a, X_b) \mid R_a] = E[\psi(X_a, X_b) \mid R_b] = 0.
$$

Next we observe that

$$
\begin{aligned}
&E it\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_aR_b \\
&\quad = -it^3\hat{\sigma}_N^{-3}(N - 6m - 1)^2E \\
&\qquad \times \sum_{j=(a-m)\vee 1}^{a+m}\sum_{k=b-m}^{(b+m)\wedge N} g(X_j)g(X_k)\psi(X_a, X_b) \\
&\quad = -it^3\sigma_g^{-3}N^{-5/2}E\sum_{j=(a-m)\vee 1}^{a+m}\sum_{k=b-m}^{(b+m)\wedge N} g(X_j)g(X_k)\psi(X_a, X_b) \\
&\qquad + O(|t|^3N^{-7/2}),
\end{aligned} \tag{A.3}
$$

as $N \to \infty$ uniformly in a, b and t. Furthermore it follows from (4.2) that

$$
\begin{aligned}
\text{(A.4)}\qquad & E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\
&\quad + R_a(e^{R_b} - 1 - R_b) + R_b(e^{R_a} - 1 - R_a)]| \\
&\quad \leq 6E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_aR_b^{3/q}| + 2E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_bR_a^{3/q}| \\
&\quad \leq 6|t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}[(E|R_a|^q)^{1/q}(E|R_b|^3)^{1/q} \\
&\quad + (E|R_b|^q)^{1/q}(E|R_a|^3)^{1/q}] \\
&\quad = O(|t|^{2+3/q}N^{-2-3/(2q)}),
\end{aligned}
$$

as $N \to \infty$ uniformly in a, b and t. Lemma A.2 now follows from (A.2), (A.3) and (A.4). □

LEMMA A.3. *Let r be a fixed positive integer, $5/3 < p \leq 2$ and $1 \leq a < b \leq N$ with $b - a > 3m$. Then*

$$
\left|Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\right| = O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1})
$$

and

$$
\begin{aligned}
&\left|Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(r)}}\right| \\
&\qquad = O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1}),
\end{aligned}
$$

as $N \to \infty$ uniformly in a, b and t.

PROOF. Let R_a and R_b be defined as in (A.1). We observe that

$$
\begin{aligned}
\text{(A.5)}\qquad & \left|Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\right| \\
&\quad = \Bigg|Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\
&\qquad + R_a(e^{R_b} - 1 - R_b) \\
&\qquad + R_b(e^{R_a} - 1 - R_a) + R_aR_b]\prod_{l=2}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\Bigg| \\
&\quad \leq 9E\left|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_aR_b\prod_{l=2}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\right|.
\end{aligned}
$$

The last inequality uses (4.2). By Hölder's inequality, the r.h.s. of (A.5) is less than or equal to

$$
\begin{aligned}
\text{(A.6)}\qquad & 9|t|\hat{\sigma}_N^{-1}\left\{E\left|\psi(X_a, X_b){\prod}'[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\right|^p\right\}^{1/p} \\
&\quad \times \left\{E\left|R_aR_b{\prod}''[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]\right|^q\right\}^{1/q},
\end{aligned}
$$

where $p^{-1} + q^{-1} = 1$, $\prod'$ denotes the product over all even integers l, $2 \le l \le r$ and $\prod''$ denotes the product over all odd integers l, $3 \le l \le r$. By virtue of m-dependence, the r.h.s. of (A.6) is bounded by

$$\begin{aligned}
\text{(A.7)} \qquad & 9|t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}(E|R_aR_b|^q)^{1/q} \\
& \times \prod_{l=2}^{r}[E|e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})} - 1|^3]^{1/3} \\
& \le 9|t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}(E|R_aR_b|^q)^{1/q} \\
& \quad \times \prod_{l=2}^{r}[E|t\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})|^3]^{1/3}.
\end{aligned}$$

Since $[E|t\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})|^3]^{1/3} = O(|t|N^{-1/2})$, as $N \to \infty$ uniformly over $1 \le a < b \le N$, $2 \le l \le r$ and t, it follows from (A.7) that

$$\left| Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})} - 1]\right| = O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1}).$$

This proves the first statement of Lemma A.3. The proof of the second statement is similar and is omitted. □

LEMMA A.4. *Let $1 \le a < b \le N$. Then with the notation of Proposition 4.2, there exists a constant K such that*

$$|E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}]| \le N^{-1},$$

for sufficiently large N uniformly over $1 \le a < b \le N$ and $N^\varepsilon \le |t| \le N^{1/2}\log N$.

PROOF. We observe that

$$\begin{aligned}
\text{(A.8)} \quad & E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}] \\
& = E\left[\int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=1}^{n} g(X_j)}\prod_l{}^* dF(\xi_{l(m+1)}) \mid \mathcal{A}_{a,b,n}\right],
\end{aligned}$$

where $F(\xi_{l(m+1)})$ denotes the distribution function of the random variable $\xi_{l(m+1)}$ and $\prod_l^*$ denotes the product over all positive odd integers l satisfying $l(m+1) \notin [a-m, a+2m] \cup [b-m, b+2m] \cup [n+1-m, \infty)$. Thus the absolute value of the r.h.s. of (A.8) is bounded by

$$\begin{aligned}
\text{(A.9)} \ & E\left[\prod_l{}^*\left|\int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=l(m+1)-m}^{l(m+1)} g(X_j)}dF(\xi_{l(m+1)})\right| |\mathcal{A}_{a,b,n}\right] \\
& = \prod_l{}^* E\left[\left|\int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=l(m+1)-m}^{l(m+1)} g(X_j)}dF(\xi_{l(m+1)})\right| |\mathcal{A}_{a,b,n}\right] \\
& = \{E|E[e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=1}^{m+1} g(X_j)} \mid \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]|\}^{k_0},
\end{aligned}$$

where k_0 equals the number of terms in the product $\prod_l^*$. The second (last) equality uses the independence (stationarity) of the ξ_j's respectively. Now we consider two cases.

Case 1. Suppose that $N^{1/2} \leq |t| \leq N^{1/2} \log N$. Then for sufficiently large N, n satisfies

$$\left\lfloor \frac{n-2m}{2(m+1)} \right\rfloor - 3 = \left\lceil -\frac{\log N}{\log(1-\gamma)} \right\rceil .$$

Since

$$k_0 \geq \left\lfloor \frac{n-2m}{2(m+1)} \right\rfloor - 3,$$

and it follows from (4.10) and (A.9) that

$$|E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}]| \leq (1-\gamma)^{\lfloor (n-2m)/[2(m+1)] \rfloor - 3},$$

whenever $(N-6m-1)\hat{\sigma}_N^{-1}|t| \geq 1/(2\sigma_g)$. Thus we conclude that

$$|E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}]| \leq N^{-1},$$

for sufficiently large N uniformly over $1 \leq a < b \leq N$ and $N^{1/2} \leq |t| \geq N^{1/2} \log N$.

Case 2. Suppose that $N^{\varepsilon} \leq |t| \leq N^{1/2}$. Then for sufficiently large N, $n = Kt^{-2} N \log N$. We observe from (4.11) and (A.9) that

$$|E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}]| \leq e^{-\mu k_0 t^2 (N-6m-1)^2 \hat{\sigma}_N^{-2}},$$

whenever $(N-6m-1)\hat{\sigma}_N^{-1}|t| \leq 3/(2\sigma_g)$. Now it can be easily seen that K can be chosen so that

$$|E[e^{it\hat{\sigma}_N^{-1}S(n)} \mid \mathcal{A}_{a,b,n}]| \leq 1/N,$$

for sufficiently large N uniformly over $1 \leq a < b \leq N$ and $N^{\varepsilon} \leq |t| \leq N^{1/2}$. □

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