Ann. Inst. Statist. Math.
Vol. 46, No. 1, 127–136 (1994)

LBI TESTS OF INDEPENDENCE IN BIVARIATE EXPONENTIAL DISTRIBUTIONS

MARTIN BILODEAU[1]* AND TAKEAKI KARIYA[2]

[1] *Département de mathématiques et de statistique, Université de Montréal, C.P.6128, succursale A, Montréal, Canada H3C 3J7*
[2] *The Institute of Economic Research, Hitotsubashi University, Kunitachi, Tokyo 186, Japan and Graduate School of Business, University of Chicago*

(Received June 22, 1992; revised April 7, 1993)

Abstract. The locally best invariant test for the hypothesis of independence in bivariate distributions with exponentially distributed marginals is derived. The model consists of a family of bivariate exponential distributions with probability density function

$$f_\theta(x_1, x_2; \lambda_1, \lambda_2) = \lambda_1\lambda_2 \exp[-(\lambda_1 x_1 + \lambda_2 x_2)] g(\lambda_1 x_1, \lambda_2 x_2; \theta)$$

with unknown scale parameter λ_j $(j = 1, 2)$ and association parameter θ which includes the independence situation. The locally best invariant (LBI) test is derived and the asymptotic null and nonnull distributions are also derived under some regularity conditions. The results are applied to the Gumbel (1960, *J. Amer. Statist. Assoc.*, **55**, 698–707), Frank (1979, *Aequationes Math.*, **19**, 194–226), and Cook and Johnson (1981, *J. Roy. Statist. Soc. Ser. B*, **43**, 210–218) families.

Key words and phrases: Bivariate exponential distribution, locally best invariant test, test of independence, parametric families of Gumbel (type I and II), Frank, and Cook and Johnson.

1. Introduction

The exponential distribution has had numerous applications in models involving time intervals between successive events and in life testing. Gumbel (1960) proposed some bivariate generalizations that can be used as parametric models for joint lifetimes of two dependent components. Recently, Genest and MacKay (1986) and Marshall and Olkin (1988) described general methods for the construction of bivariate distributions with given marginals. These methods can be applied

* The first author would like to thank the National Sciences and Engineering Research Council of Canada and the Fonds pour la formation de chercheurs et l'aide à la recherche of Québec for their financial support.

to construct bivariate distributions with an association parameter and marginals that are exponentially distributed.

A general family of such bivariate exponential distributions with an association parameter $\theta \in \Theta$ is expressed as the family of pdf's

$$f_\theta(x_1, x_2; \lambda_1, \lambda_2) = \lambda_1\lambda_2 \exp[-(\lambda_1 x_1 + \lambda_2 x_2)] g(\lambda_1 x_1, \lambda_2 x_2; \theta), \tag{1.1}$$

where $x_1, x_2 > 0$, $\lambda_1, \lambda_2 > 0$. Here, Θ is an interval regarded as the association parameter space such that the family (1.1) includes, possibly as a limiting case as $\theta \downarrow 0$, the independence situation

$$\lim_{\theta \downarrow 0} f_\theta(x_1, x_2; \lambda_1, \lambda_2) = \prod_{j=1}^{2} \lambda_j \exp(-\lambda_j x_j). \tag{1.2}$$

In fact, the family (1.1) with (1.2) includes as special cases the bivariate distributions specified by Gumbel (type I and II) (1960), Frank (1979) and Cook and Johnson (1981).

In this paper we consider the problem of testing the independence of x_1 and x_2

$$H : \theta = 0 \qquad \text{vs} \qquad K : \theta > 0 \tag{1.3}$$

in the general model (1.1) with an iid sample (x_{1k}, x_{2k}) $(k = 1, \ldots, n)$, where $\theta = 0$ may be viewed as (1.2) sometimes. Under some regularity conditions on g, a locally best invariant (LBI) test is derived in Section 2 and the null and nonnull asymptotic distributions of the LBI test statistic is derived in Section 3. In Section 4 the results are applied to the bivariate distributions due to Gumbel (1960), Frank (1979) and Cook and Johnson (1981) and the LBI tests of independence are obtained with the null and nonnull distribution.

2. LBI test of independence

The problem stated in Section 1 is clearly invariant under the group $G = \mathbb{R}_+ \times \mathbb{R}_+$ of scale transformations which acts on (x_{1k}, x_{2k}) and $(\theta, \lambda_1, \lambda_2)$ by

$$\begin{aligned} &(a_1, a_2) \circ (x_{1k}, x_{2k}) = (a_1 x_{1k}, a_2 x_{2k}) \quad \text{and} \\ &(a_1, a_2) \circ (\theta, \lambda_1, \lambda_2) = (\theta, a_1^{-1}\lambda_1, a_2^{-1}\lambda_2) \end{aligned} \tag{2.1}$$

where $(a_1, a_2) \in G$. A maximal invariant under G is clearly

$$\boldsymbol{y} = (y_{11}, \ldots, y_{1n}; y_{21}, \ldots, y_{2n}) \qquad \text{with} \qquad y_{jk} = x_{jk}/s_j \tag{2.2}$$

where $s_j = \sum_{k=1}^{n} x_{jk}$, $j = 1, 2$, and a maximal invariant parameter is θ. Thus, without loss of generality, assume $\lambda_1 = \lambda_2 = 1$. Also, the test sought being LBI it can be assumed without loss of generality that Θ is compact. Assume that $g(x_1, x_2; \theta)$ and $\partial g(x_1, x_2; \theta)/\partial\theta$ are continuous functions of (x_1, x_2, θ) on $\mathbb{R}_+ \times \mathbb{R}_+ \times \Theta$ with

$$\dot{g}(x_1, x_2) = \lim_{\theta \downarrow 0} \partial g(x_1, x_2; \theta)/\partial\theta. \tag{2.3}$$

Finally, assume that (2.7) below and

$$\int_0^\infty \int_0^\infty \frac{\partial}{\partial\theta}\left[\prod_{k=1}^n g(b_1 y_{1k}, b_2 y_{2k}; \theta)\right] b_1^{n-1} b_2^{n-1} \exp(-b_1 - b_2) db_1 db_2, \tag{2.4}$$

exist and are uniformly convergent on the space of $(\boldsymbol{y}, \theta)$.

THEOREM 2.1. *The LBI test of independence for* (1.3) *is the test which rejects H for large values of*

$$T_n(\boldsymbol{y}) = \frac{1}{n}\sum_{k=1}^n E_{\boldsymbol{b}}[\dot{g}(b_1 y_{1k}, b_2 y_{2k})] \tag{2.5}$$

where b_j, $j = 1, 2$, are independent Gamma variables with pdf $b_j^{n-1}\exp(-b_j)/\Gamma(n)$.

PROOF. Since an invariant test is a function of $\boldsymbol{y}$, let the distribution of $\boldsymbol{y}$ under θ be $P_\theta^{\boldsymbol{y}}$. Clearly, the null distribution $P_0^{\boldsymbol{y}}$ does not depend on any unknown parameter. Then, the density of $\boldsymbol{y}$ with respect to $P_0^{\boldsymbol{y}}$ is given by

$$dP_\theta^{\boldsymbol{y}}/dP_0^{\boldsymbol{y}} \equiv h(\boldsymbol{y} \mid \theta) = H(\boldsymbol{y} \mid \theta)/H(\boldsymbol{y} \mid 0) \tag{2.6}$$

where

$$\begin{aligned} H(\boldsymbol{y} \mid \theta) &= \int_0^\infty \int_0^\infty a_1^{n-1} a_2^{n-1} \left\{\prod_{k=1}^n f_\theta(a_1 x_{1k}, a_2 x_{2k}; 1, 1)\right\} da_1 da_2 \\ &= s_1^{-n} s_2^{-n} \int_0^\infty \int_0^\infty b_1^{n-1} b_2^{n-1} \exp[-(b_1 + b_2)] \\ &\qquad \cdot \left\{\prod_{k=1}^n g(b_1 y_{1k}, b_2 y_{2k}; \theta)\right\} db_1 db_2 \end{aligned} \tag{2.7}$$

(see, e.g., Wijsman (1967)). Therefore, the power function of an invariant test ϕ is

$$\pi(\phi, \theta) = \int \phi(\boldsymbol{y}) h(\boldsymbol{y} \mid \theta) dP_0^{\boldsymbol{y}}. \tag{2.8}$$

Since $h(\boldsymbol{y} \mid \theta)$ and $\partial h(\boldsymbol{y} \mid \theta)/\partial\theta$ are continuous functions of $(\boldsymbol{y}, \theta)$ and since the space of $(\boldsymbol{y}, \theta)$ is compact,

$$\partial\pi(\phi, \theta)/\partial\theta = \int \phi(\boldsymbol{y}) \partial h(\boldsymbol{y} \mid \theta)/\partial\theta dP_0^{\boldsymbol{y}}. \tag{2.9}$$

Therefore, the test function $\phi(\boldsymbol{y})$ that maximizes the slope of $\pi(\phi, \theta)$ at 0 is the test which rejects H whenever $\partial h(\boldsymbol{y} \mid 0)/\partial\theta > k$. Note that under H, $H(\boldsymbol{y} \mid 0) = \Gamma^2(n)/s_1^n s_2^n$. Hence, differentiating $H(\boldsymbol{y} \mid \theta)$ under the integral sign we obtain via (2.4)

$$\partial h(\boldsymbol{y} \mid 0)/\partial\theta = \sum_{k=1}^{n} E_{\boldsymbol{b}} \left[\dot{g}(b_1 y_{1k}, b_2 y_{2k})\right], \tag{2.10}$$

where $\boldsymbol{b} = (b_1, b_2)'$ and b_j's are independent Gamma(n) variables. The uniqueness of the LBI test comes from the necessary condition of the Neyman-Pearson lemma, completing the proof. □

3. Distribution of the LBI test

A difficulty in deriving the asymptotic distribution of T_n is that even though the summands in (2.5) are identically distributed, they are not independent. The asymptotic distribution of T_n under θ is derived under the following assumptions and proved in the appendix. Let $I_{j'k'}^{jk}$ denote the indicator function

$$I_{j'k'}^{jk} = \begin{cases} 1, & \text{if } (j,k) = (j',k'); \\ 0, & \text{otherwise.} \end{cases}$$

ASSUMPTIONS.

(i) $\dot{g}(x_1, x_2)$ is twice continuously differentiable with partial derivatives denoted by $\dot{g}_j$ and $\dot{g}_{jk}$.

(ii) The expectations $E_\theta[\dot{g}(x_1, x_2)]$, $E_\theta[x_j \dot{g}(x_1, x_2)]$ and $E_\theta[x_j \dot{g}_j(x_1, x_2)]$ are finite $(j = 1, 2)$.

(iii) There exists a function

$$G_{jk}(x_1, x_2) = \sum_{\alpha=-a}^{M} \sum_{\beta=-b}^{N} A_{\alpha\beta}^{jk} x_1^\alpha x_2^\beta,$$

where $a = 2I_{11}^{jk} + I_{12}^{jk} + I_{21}^{jk}$ and $b = 2I_{22}^{jk} + I_{12}^{jk} + I_{21}^{jk}$ such that $|\dot{g}_{jk}(x_1, x_2)| \leq G_{jk}(x_1, x_2)$ and $E_\theta[x_j x_k G_{jk}(x_1, x_2)] < \infty$, $(j, k = 1, 2)$.

THEOREM 3.1. *Suppose Assumptions* (i)–(iii) *hold. Define*

$$\zeta_j(\theta) = E_\theta[x_j \dot{g}_j(x_1, x_2)], \qquad j = 1, 2,$$
$$\mu(\theta) = E_\theta[\dot{g}(x_1, x_2)].$$

Then, $\sqrt{n}[T_n - \mu(\theta)]$ has a $N(0, v(\theta))$ limiting distribution where

$$v(\theta) = \mathrm{Var}_\theta[\dot{g}(x_1, x_2) - x_1\zeta_1(\theta) - x_2\zeta_2(\theta)].$$

Consequently, the asymptotic null distribution of $\sqrt{n}[T_n - \mu(0)]$ is $N(0, v(0))$. The proof of Theorem 3.1 in the appendix consists in writing $\sqrt{n}[T_n - \mu(\theta)] =$

$\sqrt{n}T_{n0} + \sqrt{n}(R_{1n} + R_{2n})$ where T_{n0} is an average of iid variables with mean 0 and variance $v(\theta)$. Assumptions (i)–(ii) ensure that $\sqrt{n}T_{n0} \to N(0, v(\theta))$ in distribution and R_{1n} is $o_p(n^{-1/2})$. As for Assumption (iii), it is a sufficient condition for R_{2n} to be $o_p(n^{-1/2})$. Although stated for fixed θ, Theorem 3.1 remains valid for $\theta_n = \omega n^{-1/2}$ as well.

COROLLARY 3.1. *If $\mu(\theta)$ is differentiable at $\theta = 0$ and $v(\theta)$ is continuous at $\theta = 0$, then the asymptotic nonnull distribution of $\sqrt{n}[T_n - \mu(0)]$ with respect to contiguous alternatives $\theta_n = \omega/\sqrt{n}$, $\omega > 0$, is $N(\omega\mu'(0), v(0))$ where $\mu'(0) = \partial\mu(0)/\partial\theta$.*

PROOF. From $\sqrt{n}[T_n - \mu(0)] = \sqrt{n}[T_n - \mu(\theta_n)] + \sqrt{n}[\mu(\theta_n) - \mu(0)]$, the result follows from the fact that $\sqrt{n}[\mu(\theta_n) - \mu(0)] \to \omega\mu'(0)$ and $v(\theta_n) \to v(0)$ as $n \to \infty$. □

4. Examples

4.1 *Gumbel's type* I *family*

The p.d.f. for the bivariate Gumbel type I distribution is

$$(4.1)\quad \begin{aligned} f_\theta(x_1, x_2; \lambda_1, \lambda_2) &= \lambda_1\lambda_2 \exp[-(\lambda_1 x_1 + \lambda_2 x_2)] \\ &\quad \cdot [1 + \theta(2\exp(-\lambda_1 x_1) - 1)(2\exp(-\lambda_2 x_2) - 1)], \end{aligned}$$

where $\theta \in \Theta = [-1, 1]$. It is clear that $\theta = 0$ corresponds to the independence. Since $\dot{g}(x_1, x_2) = (2\exp(-x_1) - 1)(2\exp(-x_2) - 1)$, the LBI test rejects H for large values of

$$T_n = \frac{1}{n}\sum_{k=1}^{n}[2(1 + y_{1k})^{-n} - 1][2(1 + y_{2k})^{-n} - 1].$$

Assumptions (i)–(iii) are satisfied as $\dot{g}$, $\dot{g}_j$ and $\dot{g}_{jk}$ are all bounded by constants. For the asymptotic distribution, the calculations of the moments in $\mu(\theta)$ and $v(\theta)$ can be done most easily using the moments equality for (4.1) with $\lambda_1 = \lambda_2 = 1$ given by

$$(4.2)\quad \begin{aligned} &E_\theta[h_1(x_1)h_2(x_2)] \\ &\quad = E_\theta[h_1(x_1)]E_\theta[h_2(x_2)] \\ &\qquad + \theta E_\theta[h_1(x_1)(2\exp(-x_1) - 1)]E_\theta[h_2(x_2)(2\exp(-x_2) - 1)], \end{aligned}$$

where h_j's are arbitrary functions such that $E_\theta|h_j(x_j)| < \infty$. These calculations yield $\mu(\theta) = \theta/9$ and $v(\theta) = (\theta^3 - 84\theta^2 + 648)/5832$. Therefore the asymptotic null distribution of $\sqrt{n}T_n$ is $N(0, 1/9)$ and the asymptotic nonnull distribution of $\sqrt{n}T_n$ with respect to $\theta_n = \omega/\sqrt{n}$, $\omega > 0$, is $N(\omega/9, 1/9)$.

4.2 *Gumbel's type* II *family*

We refer to Gumbel type II as the bivariate distribution with p.d.f.

$$(4.3)\quad f_\theta(x_1,x_2;\lambda_1,\lambda_2)=\lambda_1\lambda_2\exp[-(\lambda_1x_1+\lambda_2x_2)] \cdot\exp(-\theta\lambda_1\lambda_2x_1x_2)[(1+\theta\lambda_1x_1)(1+\theta\lambda_2x_2)-\theta],$$

where $\theta\in\Theta=[0,1]$. The independence is achieved at $\theta=0$. Here, $\dot{g}(x_1,x_2)=-1+x_1+x_2-x_1x_2$. Hence H is rejected for large values of the LBI test statistic

$$T_n=1-n\sum_{k=1}^{n}y_{1k}y_{2k}.$$

An equivalent test statistic rejects H for small values of $1-T_n=\bar{x}_{12}/\bar{x}_1\bar{x}_2$, where $\bar{x}_{12}=\sum_{k=1}^n x_1x_2/n$. It can be easily checked that Assumptions (i)–(iii) hold with $G_{jk}(x_1,x_2)=1$, $j,k=1,2$. Calculations for $\mu(\theta)$ and $v(\theta)$ show that $\sqrt{n}[T_n-\mu(\theta)]\to N(0,v(\theta))$ in distribution where

$$\mu(\theta)=1-J_1(\theta),$$
$$v(\theta)=2J_1^3(\theta)+3J_1^2(\theta)+4[2\theta J_1^2(\theta)+(1-\theta)J_1(\theta)-1]/\theta^2$$

and

$$J_1(\theta)=\int_0^\infty\frac{\exp(-x)}{(1+\theta x)}dx.$$

Note that as $\theta\to0$, $J_1(\theta)\to1$, $J_1'(\theta)\to-1$ and $J_1''(\theta)\to4$. Since $\mu(0)=0$ and $v(\theta)\to1$ as $\theta\to0$ via de l'Hospital's rule, the asymptotic null and nonnull distribution of $\sqrt{n}T_n$ are $N(0,1)$ and $N(\omega,1)$, where $\theta_n=\omega/\sqrt{n}$.

4.3 *Frank's family*

A family of distributions whose properties in survival models are thoroughly described in Genest (1987) is in our context of exponentially distributed marginals

$$(4.4)\quad f_\theta(x_1,x_2;\lambda_1,\lambda_2)=\lambda_1\lambda_2\exp[-(\lambda_1x_1+\lambda_2x_2)] \cdot\frac{\theta\ln(1+\theta)(1+\theta)^{F(\lambda_1x_1)+F(\lambda_2x_2)}}{[\theta+((1+\theta)^{F(\lambda_1x_1)}-1)((1+\theta)^{F(\lambda_2x_2)}-1)]^2},$$

$\theta\in\Theta=(-1,\infty)$, where $F(x)=1-\exp(-x)$. The lower and upper Fréchet bounds result as $\theta\to\infty$ and $\theta\to-1$ respectively, and as $\theta\to0$, (4.4) reduces to the independence situation. A Taylor series of $\partial g(x_1,x_2;\theta)/\partial\theta$ around $\theta=0$ gives

$$\partial g(x_1,x_2;\theta)/\partial\theta=-\frac{1}{2}[1-2F(x_1)-2F(x_2)+4F(x_1)F(x_2)]+O(\theta).$$

Hence, the LBI test, after evaluation of the expectations with respect to the Gamma(n) variables, is

$$T_n=-\frac{1}{2}\frac{1}{n}\sum_{k=1}^{n}[2(1+y_{1k})^{-n}-1][2(1+y_{2k})^{-n}-1].$$

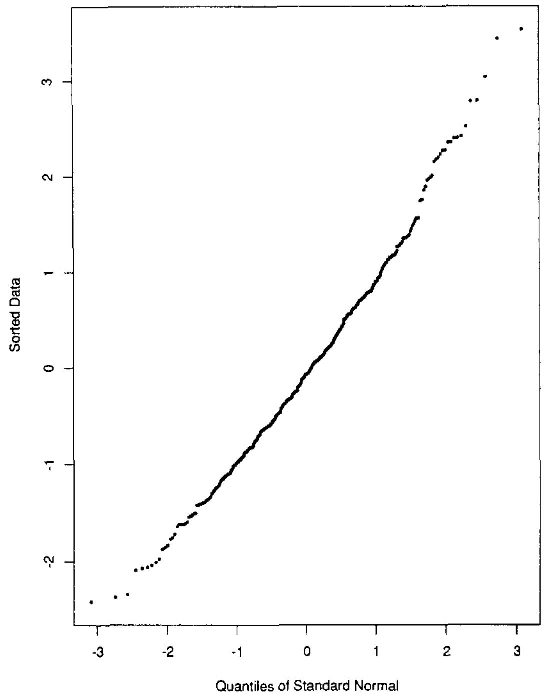

Fig. 1.

Since $T_n^* \equiv -2T_n$ is the LBI statistic for Gumbel type I, it follows that the LBI test rejects for small values of T_n^* whose null asymptotic distribution is $\sqrt{n}T_n^* \to N(0, 1/9)$.

4.4 *The family of Cook and Johnson*

As a last example, the family of Cook and Johnson (1981) used in bivariate survival analysis by Oakes (1982) has a p.d.f.

$$(4.5)\qquad f_\theta(x_1, x_2; \lambda_1, \lambda_2) = \frac{\lambda_1\lambda_2 \exp[-(\lambda_1 x_1 + \lambda_2 x_2)](1+\theta)}{[F(\lambda_1 x_1)F(\lambda_2 x_2)]^{1+\theta}[F(\lambda_1 x_1)^{-\theta} + F(\lambda_2 x_2)^{-\theta} - 1]^{2+1/\theta}},$$

$\theta > 0$, where $F(x) = 1 - \exp(-x)$. As $\theta \to \infty$, the Fréchet upper bound is reached and as $\theta \downarrow 0$, the marginals become independent exponential variables. It can be shown that

$$(4.6)\qquad \dot{g}(x_1, x_2) = [1 + \ln F(x_1)][1 + \ln F(x_2)]$$

and

$$E_b[\ln F(b_1 y_{1k})] = -w_n(y_{1k})$$

where

$$w_n(y) = \sum_{j=1}^{\infty} \frac{1}{j} \frac{1}{(1+jy)^n}, \qquad 0 < y < 1.$$

Hence, the LBI test rejects for large values of

$$T_n = \frac{1}{n}\sum_{k=1}^{n}[1 - w_n(y_{1k})][1 - w_n(y_{2k})]. \tag{4.7}$$

Assumptions (i)–(ii) are satisfied but there does not seem to be functions $G_{jk}(x_1, x_2)$ such that $|\dot{g}_{jk}(x_1, x_2)| \leq G_{jk}(x_1, x_2)$ and $E_\theta[x_j x_k G_{jk}(x_1, x_2)] < \infty$ as in Assumption (iii). If it could be shown, otherwise than by verifying Assumption (iii), that R_{2n} in (A.4) is $o_p(n^{-1/2})$ then it would follow from moment calculations that, under H, $\sqrt{n}T_n \to N(0, 1)$ in distribution as $\mu(0) = 0$ and $v(0) = 1$. A simulation when $n = 50$ of 500 values of $z \equiv \sqrt{n}T_n$ resulted in the normal probability plot below which supports the $N(0, 1)$ asymptotic null distribution.

Appendix

PROOF OF THEOREM 3.1. Define

$$T_{n0} = \frac{1}{n}\sum_{k=1}^{n}[\dot{g}(x_{1k}, x_{2k}) - x_{1k}\zeta_1(\theta) - x_{2k}\zeta_2(\theta) + \zeta_1(\theta) + \zeta_2(\theta) - \mu(\theta)]. \tag{A.1}$$

Since T_{n0} is an average of iid variables with mean 0, it follows that $n^{1/2}T_{n0} \to N(0, v(\theta))$ in distribution. Thus, it suffices to show that $n^{1/2}[T_n - \mu(\theta) - T_{n0}]$ is $o_p(1)$. Let $L_j = ny_{jk}/x_{jk} = n/s_j$ $(j = 1, 2)$. Since $L_j \to 1$ a.s. and $b_j/n \to 1$ a.s. as $b_j \sim \text{Gamma}(n)$, expand $\dot{g}(b_1 y_{1k}, b_2 y_{2k})$ in a neighborhood of $b_j L_j/n = 1$ as

$$\begin{aligned} \dot{g}\left(\frac{b_1}{n}L_1 x_{1k}, \frac{b_2}{n}L_2 x_{2k}\right) &= \dot{g}(x_{1k}, x_{2k}) + \left(\frac{b_1}{n}L_1 - 1\right) x_{1k}\dot{g}_1(x_{1k}, x_{2k}) \\ &\quad + \left(\frac{b_2}{n}L_2 - 1\right) x_{2k}\dot{g}_2(x_{1k}, x_{2k}) + E_{nk} \end{aligned} \tag{A.2}$$

with

$$\begin{aligned} E_{nk} &= \frac{1}{2}\left(\frac{b_1}{n}L_1 - 1\right)^2 x_{1k}^2\dot{g}_{11}(\eta_{1k}x_{1k}, \eta_{2k}x_{2k}) \\ &\quad + \left(\frac{b_1}{n}L_1 - 1\right)\left(\frac{b_2}{n}L_2 - 1\right) x_{1k}x_{2k}\dot{g}_{12}(\eta_{1k}x_{1k}, \eta_{2k}x_{2k}) \\ &\quad + \frac{1}{2}\left(\frac{b_2}{n}L_2 - 1\right)^2 x_{2k}^2\dot{g}_{22}(\eta_{1k}x_{1k}, \eta_{2k}x_{2k}), \end{aligned} \tag{A.3}$$

where $\eta_{jk} = \gamma_k(b_j/n)L_j + (1 - \gamma_k) = 1 + \gamma_k((b_j/n)L_j - 1)$, $0 \leq \gamma_k \leq 1$. Then, using $L_j \sum_{k=1}^{n} x_{jk}/n = 1$, $j = 1, 2$, T_n can be expressed as

$$T_n - \mu(\theta) = T_{n0} + R_{1n} + R_{2n}, \tag{A.4}$$

where T_{n0} is given by (A.1),

$$R_{1n} = \frac{1}{n}\sum_{k=1}^{n} E_b\left[\left(\frac{b_1}{n}L_1 - 1\right) x_{1k}\dot{g}_1(x_{1k}, x_{2k}) + \left(\frac{b_2}{n}L_2 - 1\right) x_{2k}\dot{g}_2(x_{1k}, x_{2k}) - (L_1 - 1)x_{1k}\zeta_1(\theta) - (L_2 - 1)x_{2k}\zeta_2(\theta)\right]$$

and

$$R_{2n} = \frac{1}{n}\sum_{k=1}^{n} E_b[E_{nk}].$$

To show $R_{1n} = o_p(n^{-1/2})$, note the equivalent expression

$$\begin{aligned} R_{1n} &= (L_1 - 1)\frac{1}{n}\sum_{k=1}^{n}[x_{1k}\dot{g}_1(x_{1k}, x_{2k}) - x_{1k}\zeta_1(\theta)] \\ &\quad + (L_2 - 1)\frac{1}{n}\sum_{k=1}^{n}[x_{2k}\dot{g}_2(x_{1k}, x_{2k}) - x_{2k}\zeta_2(\theta)]. \end{aligned}$$

Hence, since $n^{1/2}(L_j - 1)$ is $O_p(1)$, $j = 1, 2$, and $(1/n)\sum_{k=1}^{n}[x_{jk}\dot{g}_j(x_{1k}, x_{2k}) - x_{jk}\zeta_j(\theta)]$ is $o_p(1)$ as it is an average of iid variables with mean 0, it follows that $R_{1n} = o_p(n^{-1/2})$. Next, to show $R_{2n} = o_p(n^{-1/2})$, note

$$\eta_{1k}^{\alpha} \le \begin{cases} (1 + b_1L_1/n), & \alpha = 1, \\ (1 + n/(b_1L_1)), & \alpha = -1, \end{cases}$$

and $E_b(\eta_{2k}^{\beta}) \le (1 + 2L_2)^{\beta}$ $(\beta = 0, 1, \ldots, 2N)$ for n large enough. Hence, by Assumption (iii), the first term in R_{2n} is bounded above as

$$\begin{aligned} \text{(A.5)} \qquad & \frac{1}{n}\left|\sum_{k=1}^{n} E_b\left[\left(\frac{b_1}{n}L_1 - 1\right)^2 x_{1k}^2\dot{g}_{11}(\eta_{1k}x_{1k}, \eta_{2k}x_{2k})\right]\right| \\ &\le \frac{1}{n}\sum_{k=1}^{n} x_{1k}^2 \sum_{\alpha=-2}^{M}\sum_{\beta=0}^{N} A_{\alpha\beta}^{11} x_{1k}^{\alpha} x_{2k}^{\beta} \\ &\quad \cdot E_b^{1/2}\left[\left(\frac{b_1}{n}L_1 - 1\right)^4 \eta_{1k}^{2\alpha}\right] E_b^{1/2}[\eta_{2k}^{2\beta}] \\ &\le \left[\frac{1}{n}\sum_{k=1}^{n} x_{1k}^2 G_{11}(x_{1k}, x_{2k})\right] K(n, L_1)(1 + 2L_2)^N, \end{aligned}$$

where $K(n, L_1)$ is an upper bound such that

$$\text{(A.6)} \qquad E_b^{1/2}\left[\left(\frac{b_1}{n}L_1 - 1\right)^4 \eta_{1k}^{2\alpha}\right]$$

$$
\begin{aligned}
&\leq \max\left\{ E_b^{1/2}\left[\left(\frac{b_1}{n}L_1 - 1\right)^4 \left(1+\frac{n}{b_1L_1}\right)^4\right],\right.\\
&\qquad\left. E_b^{1/2}\left[\left(\frac{b_1}{n}L_1 - 1\right)^4 (1+b_1L_1/n)^{2M}\right]\right\}\\
&\equiv K(n,L_1), \qquad (\alpha = -2,-1,0,\dots,M).
\end{aligned}
$$

The factor $K(n,L_1)$ is $o_p(n^{-1/2})$ as can be seen from

$$
E_b\left[\left(\frac{b_1}{n}L_1 - 1\right)^4 (1+b_1L_1/n)^{2M}\right] = \sum_{i=0}^{2M}\binom{2M}{i}L_1^i E_b\left[\frac{b_1^i}{n^i}\left(\frac{b_1}{n}L_1 - 1\right)^4\right],
$$

where $E_b(b_1^i/n^i) \to 1$, $i \geq 0$, and $(L_1 - 1)^4$ is $o_p(n^{-1})$. The other term in the $\max\{\ \}$ expression in (A.6) can be dealt with in the same way. Hence, $K(n,L_1)$ is $o_p(n^{-1/2})$.

Since the other two factors $(1/n)\sum_{k=1}^n x_{1k}^2 G_{11}(x_{1k},x_{2k})$ and $(1+2L_2)^N$ in (A.5) are $O_p(1)$, it follows that the first term in R_{2n} is $o_p(n^{-1/2})$. Similarly, the other two terms in R_{2n} can be shown to be $o_p(n^{-1/2})$. Therefore, from (A.4) $n^{1/2}[T_n - \mu(\theta)] = n^{1/2}T_{n0} + o_p(1)$ which completes the proof. □

References

Cook, R. D. and Johnson, M. E. (1981). A family of distributions for modelling nonelliptically symmetric multivariate data, *J. Roy. Statist. Soc. Ser. B*, **43**, 210–218.

Frank, M. J. (1979). On the simultaneous associativity of $F(x,y)$ and $x+y-F(x,y)$, *Aequationes Math.*, **19**, 194–226.

Genest, C. (1987). Frank's family of bivariate distributions, *Biometrika*, **74**, 549–555.

Genest, C. and MacKay, R. J. (1986). Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données, *Canad. J. Statist.*, **14**, 145–159.

Gumbel, E. J. (1960). Bivariate exponential distributions, *J. Amer. Statist. Assoc.*, **55**, 698–707.

Marshall, B. W. and Olkin, I. (1988). Families of multivariate distributions, *J. Amer. Statist. Assoc.*, **83**, 834–841.

Oakes, D. (1982). A model for association in bivariate survival data, *J. Roy. Statist. Soc. Ser. B*, **44**, 414–442.

Wijsman, R. A. (1967). Cross-sections of orbits and their applications to densities of maximal invariants, *Proc. Fifth Berkeley Symp. on Math. Statist. Prob.*, Vol. 1, 389–400, Univ. of California Press, Berkeley.