Ann. Inst. Statist. Math.
Vol. 46, No. 1, 95–116 (1994)

DOUBLE SHRINKAGE ESTIMATION OF RATIO OF SCALE PARAMETERS

TATSUYA KUBOKAWA

Department of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

(Received November 9, 1992; revised May 26, 1993)

Abstract. The problems of estimating ratio of scale parameters of two distributions with unknown location parameters are treated from a decision-theoretic point of view. The paper provides the procedures improving on the usual ratio estimator under strictly convex loss functions and the general distributions having monotone likelihood ratio properties. In particular, *double shrinkage improved estimators* which utilize both of estimators of two location parameters are presented. Under order restrictions on the scale parameters, various improvements for estimation of the ratio and the scale parameters are also considered. These results are applied to normal, lognormal, exponential and pareto distributions. Finally, a multivariate extension is given for ratio of covariance matrices.

Key words and phrases: Point estimation, ratio of variances, shrinkage estimation, inadmissibility, Stein's truncated rule, monotone likelihood ratio property, normal, exponential, noncentral chi-square distributions, ratio of covariance matrices.

1. Introduction

Let S_1, S_2, X and Y be independent random variables where S_1/σ_1^2, S_2/σ_2^2 have chi-square distributions $\chi^2_{m_1}$, $\chi^2_{m_2}$, respectively, with m_1, m_2 degrees of freedom and X, Y have multivariate normal distributions $\mathrm{N}_p(\mu_1, \sigma_1^2 I_p)$, $\mathrm{N}_q(\mu_2, \sigma_2^2 I_q)$, respectively with unknown mean vectors μ_1, μ_2. Such a model appears in linear regression models. Suppose that we want to estimate the ratio of variances $\rho = \sigma_2^2/\sigma_1^2$ by estimator $\delta = \delta(X, Y, S_1, S_2)$ relative to the quadratic loss function $L(\delta, \rho) = (\delta/\rho - 1)^2$. Every estimator will be evaluated by the risk function $R(\omega, \delta) = E_\omega[(\delta/\rho - 1)^2]$ for unknown parameters ω.

Among estimators cS_2/S_1, the best constant c_0 is given by $c_0 = (m_1-4)/(m_2+2)$, that is, the estimator $\delta_0 = c_0 S_2/S_1$, is the best of the class. It is of interest to obtain estimators improving on δ_0 by using the information contained in X and Y. The statistic X or Y may be specially useful when p or q is not small. Applying the methods of Stein (1964) and Brown (1968) in estimation of a variance, Gelfand

and Dey (1988) proposed various improved shrinkage estimators. Of these, Stein type truncated estimators are given by

$$\delta_1 = \max\left\{\delta_0, \frac{m_1+p-4}{m_2+2}\frac{S_2}{S_1+\|X\|^2}\right\}, \tag{1.1}$$

$$\delta_2 = \min\left\{\delta_0, \frac{m_1-4}{m_2+q+2}\frac{S_2+\|Y\|^2}{S_1}\right\}. \tag{1.2}$$

For the estimation of the variance, see also Brewster and Zidek (1974), Nagata (1989), Maatta and Casella (1990) and Goutis and Casella (1991). Since δ_1 and δ_2 are based on one of $\|X\|^2$ and $\|Y\|^2$, it is of great interest to find an improved estimator employing both of $\|X\|^2$ and $\|Y\|^2$ and we shall call such a procedure a *double shrinkage improved estimator*, which, however, has not been obtained yet so far as I know. The difficulty may be due to the fact that the directions of shrinkage of estimators δ_1 and δ_2 are opposite.

The main purpose of this paper is to present double shrinkage improved estimators. For this, definite integral argument given by Kubokawa (1994) and Takeuchi (1991) is heavily exploited throughout the paper. The innovative idea of this method is to express the risk difference by a definite integral. Kubokawa (1994) has used the method to construct classes of improved estimators, including Brewster-Zidek type smooth and Stein type truncated procedures, for a normal mean vector and a scale parameter. Also the method has been utilized by Kubokawa *et al.* (1993*a*) and Kubokawa and Saleh (1993) for the problems of estimating variance components in mixed linear models and of estimating noncentrality parameters of noncentral chi-square and F distributions.

In Section 2, we provide two kinds of domination results. It is first shown that δ_0 is improved on by $\delta_\phi = \phi(\|X\|^2/S_1)S_2/S_1$ if

(a) $\phi(w_1)$ is nonincreasing and $\lim_{w_1\to\infty}\phi(w_1) = c_0$,

(b) $\phi(w_1) \le \phi_0(w_1)$ where

$$\phi_0(w_1) = \frac{m_1+p-4}{m_2+2}\frac{\int_0^{w_1} x^{p/2-1}/(1+x)^{(m_1+p)/2-1}dx}{\int_0^{w_1} x^{p/2-1}/(1+x)^{(m_1+p)/2-2}dx}. \tag{1.3}$$

It is seen that the conditions (a) and (b) are satisfied by $\phi_0(w_1)$ and $\phi_T(w_1) = \max\{m_1-4, (m_1+p-4)/(1+w_1)\}/(m_2+2)$, which yield the smooth estimator δ_{ϕ_0} and the Stein type truncated rule δ_1. For our main subject, we next consider the estimator $\delta_{\phi,\psi} = \{\phi(\|X\|^2/S_1)+\psi(\|Y\|^2/S_2)\}S_2/S_1$, and demonstrate that δ_ϕ is further dominated by $\delta_{\phi,\psi}$ if

(a) $\psi(z_2)$ is nondecreasing and $\lim_{z_2\to\infty}\psi(z_2) = 0$,

(b) $\psi(z_2) \ge \psi_0(z_2)$ where

$$\psi_0(z_2) = \frac{m_1-4}{m_2+q+2}\frac{\int_0^{z_2} x^{q/2-1}/(1+x)^{(m_2+q)/2+1}dx}{\int_0^{z_2} x^{q/2-1}/(1+x)^{(m_2+q)/2+2}dx} - c_0, \tag{1.4}$$

(c) $\phi(w_1) \ge c_0$.

It is seen that these conditions (a), (b) are satisfied by $\psi_0(z_2)$ and $\psi_T(z_2) = \min\{0, (m_1-4)(1+z_2)/(m_2+q+2) - c_0\}$. In particular, δ_1 can be improved on by

$$\delta_3 = \delta_{\phi_T,\psi_T} = \delta_1 + \min\left\{0, \frac{m_1-4}{m_2+q+2}\frac{S_2+\|Y\|^2}{S_1} - \delta_0\right\}. \tag{1.5}$$

The second term in the r.h.s. of (1.5) may be interpreted as an adjustment factor for over-shrinkage of δ_1. Also δ_3 is rewritten as

$$\begin{aligned} \delta_3 &= \delta_1 + \delta_2 - \delta_0 \\ &= \delta_2 + \max\left\{0, \frac{m_1+p-4}{m_2+2}\frac{S_2}{S_1+\|X\|^2} - \delta_0\right\}, \end{aligned} \tag{1.6}$$

and the second term in the r.h.s. of (1.6) may be viewed as adjustment for over-shrinkage of δ_2. Finally, a symmetry consideration shows that δ_0 is dominated by δ_1 and δ_2, both of which are further dominated by δ_3. In this way, we can get a double shrinkage improved estimator.

In Section 2, the above results are generalized to the cases of the strictly convex loss functions and the distributions with monotone likelihood ratio properties, including normal, lognormal, exponential and pareto distributions. In Sections 3 and 4, it is a priori supposed that there exist order restrictions between σ_1^2 and σ_2^2. Then various types of improved estimators of the ratio ρ and the variances σ_1^2, σ_2^2 are provided. As a multivariate extension, the problem of estimating the ratio of covariance matrices of two multivariate normal distributions are discussed in Section 5.

2. Point estimation of ratio of scale parameters

Let S_1, S_2, T_1 and T_2 be independent random variables where for $i = 1, 2$, $v_i = S_i/\sigma_i$ and $u_i = T_i/\sigma_i$ have densities

$$g_i(v_i)I_{[v_i>0]} \quad \text{and} \quad h_i(u_i;\lambda_i)I_{[u_i>k_i(\lambda_i)]} \tag{2.1}$$

for unknown real parameter λ_i, real function $k_i(\lambda_i)$, $k_i(0) = 0$, and the indicator function $I_{[\cdot]}$. Then we want to estimate the ratio of the scale parameters $\rho = \sigma_2/\sigma_1$ by an estimator $\delta = \delta(S_1, S_2, T_1, T_2)$ relative to the loss function $L(\delta/\rho)$ where $L(t)$ is a strictly convex function with $L(1) = 0$, that is, the derivative $L'(t)$ is strictly increasing for $t > 0$. To establish domination results in this paper, we assume the existence of the following expectations:

$$\begin{gathered} E\left[\left|L'\left(c\frac{v_2}{v_1}\right)\right|\frac{v_2}{v_1}\right], \quad E\left[\left|L'\left(c\frac{v_2}{v_1}\right)\right|v_2h_1(dv_1)\right], \\ E\left[\left|L'\left(c\frac{v_2}{v_1}\right)\right|\frac{v_2^2}{v_1}h_2(dv_2)\right], \quad E\left[L''\left(c\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1^2}\right], \\ E\left[L''\left(c\frac{v_2}{v_1}\right)v_2h_1(dv_1)\right] \quad \text{and} \quad E\left[L''\left(c\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1}h_2(dv_2)\right] \end{gathered}$$

for nonnegative constants c, d and $h_i(u_i) = h_i(u_i; 0)$.

A usual estimator of ρ is of the form cS_2/S_1 and the best constant c_0 is given by a solution of the equation

$$\int_0^\infty \int_0^\infty L'\left(c_0 \frac{v_2}{v_1}\right) \frac{v_2}{v_1} g_1(v_1) g_2(v_2) dv_1 dv_2 = 0, \tag{2.2}$$

for $v_i = S_i/\sigma_i$. The uniqueness of c_0 follows from the strict convexity of L. For improving on the estimator $\delta_0 = c_0 S_2/S_1$, consider a class of estimators

$$\delta_\phi = \begin{cases} \phi(W_1)\dfrac{S_2}{S_1} & \text{if } W_1 > 0, \\ c_0 \dfrac{S_2}{S_1} & \text{otherwise} \end{cases} \tag{2.3}$$

for $W_1 = T_1/S_1$ and positive and absolutely continuous function $\phi(\cdot)$. Assume that

(A.1) $H_1(x; \lambda_1)/H_1(x)$ is nondecreasing in $x > 0$,

where $H_1(x; \lambda_1) = \int_0^x h_1(u; \lambda_1) I_{[u \geq k_1(\lambda_1)]} du$ and $H_1(x) = \int_0^x h_1(u) du$. Note that (A.1) is guaranteed if

(A.1′) $h_1(x; \lambda_1)/h_1(x)$ is nondecreasing in $x > \max(0, k_1(\lambda_1))$.

Based on the following lemma, we can get the theorem.

LEMMA 2.1. *For positive functions $g(x)$ and $h(x)$, assume that $h(x)/g(x)$ is increasing. If $K(x)$ is a function such that $K(x) < 0$ for $x < x_0$ and $K(x) > 0$ for $x > x_0$, then*

$$\int_0^\infty K(x) \frac{h(x)}{g(x)} dx \geq \frac{h(x_0)}{g(x_0)} \int_0^\infty K(x) dx,$$

where the equality holds if and only if $h(x)/g(x)$ is a constant almost everywhere.

THEOREM 2.1. *Assume* (A.1) *and the following conditions:*

(a) *$\phi(w_1)$ is nonincreasing and* $\lim_{w_1 \to \infty} \phi(w_1) = c_0$,

(b) $\int_0^\infty \int_0^\infty L'(\phi(w_1)v_2/v_1)(v_2/v_1) g_1(v_1) g_2(v_2) H_1(w_1 v_1) dv_1 dv_2 \leq 0$.

Then δ_ϕ dominates δ_0.

PROOF. By the definite integral argument given by Kubokawa (1994) and Takeuchi (1991), the risk difference is written as

$$\begin{aligned} (2.4)\quad & R(\omega, \delta_0) - R(\omega, \delta_\phi) \\ &= E\left[\left\{L\left(c_0 \frac{v_2}{v_1}\right) - L\left(\phi\left(\frac{u_1}{v_1}\right)\frac{v_2}{v_1}\right)\right\} I_{[u_1 > 0]}\right] \\ &= E\left[\int_1^\infty \frac{d}{dt} L\left(\phi\left(t\frac{u_1}{v_1}\right)\frac{v_2}{v_1}\right) dt I_{[u_1>0]}\right] \\ &= E\left[\int_1^\infty L'\left(\phi\left(t\frac{u_1}{v_1}\right)\frac{v_2}{v_1}\right)\frac{u_1}{v_1}\frac{v_2}{v_1}\phi'\left(t\frac{u_1}{v_1}\right) dt I_{[u_1>0]}\right] \end{aligned}$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_1^\infty L'\left(\phi(w_1)\frac{v_2}{v_1}\right)\frac{w_1}{t^2}v_2\phi'(w_1)I_{[w_1>\max(0,k_1(\lambda_1)t/v_1)]}$$
$$\cdot g_1(v_1)g_2(v_2)h_1\left(\frac{w_1}{t}v_1;\lambda_1\right)dtdv_1dv_2dw_1$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^{w_1}\phi'(w_1)L'\left(\phi(w_1)\frac{v_2}{v_1}\right)v_2I_{[v_1x>k_1(\lambda_1)]}$$
$$\cdot g_1(v_1)g_2(v_2)h_1(v_1x;\lambda_1)dxdv_1dv_2dw_1$$
$$= \int_0^\infty\int_0^\infty\phi'(w_1)\int_0^\infty L'\left(\phi(w_1)\frac{v_2}{v_1}\right)v_2g_2(v_2)dv_2$$
$$\cdot\frac{1}{v_1}g_1(v_1)H_1(w_1v_1;\lambda_1)dv_1dw_1.$$

Here the fourth and the fifth equalities of (2.4) can be shown by making the transformations $w_1 = (t/v_1)u_1$ and $x = w_1/t$, respectively. From the strict convexity of L, it can be seen that $\int_0^\infty L'(\phi(w_1)v_2/v_1)v_2g_2(v_2)dv_2$ is nonincreasing in v_1. Hence from the assumptions (A.1) and (a), using Lemma 2.1 gives

$$\begin{aligned}(2.5)\qquad & R(\omega,\delta_0)-R(\omega,\delta_\phi)\\ &\ge\int_0^\infty\frac{H_1(w_1v_1^*;\lambda_1)}{H_1(w_1v_1^*)}\phi'(w_1)\int_0^\infty\int_0^\infty L'\left(\phi(w_1)\frac{v_2}{v_1}\right)\\ &\quad\cdot v_2g_2(v_2)dv_2\frac{1}{v_1}g_1(v_1)H_1(w_1v_1)dv_1dw_1,\end{aligned}$$

where $v_1^* = v_1^*(w_1)$ is a point such that $\int_0^\infty L'(\phi(w_1)v_2/v_1^*)v_2g_2(v_2)dv_2 = 0$. From the condition (b), the r.h.s. of (2.5) is nonnegative, which proves Theorem 2.1. □

Define $\phi_0(w_1)$ and $\phi_1(w_1)$, respectively, by unique solutions of the equations

$$(2.6)\qquad \int_0^\infty\int_0^\infty L'\left(\phi_0(w_1)\frac{v_2}{v_1}\right)\frac{v_2}{v_1}g_1(v_1)g_2(v_2)H_1(w_1v_1)dv_1dv_2 = 0,$$

$$(2.7)\qquad \int_0^\infty\int_0^\infty L'\left(\phi_1(w_1)\frac{v_2}{v_1}\right)v_2g_1(v_1)g_2(v_2)h_1(w_1v_1)dv_1dv_2 = 0,$$

and let $\phi_T(w_1) = \max\{c_0, \phi_1(w_1)\}$. It is easily checked that $\phi_0(w_1)$ satisfies the conditions (a), (b) if the following assumption holds:

(A.2) $H_1(c_1x)/H_1(c_2x)$ is nondecreasing in x for $0 < c_1 < c_2$.

To guarantee that $\phi_T(w_1)$ satisfies (b), we need to assume that

(A.2′) $h_1(c_1x)/h_1(c_2x)$ is nondecreasing in x for $0 < c_1 < c_2$,

which implies (A.2) and that $xh_1(x)/H_1(x)$ is decreasing. Using this fact and Lemma 2.1, we observe that

$$\begin{aligned}(2.8)\qquad 0 &= \int_0^\infty\int_0^\infty L'\left(\phi_1(w_1)\frac{v_2}{v_1}\right)\frac{v_2}{v_1}g_1(v_1)g_2(v_2)\frac{v_1h_1(w_1v_1)}{H_1(w_1v_1)}\\ &\quad\cdot H_1(w_1v_1)dv_1dv_2\\ &\ge d(w_1)\int_0^\infty\int_0^\infty L'\left(\phi_1(w_1)\frac{v_2}{v_1}\right)\frac{v_2}{v_1}g_1(v_1)g_2(v_2)H_1(w_1v_1)dv_1dv_2\end{aligned}$$

for some positive function $d(w_1)$. The inequality (2.8) shows that $\phi_1(w_1) \le \phi_0(w_1)$ or $\phi_T(w_1) \le \phi_0(w_1)$, so that (b) holds for $\phi_T(w_1)$. For verifying (a), suppose that $\phi_1(x) < \phi_1(y)$ for $x < y$. Then,

$$
\begin{aligned}
(2.9) \qquad 0 &= \int_0^\infty \int_0^\infty L'\left(\phi_1(x)\frac{v_2}{v_1}\right) v_2 g_1(v_1) g_2(v_2) h_1(xv_1) dv_1 dv_2 \\
&< \int_0^\infty \int_0^\infty L'\left(\phi_1(y)\frac{v_2}{v_1}\right) v_2 g_1(v_1) g_2(v_2) \frac{h_1(xv_1)}{h_1(yv_1)} h_1(yv_1) dv_1 dv_2 \\
&\le e(x,y) \int_0^\infty \int_0^\infty L'\left(\phi_1(y)\frac{v_2}{v_1}\right) v_2 g_1(v_1) g_2(v_2) h_1(yv_1) dv_1 dv_2 \\
&= 0,
\end{aligned}
$$

for some positive function $e(x,y)$, which yields a contradiction. Hence $\phi_1(w_1)$ is nonincreasing, so that, together with $\phi_T(w_1) \le \phi_0(w_1)$, we see that $\phi_T(w_1)$ satisfies (a). In this way, we get two types of improved estimators δ_{ϕ_0} and δ_{ϕ_T}.

COROLLARY 2.1. *Under* (A.1) *and* (A.2), *the estimator* δ_{ϕ_0} *dominates* δ_0.

COROLLARY 2.2. *Under* (A.1) *and* (A.2′), *the estimator* δ_{ϕ_T} *dominates* δ_0.

Now we consider to improve on δ_ϕ by using the statistic T_2, which is the main purpose of this paper. The estimator we look into is

$$
(2.10) \qquad \delta_{\phi,\psi,\gamma} = \begin{cases} \{\phi(W_1) + \psi(Z_2)\} S_2/S_1 & \text{if } W_1 > 0,\ Z_2 > 0 \\ \phi(W_1) S_2/S_1 & \text{if } W_1 > 0,\ Z_2 \le 0 \\ \gamma(Z_2) S_2/S_1 & \text{if } W_1 \le 0,\ Z_2 > 0 \\ c_0 S_2/S_1 & \text{otherwise,} \end{cases}
$$

for $Z_2 = T_2/S_2$ and positive and absolutely continuous functions ψ, γ. Suppose that

(A.3) $H_2(x;\lambda_2)/H_2(x)$ is nondecreasing in $x > 0$,

where $H_2(x;\lambda_2)$, $H_2(x)$ and $h_2(u)$ are defined similarly to the case of $h_1(x;\lambda_1)$. (A.3) holds if

(A.3′) $h_2(x;\lambda_2)/h_2(x)$ is nondecreasing in $x > \max(0, k_2(\lambda_2))$.

THEOREM 2.2. *Assume* (A.3) *and the following conditions*:

(a) $\psi(z_2)$ *is nondecreasing and* $\lim_{z_2\to\infty} \psi(z_2) = 0$,

(b) $\int_0^\infty \int_0^\infty L'([c_0 + \psi(z_2)]v_2/v_1)(v_2/v_1) g_1(v_1) g_2(v_2) H_2(z_2 v_2) dv_1 dv_2 \ge 0$,

(c) $\phi(w_1) \ge c_0$,

(d) $\gamma(z_2)$ *satisfies the above conditions* (a) *and* (b) *for* $\gamma(z_2) = \psi(z_2) + c_0$.

Then $\delta_{\phi,\psi,\gamma}$ *dominates* δ_ϕ.

PROOF. First we notice that for $u_2 = T_2/\sigma_2$,

$$
\begin{aligned}
&R(\omega, \delta_\phi) - R(\omega, \delta_{\phi,\psi,\gamma}) \\
&\quad = E\left[\left\{ L\left(\phi\left(\frac{u_1}{v_1}\right)\frac{v_2}{v_1}\right) - L\left(\left[\phi\left(\frac{u_1}{v_1}\right) + \psi\left(\frac{u_2}{v_2}\right)\right]\frac{v_2}{v_1}\right)\right\} I_{[w_1>0,z_2>0]}\right] \\
&\qquad + E\left[\left\{ L\left(c_0\frac{v_2}{v_1}\right) - L\left(\gamma\left(\frac{u_2}{v_2}\right)\frac{v_2}{v_1}\right)\right\} I_{[w_1\le 0,z_2>0]}\right] \\
&\quad \equiv \Delta_1 + \Delta_2.
\end{aligned}
$$

By the same arguments as in the proof of Theorem 2.1,

$$
\begin{aligned}
(2.11)\quad \Delta_1 &= E\left[\int_1^\infty \frac{d}{dt}L\left(\left[\phi\left(\frac{u_1}{v_1}\right)+\psi\left(t\frac{u_2}{v_2}\right)\right]\frac{v_2}{v_1}\right)dtI_{[w_1>0,z_2>0]}\right]\\
&= E\left[\int_1^\infty L'\left(\left[\phi\left(\frac{u_1}{v_1}\right)+\psi\left(t\frac{u_2}{v_2}\right)\right]\frac{v_2}{v_1}\right)\frac{u_2}{v_1}\psi'\left(t\frac{u_2}{v_2}\right)dtI_{[w_1>0,z_2>0]}\right]\\
&\geq E\left[\int_1^\infty L'\left(\left[c_0+\psi\left(t\frac{u_2}{v_2}\right)\right]\frac{v_2}{v_1}\right)\frac{u_2}{v_1}\psi'\left(t\frac{u_2}{v_2}\right)dtI_{[w_1>0,z_2>0]}\right],
\end{aligned}
$$

where the inequality in (2.11) follows from the fact that $\psi' \geq 0$, $\phi \geq c_0$ and $L'(t)$ is increasing. Making the transformations $z_2 = (t/v_2)u_2$ and $x = z_2/t$ gives

$$
\begin{aligned}
(2.12)\quad \Delta_1 &\geq \int_0^\infty\int_0^\infty\int_0^\infty\int_1^\infty L'\left([c_0+\psi(z_2)]\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1}\frac{z_2}{t^2}\psi'(z_2)g_1(v_1)g_2(v_2)\\
&\quad\cdot h_2\left(\frac{z_2}{t}v_2;\lambda_2\right)I_{[z_2>\max(0,k_2(\lambda_2)t/v_2)]}dtdv_1dv_2dz_2E[I_{[u_1>0]}]\\
&= \int_0^\infty \psi'(z_2)\int_0^\infty\int_0^\infty L'\left([c_0+\psi(z_2)]\frac{v_2}{v_1}\right)\frac{v_2}{v_1}\\
&\quad\cdot g_1(v_1)g_2(v_2)H_2(z_2v_2;\lambda_2)dv_1dv_2dz_2E[I_{[u_1>0]}].
\end{aligned}
$$

Similar to (2.5), it can be verified that the r.h.s. of the equality in (2.12) is non-negative from the assumption (A.3) and the condition (b). Similarly, it can be shown that $\Delta_2 \geq 0$ under the condition (d), and Theorem 2.2 is proved. □

Let $\psi_0(z_2)$ and $\psi_1(z_2)$ be unique solutions of the equations

$$
(2.13)\quad \int_0^\infty\int_0^\infty L'\left([c_0+\psi_0(z_2)]\frac{v_2}{v_1}\right)\frac{v_2}{v_1}g_1(v_1)g_2(v_2)H_2(z_2v_2)dv_1dv_2 = 0,
$$

$$
(2.14)\quad \int_0^\infty\int_0^\infty L'\left([c_0+\psi_1(z_2)]\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1}g_1(v_1)g_2(v_2)h_2(z_2v_2)dv_1dv_2 = 0,
$$

and let $\psi_T(z_2) = \min\{0, \psi_1(z_2)\}$. Also let $\gamma_0(z_2)$ and $\gamma_T(z_2)$ be defined by $\gamma_0 = c_0 + \psi_0$ and $\gamma_T = c_0 + \psi_T$. Then it is easily checked that $\psi_0(z_2)$ satisfies (a), (b) of Theorem 2.2 if the following assumption holds:

(A.4) $H_2(c_1x)/H_2(c_2x)$ is nondecreasing in x for $0 < c_1 < c_2$.

Also the same arguments as in (2.8) and (2.9) can be used to check $\psi_T(z_2)$ satisfies (a), (b) under the assumption:

(A.4′) $h_2(c_1x)/h_2(c_2x)$ is nondecreasing in x for $0 < c_1 < c_2$.

COROLLARY 2.3. *Under* (A.3) *and* (A.4), *the estimator* $\delta_{\phi,\psi_0,\gamma_0}$ *dominates* δ_ϕ *if* $\phi(w_1) \geq c_0$.

COROLLARY 2.4. *Under* (A.3) *and* (A.4′), *the estimator* $\delta_{\phi,\psi_T,\gamma_T}$ *dominates* δ_ϕ *if* $\phi(w_1) \geq c_0$.

When the statistics T_1 and T_2 are used in reverse order, quite similar results hold. That is, the estimator

$$\delta_\phi^* = \begin{cases} \phi(Z_2)\dfrac{S_2}{S_1} & \text{if } Z_2 > 0, \\ c_0\dfrac{S_2}{S_1} & \text{otherwise} \end{cases} \tag{2.15}$$

is considered for dominating δ_0, and for the further improvement, the estimator

$$\delta_{\phi,\psi,\gamma}^* = \begin{cases} \{\phi(Z_2)+\psi(W_1)\}S_2/S_1 & \text{if } W_1>0,\ Z_2>0 \\ \gamma(W_1)S_2/S_1 & \text{if } W_1>0,\ Z_2\le 0 \\ \phi(Z_2)S_2/S_1 & \text{if } W_1\le 0,\ Z_2>0 \\ c_0S_2/S_1 & \text{otherwise} \end{cases} \tag{2.16}$$

is taken. Then we can get the following theorems.

THEOREM 2.3. *Assume* (A.3) *and the following conditions*:
(a) $\phi(z_2)$ *is nondecreasing and* $\lim_{z_2\to\infty}\phi(z_2)=c_0$,
(b) $\int_0^\infty\int_0^\infty L'(\phi(z_2)v_2/v_1)(v_2/v_1)g_1(v_1)g_2(v_2)H_2(z_2v_2)dv_1dv_2 \ge 0$.
Then δ_ϕ^* *dominates* δ_0.

THEOREM 2.4. *Assume* (A.1) *and the following conditions*:
(a) $\psi(w_1)$ *is nonincreasing and* $\lim_{w_1\to\infty}\phi(w_1)=0$,
(b) $\int_0^\infty\int_0^\infty L'([c_0+\psi(w_1)]v_2/v_1)(v_2/v_1)g_1(v_1)g_2(v_2)H_1(w_1v_1)dv_1dv_2 \le 0$,
(c) $\phi(z_2)\le c_0$,
(d) $\gamma(w_1)$ *satisfies the above conditions* (a) *and* (b) *for* $\gamma(w_1)=\psi(w_1)+c_0$.
Then $\delta_{\phi,\psi,\gamma}^*$ *dominates* δ_ϕ.

We conclude this section with the following two examples.

Example 2.1. (Normal distribution) Let X, Y, S_1 and S_2 be independent random variables such that $X\sim \mathrm{N}_p(\mu_1,\sigma_1^2I_p)$, $Y\sim \mathrm{N}_q(\mu_2,\sigma_2^2I_q)$, $S_1/\sigma_1^2\sim\chi_{m_1}^2$ and $S_2/\sigma_2^2\sim\chi_{m_2}^2$. When we want to estimate $\rho=\sigma_2^2/\sigma_1^2$ under the loss $(\delta/\rho-1)^2$, the best of estimators cS_2/S_1 is given by $\delta_0=c_0S_2/S_1$ with $c_0=(m_1-4)/(m_2+2)$. The improvements on δ_0 were studied by Gelfand and Dey (1988). Since the assumptions (A.1)–(A.4′) are satisfied, Theorems 2.1 and 2.2 can be applied. The functions $\phi_0(w_1)$ and $\psi_0(z_2)$ defined in Theorems 2.1 and 2.2, respectively, are expressed as

$$\phi_0(w_1)=\frac{m_1+p-4}{m_2+2}\frac{\int_0^{w_1}x^{p/2-1}/(1+x)^{(m_1+p)/2-1}dx}{\int_0^{w_1}x^{p/2-1}/(1+x)^{(m_1+p)/2-2}dx},$$

$$\psi_0(z_2)=\frac{m_1-4}{m_2+q+2}\frac{\int_0^{z_2}x^{q/2-1}/(1+x)^{(m_2+q)/2+1}dx}{\int_0^{z_2}x^{q/2-1}/(1+x)^{(m_2+q)/2+2}dx}-\frac{m_1-4}{m_2+2}.$$

Also $\phi_T(w_1)$ and $\psi_T(z_2)$ are given by

$$\phi_T(w_1) = \max\left\{\frac{m_1-4}{m_2+2}, \frac{m_1+p-4}{m_2+2}\frac{1}{1+w_1}\right\},$$
$$\psi_T(z_2) = \min\left\{0, \frac{m_1-4}{m_2+q+2}(1+z_2) - \frac{m_1-4}{m_2+2}\right\}.$$

Hence for instance, δ_0 is dominated by

$$\delta_1 = \delta_{\phi_T} = \max\left\{\delta_0, \frac{m_1+p-4}{m_2+2}\frac{S_2}{S_1+\|X\|^2}\right\}, \tag{2.17}$$

which is further improved by

$$\begin{aligned}\delta_3 = \delta_{\phi_T,\psi_T} &= \delta_1 + \min\left\{0, \frac{m_1-4}{m_2+q+2}\frac{S_2+\|Y\|^2}{S_1} - \delta_0\right\} \\ &= \delta_1 + \delta_2 - \delta_0,\end{aligned} \tag{2.18}$$

where

$$\delta_2 = \min\left\{\delta_0, \frac{m_1-4}{m_2+q+2}\frac{S_2+\|Y\|^2}{S_1}\right\}. \tag{2.19}$$

On the other hand, applying Theorems 2.3 and 2.4 gives that δ_0 is dominated by δ_2, being improved on by δ_3. Hence δ_3 is better than both of δ_1 and δ_2. Since $\delta_3 = \delta_1 + (\delta_2 - \delta_0) = \delta_2 + (\delta_1 - \delta_0)$ with $\delta_2 - \delta_0 < 0$ and $\delta_1 - \delta_0 > 0$, the terms $(\delta_2 - \delta_0)$ and $(\delta_1 - \delta_0)$ may be interpreted as adjustment factors for over-shrinkage of δ_1 and δ_2, respectively. It should be noted that similar results can be provided for a lognormal distribution.

Example 2.2. (Exponential distribution) Let $(X_1, \dots, X_{n_1})$ and $(Y_1, \dots, Y_{n_2})$ be two independent random samples from exponential distributions with density functions $\sigma_1^{-1}\exp\{-(x-\mu_1)/\sigma_1\}I_{[x\geq\mu_1]}$ and $\sigma_2^{-1}\exp\{-(y-\mu_2)/\sigma_2\}I_{[y\geq\mu_2]}$, respectively, for unknown parameters μ_1, μ_2, σ_1 and σ_2. Arnold (1970), Brewster (1974), Nagata (1991) and Kubokawa (1994) have dealt with estimation of the scale parameter. It is here supposed that we want to estimate the ratio of the scale parameters $\rho = \sigma_2/\sigma_1$ under the loss $(\delta/\rho - 1)^2$. This problem was studied by Madi and Tsui (1990) from a decision-theoretic point of view. For order statistics $X_{(j)}$'s, put $S_1 = \sum_{j=2}^{n_1}(X_{(j)} - X_{(1)})$ and $T_1 = n_1 X_{(1)}$, and let S_2, T_2 be defined similarly. Then for $i = 1, 2$, S_i and T_i are independent and S_i, $T_i - n_i\mu_i$ have Gamma$(n_i - 1, \sigma_i)$—, Gamma$(1, \sigma_i)$—distributions, respectively. In this case, $\lambda_i = n_i\mu_i/\sigma_i$, $k_i(\lambda_i) = \lambda_i$, $g_i(v_i) = [\Gamma(n_i-1)]^{-1}v_i^{n_i-2}\exp(-v_i)I_{[v_i>0]}$ and $h_i(u_i;\lambda_i) = \exp\{-(u_i - \lambda_i)\}I_{[u_i>\lambda_i]}$. Since $H_i(x) = 1 - \exp(-x)$ and $h_i(x) = \exp(-x)$, the assumptions (A.1)–(A.4′) can be easily checked, so that Theorems 2.1 and 2.2 are applied in order to improve on $\delta_0 = \{(n_1-3)/n_2\}S_2/S_1$. The functions $\phi_0(w_1)$ and $\psi_0(z_2)$ are written as

$$\phi_0(w_1) = \frac{n_1-3}{n_2}\frac{1-(1+w_1)^{-n_1+2}}{1-(1+w_1)^{-n_1+3}},$$
$$\psi_0(z_2) = \frac{n_1-3}{n_2}\frac{1-(1+z_2)^{-n_2}}{1-(1+z_2)^{-n_2-1}} - \frac{n_1-3}{n_2}.$$

Also $\phi_T(w_1)$ and $\psi_T(z_2)$ are given by

$$\phi_T(w_1) = \max\left\{\frac{n_1-3}{n_2}, \frac{n_1-2}{n_2}\frac{1}{1+w_1}\right\},$$
$$\psi_T(z_2) = \min\left\{0, \frac{n_1-3}{n_2+1}(1+z_2) - \frac{n_1-3}{n_2}\right\}.$$

Hence δ_0 is, for example, dominated by

$$\delta_1 = \delta_{\phi_T} = \begin{cases} \max\left\{\delta_0, \dfrac{n_1-2}{n_2}\dfrac{S_2}{S_1+T_1}\right\}, & \text{if } T_1 > 0, \\ \delta_0, & \text{otherwise,} \end{cases} \tag{2.20}$$

which is further improved on by

$$\delta_3 = \delta_{\phi_T,\psi_T} = \delta_1 + \delta_2 - \delta_0, \tag{2.21}$$

where

$$\delta_2 = \begin{cases} \min\left\{\delta_0, \dfrac{n_1-3}{n_2+1}\dfrac{S_2+T_2}{S_1}\right\}, & \text{if } T_2 > 0, \\ \delta_0, & \text{otherwise.} \end{cases} \tag{2.22}$$

On the other hand, applying Theorems 2.3 and 2.4 gives that δ_3 is better than δ_2. It should be noted that similar results hold for a pareto distribution.

3. Estimation of ratio of ordered scale parameters

In the variance components models and other statistical models with applications, we sometimes recognize the existence of order restrictions between unknown parameters. In this section, it is a priori supposed that there exists an order restriction of either of $\sigma_1 < \sigma_2$ or $\sigma_1 > \sigma_2$ between the scales σ_1 and σ_2, and in each case, we want to find superior estimators of the ratio of the scales $\rho = \sigma_2/\sigma_1$.

3.1 *The case of $\rho > 1$*

We first treat the case of $\rho > 1$. For improving on $\delta_0 = c_0 S_2/S_1$, consider the estimator

$$\delta_\phi = \phi\left(\frac{S_2}{S_1}\right)\frac{S_2}{S_1}. \tag{3.1}$$

THEOREM 3.1. *For $w = S_2/S_1$, assume the following conditions:*
(a) *$\phi(w)$ is nonincreasing and* $\lim_{w\to\infty}\phi(w) = c_0$,
(b) $\int_0^w L'(\phi(w)x)\int_0^\infty v_1 x g_1(v_1) g_2(v_1 x) dv_1 dx \le 0$.
Then δ_ϕ given by (3.1) *dominates δ_0.*

Let $\phi_0(w)$ be a solution of the equation

$$\int_0^w L'(\phi_0(w)x)H(x)dx = 0, \tag{3.2}$$

where $H(x) = \int_0^\infty xv_1g_1(v_1)g_2(v_1x)dv_1$. Since $x \leq w$, we observe that $0 \leq L'(\phi_0(w)w)\int_0^w H(x)dx$, which means that

$$L'(\phi_0(w)w) \geq 0. \tag{3.3}$$

Differentiating (3.2) with respect to w yields

$$L'(\phi_0(w)w)H(w) + \phi_0'(w)\int_0^w L''(\phi_0(w)x)xH(x)dx = 0,$$

so that from (3.3), $\phi_0(w)$ is decreasing. Also $\lim_{w\to\infty}\phi_0(w) = c_0$. Hence $\phi_0(w)$ satisfies (a) and (b) of Theorem 3.1 and we get the improved estimator δ_{ϕ_0}. On the other hand, from (3.3), $\phi_0(w) \geq 1/w$. Putting $\phi_T(w) = \max(c_0, 1/w)$, we can see that (a), (b) hold for $\phi_T(w)$, getting another improved estimator $\delta_{\phi_T} = \max\{\delta_0, 1\}$.

For further improvement, the estimator we treat is of the form

$$\delta_{\phi,\psi} = \begin{cases} \left\{\phi\left(\frac{S_2}{S_1}\right) + \psi\left(\frac{T_2}{S_2}\right)\right\}\frac{S_2}{S_1} & \text{if } T_2 > 0, \\ \delta_\phi, & \text{otherwise.}\end{cases} \tag{3.4}$$

THEOREM 3.2. *Assume* (A.3) *and the following conditions*:
(a) $\psi(z_2)$ *is nondecreasing and* $\lim_{z_2\to\infty}\psi(z_2) = 0$,
(b) $\int_0^\infty\int_0^\infty L'([c_0 + \psi(z_2)]v_2/v_1)(v_2/v_1)g_1(v_1)g_2(v_2)H_2(z_2v_2)dv_1dv_2 \geq 0$,
(c) $\phi(w) \geq c_0$.
Then $\delta_{\phi,\psi}$ *given by* (3.4) *dominates* δ_ϕ *given by* (3.1).

Let $\psi_0(z_2)$ and $\psi_1(z_2)$ be solutions of the equations (2.13) and (2.14) and let $\psi_T(z_2) = \min\{0, \psi_1(z_2)\}$. Then $\psi_0(z_2)$ and $\psi_T(z_2)$ satisfy (a), (b) of Theorem 3.2 under (A.4$'$). It does not seem easy to derive a superior rule to $\delta_{\phi,\psi}$ by use of the statistic T_1.

PROOF OF THEOREM 3.1. The risk difference is written as

$$\begin{aligned} R(\omega,\delta_0) - R(\omega,\delta_\phi) &= E\left[L\left(c_0\frac{v_2}{v_1}\right) - L\left(\phi\left(\frac{v_2}{v_1}\rho\right)\frac{v_2}{v_1}\right)\right] \\ &= E\left[\int_1^\infty \frac{d}{dt}\left\{L\left(\phi\left(\frac{v_2}{v_1}\rho t\right)\frac{v_2}{v_1}\right)\right\}dt\right] \\ &= E\left[\int_1^\infty L'\left(\phi\left(\frac{v_2}{v_1}\rho t\right)\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1^2}\rho\phi'\left(\frac{v_2}{v_1}\rho t\right)dt\right]. \end{aligned}$$

After noting that $\phi' \leq 0$ and $\rho > 1$, making the transformations gives

$$\begin{aligned} &R(\omega,\delta_0) - R(\omega,\delta_\phi) \\ &\quad\geq E\left[\int_1^\infty L'\left(\phi\left(\frac{v_2}{v_1}t\right)\frac{v_2}{v_1}\right)\frac{v_2^2}{v_1^2}\rho\phi'\left(\frac{v_2}{v_1}\rho t\right)dt\right] \\ &\quad= \int_0^\infty\int_0^\infty\int_1^\infty L'\left(\phi(w)\frac{w}{t}\right)\frac{w^2}{t^2}\rho\phi'(w\rho)g_1(v_1)g_2\left(\frac{w}{t}v_1\right)\frac{v_1}{t}dtdv_1dw \\ &\quad= \int_0^\infty \rho\phi'(w\rho)\int_0^\infty\int_0^w L'(\phi(w)x)xv_1g_1(v_1)g_2(v_1x)dxdv_1dw, \end{aligned}$$

which is nonnegative from (a) and (b), establishing Theorem 3.1. □

PROOF OF THEOREM 3.2. Observe that

$$
\begin{aligned}
&R(\omega, \delta_\phi) - R(\omega, \delta_{\phi,\psi}) \\
&= E\left[\int_1^\infty \frac{d}{dt} L\left(\left[\phi\left(\frac{v_2}{v_1}\rho\right) + \psi\left(\frac{u_2}{v_2}t\right)\right]\frac{v_2}{v_1}\right) dt I_{[u_2>0]}\right] \\
&= E\left[\int_1^\infty L'\left(\left[\phi\left(\frac{v_2}{v_1}\rho\right) + \psi\left(\frac{u_2}{v_2}t\right)\right]\frac{v_2}{v_1}\right) \frac{u_2}{v_1}\psi'\left(\frac{u_2}{v_2}t\right) dt I_{[u_2>0]}\right] \\
&\geq E\left[\int_1^\infty L'\left(\left[c_0 + \psi\left(\frac{u_2}{v_2}t\right)\right]\frac{v_2}{v_1}\right) \frac{u_2}{v_1}\psi'\left(\frac{u_2}{v_2}t\right) dt I_{[u_2>0]}\right],
\end{aligned}
$$

which is nonnegative from the proof of Theorem 2.2. □

3.2 *The case of $\rho < 1$*

Next we state the case of $\rho < 1$.

THEOREM 3.3. *Assume that*
(a) *$\phi(w)$ is nonincreasing and $\phi(0) = c_0$,*
(b) $\int_w^\infty L'(x\phi(w)) \int_0^\infty x v_1 g_1(v_1) g_2(v_1 x) dv_1 dx \geq 0$.
Then δ_ϕ given by (3.1) *dominates δ_0.*

Let $\phi_0(w)$ be a solution of the equation

$$\int_w^\infty L'(x\phi(w)) \int_0^\infty x v_1 g_1(v_1) g_2(v_1 x) dv_1 dx = 0$$

and put $\phi_T(w) = \min(c_0, 1/w)$. From similar discussions stated below Theoren 3.1, it can be verified that $\phi_0(w)$ and $\phi_T(w)$ satisfy (a) and (b) of Theorem 3.3.

For the further improvement, we consider the estimator

$$
\delta_{\phi,\psi} = \begin{cases} \left[\phi\left(\frac{S_2}{S_1}\right) + \psi\left(\frac{T_1}{S_1}\right)\right]\frac{S_2}{S_1} & \text{if } T_1 > 0, \\ \delta_\phi & \text{otherwise.} \end{cases} \tag{3.5}
$$

THEOREM 3.4. *Assume* (A.1) *and the following conditions:*
(a) *$\psi(w_1)$ is nonincreasing and* $\lim_{w_1\to\infty} \psi(w_1) = 0$,
(b) $\int_0^\infty \int_0^\infty L'([c_0 + \psi(w_1)]v_2/v_1)(v_2/v_1) H_1(w_1 v_1) g_1(v_1) g_2(v_2) dv_1 dv_2 \leq 0$,
(c) $\phi(w) \leq c_0$.
Then $\delta_{\phi,\psi}$ given by (3.5) *dominates δ_ϕ given by* (3.1).

Let $\psi_0(w_1)$ and $\psi_1(w_1)$ be solutions of the equations

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty L'\left([c_0 + \psi_0(w_1)]\frac{v_2}{v_1}\right) \frac{v_2}{v_1} H_1(w_1 v_1) g_1(v_1) g_2(v_2) dv_1 dv_2 = 0, \\
&\int_0^\infty \int_0^\infty L'\left([c_0 + \psi_1(w_1)]\frac{v_2}{v_1}\right) v_2 h_1(w_1 v_1) g_1(v_1) g_2(v_2) dv_1 dv_2 = 0,
\end{aligned}
$$

and put $\psi_T(w_1) = \max(0, \psi_1(w_1) - c_0)$. From the arguments below Theorem 2.1, it is seen that $\psi_0(w_1)$ and $\psi_T(w_1)$ satisfy (a) and (b) of Theorem 3.4.

PROOF OF THEOREM 3.3. Similar to the proof of Theorem 3.1, the risk difference is written as

$$\begin{aligned}
&R(\omega, \delta_0) - R(\omega, \delta_\phi) \\
&\quad = -E\left[\int_0^1 \frac{d}{dt} L\left(\frac{v_2}{v_1}\phi\left(t\rho\frac{v_2}{v_1}\right)\right) dt\right] \\
&\quad = -E\left[\int_0^1 L'\left(\frac{v_2}{v_1}\phi\left(t\rho\frac{v_2}{v_1}\right)\right) \rho\frac{v_2^2}{v_1^2}\phi'\left(t\rho\frac{v_2}{v_1}\right) dt\right] \\
&\quad \geq -E\left[\int_0^1 L'\left(\frac{v_2}{v_1}\phi\left(t\frac{v_2}{v_1}\right)\right) \rho\frac{v_2^2}{v_1^2}\phi'\left(t\rho\frac{v_2}{v_1}\right) dt\right] \\
&\quad = -\int_0^\infty \rho\phi'(\rho w)\int_0^\infty \int_w^\infty L'(x\phi(w))xv_1g_1(v_1)g_2(v_1x)dxdv_1dw,
\end{aligned}$$

which is nonnegative, proving the theorem. □

PROOF OF THEOREM 3.4. Observe that

$$\begin{aligned}
&R(\omega, \delta_\phi) - R(\omega, \delta_{\phi,\psi}) \\
&\quad = E\left[\int_1^\infty \frac{d}{dt} L\left(\left[\phi\left(\rho\frac{v_2}{v_1}\right) + \psi\left(t\frac{u_1}{v_1}\right)\right]\frac{v_2}{v_1}\right) dt I_{[u_1>0]}\right] \\
&\quad = E\left[\int_1^\infty L'\left(\left[\phi\left(\rho\frac{v_2}{v_1}\right) + \psi\left(t\frac{u_1}{v_1}\right)\right]\frac{v_2}{v_1}\right)\frac{v_2}{v_1}\frac{u_1}{v_1}\psi'\left(t\frac{u_1}{v_1}\right) dt I_{[u_1>0]}\right] \\
&\quad \geq E\left[\int_1^\infty L'\left(\left[c_0 + \psi\left(t\frac{u_1}{v_1}\right)\right]\frac{v_2}{v_1}\right)\frac{v_2}{v_1}\frac{u_1}{v_1}\psi'\left(t\frac{u_1}{v_1}\right) dt I_{[u_1>0]}\right],
\end{aligned}$$

which is nonnegative from the proof of Theorem 2.1. □

4. Estimation of ordered scale parameters

Besides the estimation of the ratio, we here deal with the problem of estimating the ordered scale parameters with utilizing the discussions of the previous sections. It will be clarified that two problems of estimation of σ_1 and σ_2 have different structures in domination under the order restriction $\sigma_1 < \sigma_2$.

4.1 *Estimation of σ_1 under $\sigma_1 < \sigma_2$*

In estimation of σ_1, the best of estimators cS_1 is given by $\hat{\sigma}_1 = c_0 S_1$, where c_0 is a solution of the equation

$$\int_0^\infty L'(c_0 v_1)v_1 g_1(v_1)dv_1 = 0. \tag{4.1}$$

Taking the order restriction $\sigma_1 < \sigma_2$ into account, we find improved procedures among the estimators

$$\delta_\phi = \phi\left(\frac{S_2}{S_1}\right) S_1. \tag{4.2}$$

THEOREM 4.1. *Assume that*
(a) $\phi(w)$ *is nondecreasing and* $\lim_{w\to\infty}\phi(w) = c_0$,
(b) $\int_0^\infty L'(\phi(w)v_1)v_1 g_1(v_1)G_2(v_1 w)dv_1 \geq 0$ *where* $G_2(x) = \int_0^x g_2(y)dy$.
Then δ_ϕ *given by* (4.2) *dominates* $\hat{\sigma}_1$.

Let $\phi_0(w)$ and $\phi_1(w)$ be solutions of the equations

$$\int_0^\infty L'(\phi_0(w)v_1)v_1 g_1(v_1)G_2(v_1 w)dv_1 = 0, \tag{4.3}$$

$$\int_0^\infty L'(\phi_1(w)v_1)v_1^2 g_1(v_1)g_2(v_1 w)dv_1 = 0, \tag{4.4}$$

and put $\phi_T(w) = \min(c_0, \phi_1(w))$. To guarantee (a), (b) of Theorem 4.1, we need to assume that
(A.5) $g_2(c_1 x)/g_2(c_2 x)$ is increasing in x for $0 < c_1 < c_2$,
which implies that $xg_2(x)/G_2(x)$ is decreasing. From Lemma 2.1, notice that

$$\begin{aligned}\int_0^\infty & L'(\phi(w)v_1)v_1^2 g_1(v_1)g_2(v_1 w)dv_1 \\ &\leq \frac{v_1^* g_2(v_1^* w)}{G_2(v_1^* w)} \int_0^\infty L'(\phi(w)v_1)v_1 g_1(v_1)G_2(v_1 w)dv_1\end{aligned} \tag{4.5}$$

for $v_1^* = 1/\phi(w)$. Differentiating (4.3) with respect to w yields

$$\begin{aligned}\phi_0'(w) & \int_0^\infty L''(\phi_0(w)v_1)v_1^2 g_1(v_1)G_2(v_1 w)dv_1 \\ &+ \int_0^\infty L'(\phi_0(w)v_1)v_1^2 g_1(v_1)g_2(v_1 w)dv_1 = 0,\end{aligned}$$

so that from (4.5), $\phi_0(w)$ is nondecreasing. Since $\lim_{w\to\infty}\phi_0(w) = c_0$, $\phi_0(w)$ satisfies (a) and (b) under the assumption (A.5). From (4.5) and the monotonicity of $L'(\cdot)$, we get that $\phi_0(w) \leq \phi_1(w)$ or $\phi_0(w) \leq \phi_T(w)$, so that $\lim_{w\to\infty}\phi_T(w) = c_0$ and $\phi_T(w)$ satisfies (b). Also the monotonicity of $\phi_1(w)$ can be shown under (A.5) by the same arguments stated below the proof of Theorem 2.1. Hence two kinds of improved estimators δ_{ϕ_0} and δ_{ϕ_T} are obtained.

For further improvements, we consider the estimator

$$\delta_{\phi,\psi,\gamma_1,\gamma_2} = \begin{cases} S_1\phi\left(\frac{S_2}{S_1}\right)\psi\left(\frac{S_2}{S_1}, \frac{T_1}{S_1}, \frac{T_2}{S_1}\right) & \text{if } T_1 > 0,\ T_2 > 0 \\ S_1\phi\left(\frac{S_2}{S_1}\right)\gamma_1\left(\frac{S_2}{S_1}, \frac{T_1}{S_1}\right) & \text{if } T_1 > 0,\ T_2 \leq 0 \\ S_1\phi\left(\frac{S_2}{S_1}\right)\gamma_2\left(\frac{S_2}{S_1}, \frac{T_2}{S_1}\right) & \text{if } T_1 \leq 0,\ T_2 > 0 \\ S_1\phi\left(\frac{S_2}{S_1}\right) & \text{if } T_1 \leq 0,\ T_2 \leq 0. \end{cases} \tag{4.6}$$

For $w_2 = T_2/S_1$, let

$$H(v_1; w_1, w_2; \lambda_1, \lambda_2) = \int_0^{v_1} u \prod_{i=1}^{2} h_i(w_i u; \lambda_i) I_{[u \geq k_i(\lambda_i)/w_i]} du,$$

$H(v_1; w_1, w_2) = H(v_1; w_1, w_2; 0, 0)$ and $h(v_1; w_1, w_2) = v_1 h_1(w_1 v_1) h_2(w_2 v_1)$. Then we can show the following lemma whose proof is omitted.

LEMMA 4.1. *If the assumptions* (A.1′) *and* (A.3′) *hold, then*

$$H(v_1; w_1, w_2; \lambda_1, \lambda_2)/H(v_1; w_1, w_2)$$

is increasing in v_1. *If* (A.2′) *and* (A.4′) *hold,* $v_1 h(v_1; w_1, w_2)/H(v_1; w_1, w_2)$ *is decreasing in* v_1 *and*

$$H(v_1; c_1 w_1, c_2 w_2)/H(v_1; w_1, w_2) \qquad \text{and} \qquad h(v_1; c_1 w_1, c_2 w_2)/h(v_1; w_1, w_2)$$

are increasing in v_1 *for* $0 < c_1, c_2 \leq 1$.

THEOREM 4.2. *Assume* (A.1′), (A.3′) *and the following conditions*:
(a) $\psi(w, w_1, w_2)$ *is nondecreasing in* w_1, w_2 *and* $\lim_{t\to\infty} \psi(w, tw_1, tw_2) = 1$,
(b) $\int_0^\infty L'(v_1\phi(w)\psi(w, w_1, w_2)) v_1^2 g_1(v_1) g_2(wv_1) H(v_1; w_1, w_2) dv_1 \geq 0$,
(c) $\gamma_i(w, w_i)$ *is nondecreasing in* w_i *and* $\lim_{w_i\to\infty} \gamma_i(w, w_i) = 1$ *for* $i = 1, 2$,
(d) $\int_0^\infty L'(v_1\phi(w)\gamma_i(w, w_i)) v_1^2 g_1(v_1) g_2(wv_1) H_i(w_i v_1) dv_1 \geq 0$ *for* $i = 1, 2$,
(e) $\phi(w)\psi(w, w_1, w_2)$ *and* $\phi(w)\gamma_i(w, w_i)$ *are nondecreasing in* w *for* $i = 1, 2$.
Then $\delta_{\phi,\psi,\gamma_1,\gamma_2}$ *given by* (4.6) *dominates* δ_ϕ *given by* (4.2).

Note that the conditions (c) and (d) are quite similar to (a) and (b). Let $\psi_0(w, w_1, w_2)$ and $\psi_1(w, w_1, w_2)$ be solutions of the equations

$$\int_0^\infty L'(v_1\phi\psi_0) v_1^2 g_1(v_1) g_2(wv_1) H(v_1; w_1, w_2) dv_1 = 0,$$
$$\int_0^\infty L'(v_1\phi\psi_1) v_1^3 g_1(v_1) g_2(wv_1) h(v_1; w_1, w_2) dv_1 = 0.$$

Based on Lemma 4.1, it can be verified that $\min(1, \psi_0)$ and $\min(1, \psi_1)$ satisfy (a), (b) of Theorem 4.2 under (A.2′), (A.4′).

PROOF OF THEOREM 4.1. The risk difference is expressed as

$$\begin{aligned}
&R(\omega, \hat{\sigma}_1) - R(\omega, \delta_\phi) \\
&\quad = E\left[\int_1^\infty \frac{d}{dt} L\left(\phi\left(\rho t \frac{v_2}{v_1}\right) v_1\right) dt\right] \\
&\quad = E\left[\int_1^\infty L'\left(\phi\left(\rho t \frac{v_2}{v_1}\right) v_1\right) \rho v_2 \phi'\left(\rho t \frac{v_2}{v_1}\right) dt\right] \\
&\quad \geq \rho E\left[\int_1^\infty L'\left(\phi\left(t \frac{v_2}{v_1}\right) v_1\right) v_2 \phi'\left(\rho t \frac{v_2}{v_1}\right) dt\right] \\
&\quad = \rho \int_0^\infty \phi'(\rho w) \int_0^\infty L'(\phi(w) v_1) v_1^2 g_1(v_1) \int_0^w g_2(v_1 x) dx dv_1 dw,
\end{aligned}$$

which is nonnegative, proving Theorem 4.1. □

PROOF OF THEOREM 4.2. The risk difference is written by

$$
\begin{aligned}
&R(\omega,\delta_\phi)-R(\omega,\delta_{\phi,\psi,\gamma_1,\gamma_2})\\
&\quad= E\left[\int_1^\infty \frac{d}{dt}L\left(v_1\phi\left(\rho\frac{v_2}{v_1}\right)\psi\left(\rho\frac{v_2}{v_1},t\frac{u_1}{v_1},\rho t\frac{u_2}{v_1}\right)\right)dtI_{[u_1>0,u_2>0]}\right]\\
&\quad\quad+\sum_{i=1}^{2}E\left[\int_1^\infty \frac{d}{dt}L\left(v_1\phi\left(\rho\frac{v_2}{v_1}\right)\gamma_i\left(\rho\frac{v_2}{v_1},\rho_i t\frac{u_i}{v_1}\right)\right)dtI_{[u_i>0]}\right]\\
&\quad=\Delta_1+\sum_{i=1}^{2}\Delta_{2i}\qquad\text{(say)},
\end{aligned}
$$

where $\rho_1=1$ and $\rho_2=\rho$. Denote $\psi^{(i)}(x_1,x_2,x_3)=(\partial/\partial x_i)\psi(x_1,x_2,x_3)$. Noting that $\psi^{(2)}\geq 0$, $\psi^{(3)}\geq 0$ and $\rho\geq 1$, from (a) and (e), we observe that

$$
\begin{aligned}
\Delta_1 &= E\left[\int_1^\infty L'\left(v_1\phi\left(\rho\frac{v_2}{v_1}\right)\psi\left(\rho\frac{v_2}{v_1},t\frac{u_1}{v_1},\rho t\frac{u_2}{v_1}\right)\right)v_1\phi\left(\rho\frac{v_2}{v_1}\right)\right.\\
&\qquad\left.\cdot\left\{\frac{u_1}{v_1}\psi^{(2)}+\rho\frac{u_2}{v_1}\psi^{(3)}\right\}dtI_{[u_1>0,u_2>0]}\right]\\
&\geq E\left[\int_1^\infty L'\left(v_1\phi\left(\frac{v_2}{v_1}\right)\psi\left(\frac{v_2}{v_1},t\frac{u_1}{v_1},t\frac{u_2}{v_1}\right)\right)v_1\phi\left(\rho\frac{v_2}{v_1}\right)\right.\\
&\qquad\left.\cdot\left\{\frac{u_1}{v_1}\psi^{(2)}+\rho\frac{u_2}{v_1}\psi^{(3)}\right\}dtI_{[u_1>0,u_2>0]}\right]\\
&=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L'(v_1\phi(w)\psi(w,w_1,w_2))\{w_1\psi^{(2)}+\rho w_2\psi^{(3)}\}\\
&\qquad\cdot\phi(\rho w)v_1^2 g_1(v_1)g_2(wv_1)H(v_1;w_1,w_2;\lambda_1,\lambda_2)dv_1dwdw_1dw_2,
\end{aligned}
$$

which can be shown to be nonnegative by the same arguments as in the proof of Theorem 2.1 based on Lemmas 2.1 and 4.1. Similarly we can demonstrate that $\Delta_{2i}\geq 0$ for $i=1,2$, and the proof is complete. □

4.2 *Estimation of σ_2 under $\sigma_1<\sigma_2$*

Next we consider to estimate the larger scale parameter σ_2 under $\sigma_1<\sigma_2$. The estimation of σ_2 has a different domination structure from the case of σ_1. The best multiplier c_0 is given by

$$\int_0^\infty L'(c_0v_2)v_2g_2(v_2)dv_2=0,$$

and the estimator

$$\delta_\phi=\phi\left(\frac{S_1}{S_2}\right)S_2 \tag{4.7}$$

is considered for improving on $\hat{\sigma}_2 = c_0 S_2$.

THEOREM 4.3. *For $z = S_1/S_2$, assume that*
(a) *$\phi(z)$ is nondecreasing and $\phi(0) = c_0$,*
(b) *$\int_0^\infty L'(\phi(z)v_2)v_2 \int_{v_2 z}^\infty g_1(u)du g_2(v_2)dv_2 \leq 0$.*
Then δ_ϕ given by (4.7) *dominates $\hat{\sigma}_2$.*

The proof of Theorem 4.3 is similar to Theorems 3.3 and 4.1 and is omitted. Let $\phi_0(z)$ and $\phi_1(z)$ be solutions of the equations

$$\int_0^\infty L'(\phi_0(z)v_2)v_2 \int_{v_2 z}^\infty g_1(u)du g_2(v_2)dv_2 = 0,$$
$$\int_0^\infty L'(\phi_1(z)v_2)v_2^2 g_1(v_2 z)g_2(v_2)dv_2 = 0$$

and put $\phi_T(z) = \max(c_0, \phi_1(z))$. By the same manner as stated below Theorem 4.1, it can be checked that $\phi_0(z)$ and $\phi_T(z)$ satisfy (a), (b) of Theorem 4.3.

Next taking the estimator

$$\delta_{\phi,\psi} = \begin{cases} S_2 \left\{ \phi\left(\dfrac{S_1}{S_2}\right) + \psi\left(\dfrac{T_2}{S_2}\right) \right\} & \text{if } T_2 > 0 \\ \delta_\phi & \text{otherwise,} \end{cases} \tag{4.8}$$

leads to the further domination.

THEOREM 4.4. *Assume* (A.3) *and the following conditions:*
(a) *$\psi(z_2)$ is nondecreasing and $\lim_{z_2 \to \infty} \psi(z_2) = 0$,*
(b) *$\int_0^\infty L'([c_0 + \psi(z_2)]v_2)v_2 g_2(v_2)H_2(z_2 v_2)dv_2 \geq 0$,*
(c)*$\phi(z) \geq c_0$.*
Then $\delta_{\phi,\psi}$ given by (4.8) *dominates δ_ϕ given by* (4.7).

The theorem can be proved quite similarly to Theorem 2.2. Also a similar discussion as stated below the proof of Theorem 2.2 gives two types of improved procedures.

5. A multivariate extension

In the previous sections, the one-dimensional case is treated for estimation of ratio of scale parameters. As a multivariate extension, we here deal with estimating ratio of dispersion matrices of two multivariate normal distributions and try to provide results corresponding to those in Section 2.

Let S_1, S_2, X and Y be independent random variables such that

$$S_1 \sim \mathrm{W}_p(m, \Sigma_1), \qquad S_2 \sim \mathrm{W}_p(n, \Sigma_2), \qquad X \sim \mathrm{N}_p(\xi_1, \Sigma_1), \qquad Y \sim \mathrm{N}_p(\xi_2, \Sigma_2),$$

where $\mathrm{W}_p(m, \Sigma_1)$ designates a p-variate Wishart distribution with mean $m\Sigma_1$. Here Σ_1, Σ_2, ξ_1 and ξ_2 are unknown. Suppose that we want to estimate ratio of the dispersion matrices $\Delta = \Sigma_2\Sigma_1^{-1}$ by estimator $\hat{\Delta}$ relative to the Stein loss function

$$L(\hat{\Delta}, \Delta) = \operatorname{tr} \hat{\Delta}\Delta^{-1} - \log|\hat{\Delta}\Delta^{-1}| - p. \tag{5.1}$$

The inadmissibility results of a usual estimator of the covariance matrix with utilizing a sample mean have been shown by Sinha and Ghosh (1987), Perron (1990), Kubokawa *et al.* (1992) and so on. Of these, for estimation of the covariance matrix Σ_2, Kubokawa *et al.* (1993*b*) found out the best of estimators $aS_2 + b(Y'S_2^{-1}Y)^{-1}YY'$ for constants a and b and presented better estimators. Thereby we want to begin with obtaining the best constants a and b for estimators $\hat{\Delta}(a,b) = (aS_2 + b(Y'S_2^{-1}Y)^{-1}YY')S_1^{-1}$. A usual calculation gives the best constants

$$a_0 = \frac{m-p-1}{n+1} \quad \text{and} \quad b_0 = \frac{p(m-p-1)}{(n-p+1)(n+1)},$$

so that the best of estimators $\hat{\Delta}(a,b)$ is

$$\hat{\Delta}_0 = \left(a_0 S_2 + \frac{b_0}{Y'S_2^{-1}Y}YY'\right)S_1^{-1}. \tag{5.2}$$

Now the estimator we consider for improving on $\hat{\Delta}_0$ is of the form

$$\hat{\Delta}_\phi = \left(a_0 S_2 + \frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}YY'\right)S_1^{-1}. \tag{5.3}$$

THEOREM 5.1. *For $z_2 = Y'S_2^{-1}Y$, assume that*
(a) *$\phi(z_2)$ is nondecreasing and $\lim_{z_2\to\infty}\phi(z_2) = b_0$,*
(b) *$\phi(z_2) \geq \phi_0(z_2)$ where*

$$\phi_0(z_2) = \frac{m-p-1}{n+1}\frac{\int_0^{z_2} t^{p/2-1}/(1+t)^{(n+1)/2}dt}{\int_0^{z_2} t^{p/2-1}/(1+t)^{(n+3)/2}dt} - a_0.$$

Then $\hat{\Delta}_\phi$ dominates $\hat{\Delta}_0$.

Since $\phi_0(z_2) \leq (m-p-1)(n+1)^{-1}(1+z_2) - a_0$, putting

$$\begin{aligned}\phi_T(z_2) &= \min\left\{b_0,\ \frac{m-p-1}{n+1}(1+z_2) - a_0\right\}\\ &= \min\left\{\frac{p}{n-p+1},\ z_2\right\}\frac{m-p-1}{n+1},\end{aligned}$$

we see that $\phi_T(z_2)$ satisfies (a), (b). Also (a), (b) hold for $\phi_0(z_2)$.

Next for improving on $\hat{\Delta}_\phi$, consider the estimator

$$\hat{\Delta}_{\phi,\psi} = \left\{\psi(X'S_1^{-1}X)S_2 + \frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}YY'\right\}S_1^{-1}. \tag{5.4}$$

THEOREM 5.2. *For* $w_1 = X'S_1^{-1}X$, *assume that*
(a) $\psi(w_1)$ *is nonincreasing and* $\lim_{w_1\to\infty}\psi(w_1) = a_0$,
(b) $(p-1)\{\psi(w_1)\}^{-1} + \{\psi(w_1)+b_0\}^{-1} \geq \eta(w_1)$ *where*

$$\eta(w_1) = \frac{n}{m-p}\left\{p-1+\frac{\int_0^{w_1} t^{p/2-1}/(1+t)^{(m-1)/2}dt}{\int_0^{w_1} t^{p/2-1}/(1+t)^{(m+1)/2}dt}\right\},$$

(c) $\phi(z_2) \leq b_0$.
Then $\hat{\Delta}_{\phi,\psi}$ *given by* (5.4) *dominates* $\hat{\Delta}_\phi$ *given by* (5.3).

From the conditions (a) and (b), $\psi(w_1)$ should be contained in the interval $[a_0,\ \overline{\psi}(\eta(w_1))]$ where

$$\overline{\psi}(\eta(w_1)) = \frac{1}{2}\left(\frac{p}{\eta(w_1)}-b_0\right)+\frac{1}{2}\left\{\left(\frac{p}{\eta(w_1)}-b_0\right)^2+\frac{4(p-1)b_0}{\eta(w_1)}\right\}^{1/2}. \tag{5.5}$$

It can be demonstrated that $\overline{\psi}(\eta(w_1))$ is a decreasing function of $\eta(w_1)$ or w_1 and that $\lim_{w_1\to\infty}\overline{\psi}(\eta(w_1)) = a_0$ with $\lim_{w_1\to\infty}\eta(w_1) = np/(m-p-1)$. Hence $\overline{\psi}(\eta(w_1))$ satisfies (a) and (b). Since $\eta(w_1) \leq n(m-p)^{-1}(p+w_1)$, putting $\psi_T(w_1) = \max\{a_0,\ \overline{\psi}(n(m-p)^{-1}(p+w_1))\}$, we can see that (a), (b) hold for $\psi_T(w_1)$.

PROOF OF THEOREM 5.1. To calculate the risk function of $\hat{\Delta}_\phi$, write $X = A_1U_1$, $S_1 = A_1W^*A_1'$, $\Sigma_1 = A_1A_1'$ where $U_1 \sim \mathrm{N}_p(A_1^{-1}\xi_1, I)$ and $W^* \sim \mathrm{W}_p(m, I)$. Set $W = H_1W^*H_1'$ for an orthogonal matrix H_1 such that $H_1U_1 = (\|U_1\|, 0, \ldots, 0)'$. Note that W and U_1 are independent. Define $u_1 = \|U_1\|^2$ and $v_1 = w_{11.2} = w_{11} - W_{12}W_{22}^{-1}W_{21}$ where W is partitioned as $w_{11}(1\times 1)$, $W_{12}(1\times(p-1))$ and $W_{22}((p-1)\times(p-1))$. Then

$$u_1 \sim \chi_p^2(\lambda_1) \qquad \text{and} \qquad v_1 \sim \chi_{m-p+1}^2,$$

and we denote their densities $h_1(u_1;\lambda_1)$ and $g_1(v_1)$ for $\lambda_1 = \xi_1'\Sigma_1^{-1}\xi_1/2$. For Y and S_2, let u_2 and v_2 be defined similarly as

$$u_2 \sim \chi_p^2(\lambda_2) \qquad \text{and} \qquad v_2 \sim \chi_{n-p+1}^2$$

with densities $h_2(u_2;\lambda_2)$ and $g_2(v_2)$ for $\lambda_2 = \xi_2'\Sigma_2^{-1}\xi_2/2$.
We observe that

$$\begin{aligned}
E&\left[\mathrm{tr}\left\{\left(a_0S_2+\frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}YY'\right)S_1^{-1}\Sigma_1\Sigma_2^{-1}\right\}\right]\\
&=\frac{1}{m-p-1}E\left[\mathrm{tr}(a_0S_2\Sigma_2^{-1})+\frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}Y'\Sigma_2^{-1}Y\right]\\
&=\frac{1}{m-p-1}E\left[a_0np+v_2\phi\left(\frac{u_2}{v_2}\right)\right],
\end{aligned}$$

and that

$$E\left[\log\left|\left(a_0S_2+\frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}YY'\right)S_1^{-1}\Sigma_1\Sigma_2^{-1}\right|\right]$$
$$=E[\log|S_1^{-1}\Sigma_1||a_0S_2\Sigma_2^{-1}|]+E\left[\log\left\{1+\phi\left(\frac{u_2}{v_2}\right)/a_0\right\}\right].$$

Hence the risk difference is written by

$$R(\omega,\hat{\Delta}_0)-R(\omega,\hat{\Delta}_\phi)$$
$$=E\left[\int_1^\infty\frac{d}{dt}\left\{\frac{v_2}{m-p-1}\phi\left(t\frac{u_2}{v_2}\right)-\log\left\{1+\frac{1}{a_0}\phi\left(t\frac{u_2}{v_2}\right)\right\}\right\}dt\right]$$
$$=\int_0^\infty\phi'(z_2)\int_0^\infty\left\{\frac{v_2}{m-p-1}-\frac{1}{a_0+\phi(z_2)}\right\}g_2(v_2)$$
$$\cdot v_2\int_0^{z_2}h_2(v_2x;\lambda_2)dxdv_2dz_2,$$

which can be shown to be nonnegative by the same arguments as stated in the previous sections. □

PROOF OF THEOREM 5.2. We first note that

$$E[\mathrm{tr}\,W^{-1}\mid w_{11.2}]=E\left[\frac{1}{w_{11.2}}+\mathrm{tr}\left(W_{22}-\frac{1}{w_{11}}W_{21}W_{12}\right)^{-1}\middle|w_{11.2}\right]$$
$$=E\left[\frac{1}{w_{11.2}}+\mathrm{tr}\,W_{22}^{-1}+\frac{W_{12}W_{22}^{-2}W_{21}}{w_{11.2}}\middle|w_{11.2}\right]$$
$$=\frac{1}{m-p}\left\{\frac{m-1}{w_{11.2}}+p-1\right\}.$$

Each term in the risk function of $\hat{\Delta}_{\phi,\psi}$ is evaluated as follows:

$$E[\psi(X'S_1^{-1}X)\,\mathrm{tr}(S_2S_1^{-1}\Sigma_1\Sigma_2^{-1})]$$
$$=nE\left[\psi\left(\frac{u_1}{w_{11.2}}\right)\mathrm{tr}\,W^{-1}\right]$$
$$=\frac{n}{m-p}E\left[\psi\left(\frac{u_1}{v_1}\right)\left(\frac{m-1}{v_1}+p-1\right)\right],$$
$$E\left[\frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}\,\mathrm{tr}\,YY'S_1^{-1}\Sigma_1\Sigma_2^{-1}\right]=\frac{1}{m-p-1}E\left[v_2\phi\left(\frac{u_2}{v_2}\right)\right],$$
$$E\left[\log\left|\left\{\psi(X'S_1^{-1}X)S_2+\frac{\phi(Y'S_2^{-1}Y)}{Y'S_2^{-1}Y}YY'\right\}S_1^{-1}\Sigma_1\Sigma_2^{-1}\right|\right]$$
$$=E[\log|S_1^{-1}\Sigma_1||S_2\Sigma_2^{-1}|]$$
$$+E\left[(p-1)\log\psi\left(\frac{u_1}{v_1}\right)+\log\left(\psi\left(\frac{u_1}{v_1}\right)+\phi\left(\frac{u_2}{v_2}\right)\right)\right].$$

Hence the risk difference is represented by

$$\begin{aligned}
&R(\omega, \hat{\Delta}_\phi) - R(\omega, \hat{\Delta}_{\phi,\psi}) \\
&\quad = E\left[\int_1^\infty \frac{d}{dt}\left\{\frac{n}{m-p}\left(\frac{m-1}{v_1}+p-1\right)\psi\left(t\frac{u_1}{v_1}\right)\right.\right. \\
&\qquad \left.\left. -(p-1)\log\psi\left(t\frac{u_1}{v_1}\right) - \log\left(\psi\left(t\frac{u_1}{v_1}\right)+\phi\left(\frac{u_2}{v_2}\right)\right)\right\}dt\right] \\
&\quad \geq \int_0^\infty \psi'(w_1)\int_0^\infty \left\{\frac{n}{m-p}\left(\frac{m-1}{v_1}+p-1\right) - \frac{p-1}{\psi(w_1)} - \frac{1}{\psi(w_1)+b_0}\right\} \\
&\qquad \cdot v_1 g_1(v_1)\int_0^{w_1} h_1(v_1 x;\lambda_1)dx dv_1 dw_1,
\end{aligned}$$

which can be proved to be nonnegative and the proof is complete. □

Acknowledgements

The author is extremely thankful to a referee for his very valuable and detailed comments and helpful suggestions.

References

Arnold, B. C. (1970). Inadmissibility of the usual scale estimate for a shifted exponential distribution, *J. Amer. Statist. Assoc.*, **65**, 1260–1264.

Brewster, J. F. (1974). Alternative estimators for the scale parameter of the exponential distribution with unknown location, *Ann. Statist.*, **2**, 553–557.

Brewster, J. F. and Zidek, J. V. (1974). Improving on equivariant estimators, *Ann. Statist.*, **2**, 21–38.

Brown, L. D. (1968). Inadmissibility of the usual estimators of scale parameters in problems with unknown location and scale parameters, *Ann. Math. Statist.*, **39**, 29–48.

Gelfand, A. E. and Dey, D. K. (1988). On the estimation of a variance ratio, *J. Statist. Plann. Inference*, **19**, 121–131.

Goutis, C. and Casella, G. (1991). Improved invariant confidence intervals for a normal variance, *Ann. Statist.*, **19**, 2015–2031.

Kubokawa, T. (1994). A unified approach to improving equivariant estimators, *Ann. Statist.* (to appear).

Kubokawa, T. and Saleh, A. K. Md. E. (1993). On improved positive estimators of variance components, *Statist. Decisions* (to appear).

Kubokawa, T., Robert, C. and Saleh, A. K. Md. E. (1992). Empirical Bayes estimation of the covariance matrix of a normal distribution with unknown mean under an entropy loss, *Sankhyā Ser. A*, **54**, 402–410.

Kubokawa, T., Robert, C. P. and Saleh, A. K. Md. E. (1993*a*). Estimation of noncentrality parameters, *Canad. J. Statist.*, **21**, 45–57.

Kubokawa, T., Honda, T., Morita, K. and Saleh, A. K. Md. E. (1993*b*). Estimating a covariance matrix of a normal distribution with unknown mean, *J. Japan Statist. Soc.*, **23**, 131–144.

Maatta, J. M. and Casella, G. (1990). Developments in decision-theoretic variance estimation, *Statist. Sci.*, **5**, 90–120.

Madi, M. and Tsui, K.-W. (1990). Estimation of the ratio of the scale parameters of two exponential distributions with unknown location parameters, *Ann. Inst. Statist. Math.*, **42**, 77–87.

Nagata, Y. (1989). Improvements of interval estimations for the variance and the ratio of two variance, *J. Japan Statist. Soc.*, **19**, 151–161.

Nagata, Y. (1991). Improved interval estimation for the scale parameter of exponential distribution with an unknown location-scale parameter, *Quality, Journal of the Japanese Society of Quality Control*, **21**, 5–10 (in Japanese).

Perron, F. (1990). Equivariant estimators of the covariance matrix, *Canad. J. Statist.*, **18**, 179–182.

Sinha, B. K. and Ghosh, M. (1987). Inadmissibility of the best equivariant estimators of the variance-covariance matrix, the precision matrix, and the generalized variance under entropy loss, *Statist. Decisions*, **5**, 201–227.

Stein, C. (1964). Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean, *Ann. Inst. Statist. Math.*, **16**, 155–160.

Takeuchi, K. (1991). Personal communications.