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SEQUENTIAL ESTIMATION OF A PARAMETER OF AN EXPONENTIAL DISTRIBUTION

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Abstract. We consider the problem of minimum risk point estimation for the parameter $\theta = a\mu + b\sigma$ of the exponential distribution with unknown location parameter μ and scale parameter σ when the loss function is squared error plus linear cost. In this paper, we propose a sequential estimator of θ and show that the associated risk is asymptotically one cost less than that given by Ghosh and Mukhopadhyay (1989, *South African Statist. J.*, **23**, 251–268).

Key words and phrases: Two-parameter exponential, asymptotically unbiased estimator, sequential estimator, uniform integrability, second order.

1. Introduction

Let $X_1, X_2, \dots$ be independent and identically distributed (iid) random variables with the common probability density function (pdf)

$$f_{\mu,\sigma}(x) = \sigma^{-1} \exp\{-(x-\mu)/\sigma\} I(x \geq \mu) \tag{1.1}$$

where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are assumed to be unknown parameters, and $I(A)$ denotes the indicator function of A. In this paper we consider the problem of sequential point estimation for the parameter $\theta = a\mu + b\sigma$ with $1 \leq a < \infty$ and $0 < b < \infty$ being known constants. As the loss function, we consider the squared error plus the linear cost of sampling. Mukhopadhyay (1987) treated the case where $a = b = 1$, that is, the mean of the distribution (1.1). Mukhopadhyay and Ekwo (1987) discussed this problem for the scale parameter σ. When we take $a = 1$ and $b = -\log(1-p)$ for a given $p \in (0,1)$ θ is the p-th percentile of (1.1). In the case of this paper, Ghosh and Mukhopadhyay (1989) have developed the second order expansions for the risk and the regret. In this paper, we modify the sequential estimator of θ proposed by Ghosh and Mukhopadhyay (1989) and show that the risk associated with this modified estimator is one cost less than that given by Ghosh and Mukhopadhyay (1989) when the cost is sufficiently small. In Section 2 the results are presented. Proofs of the results are given in Section 3.

2. The results

In this section, we propose a modified sequential estimator of $\theta = a\mu + b\sigma$ ($1 \le a < \infty$, $0 < b < \infty$) where μ and σ are assumed to be unknown and show that the risk associated with the modified sequential estimator is asymptotically one cost less than that given by Ghosh and Mukhopadhyay (1989).

Let $\theta_n = \theta_n(X_1, \ldots, X_n)$ be an estimator of θ based on a sample $X_1, \ldots, X_n$ with the pdf $f_{\mu,\sigma}(x)$ of (1.1). We suppose that the loss incurred in estimating θ by θ_n is given by $L_n(c) = A(\theta_n - \theta)^2 + cn$ where A and c are known positive constants and c plays the role of cost per unit sampling. We call $R_n(c) = E\{L_n(c)\}$ the risk associated with θ_n. Let $T_n = \min\{X_1, \ldots, X_n\}$ and $s_n = (n-1)^{-1}\sum_{i=1}^{n}(X_i - T_n)$ for $n \ge 2$. Ghosh and Mukhopadhyay (1989) considered the unbiased estimator θ_n of θ defined by

$$\theta_n = a(T_n - n^{-1}s_n) + bs_n. \tag{2.1}$$

The risk associated with θ_n is

$$R_n(c) = A\sigma^2\{b^2 n^{-1} + (a-b)^2 n^{-1}(n-1)^{-1}\} + cn$$

which is minimized by $n = n_0$ assuming for simplicity that n_0 is an integer. On the other hand, $\bar{R}_n(c) = A\sigma^2 b^2 n^{-1} + cn$ is minimized if we take $n = n^* = (Ab^2/c)^{1/2}\sigma$ where n^* is assumed to be an integer. Then $R_n(c)$ is approximately equal to $\bar{R}_n(c)$ for sufficiently large n. By the way σ is unknown, so there does not exist any fixed sample procedure which minimizes $\bar{R}_n(c)$. Thus we consider the following stopping rule. Let

$$N = N_c = \inf\{n \ge m : n \ge (Ab^2/c)^{1/2}s_n\} \tag{2.2}$$

where m (≥ 2) is the starting sample size.

We shall first give the result concerning the bias of the sequential estimator θ_N of θ where θ_n and N are given in (2.1) and (2.2), respectively.

PROPOSITION 2.1. *If $m \ge 3$ then we have*

$$E(\theta_N) = \theta - (c/A)^{1/2} + o(c^{1/2}) \quad as \quad c \to 0. \tag{2.3}$$

Now, taking account of Proposition 2.1 we propose the sequential estimator $\hat{\theta}_N$ of θ defined by $\hat{\theta}_N = \theta_N + (c/A)^{1/2}$. From (2.3) $\hat{\theta}_N$ is a second order asymptotically unbiased estimator of θ. Let $R_N^*(c) = AE\{(\hat{\theta}_N - \theta)^2\} + cE(N)$ which is the risk associated with $\hat{\theta}_N$. We note that $R_{n_0}(c) = \min\{R_n(c) : n \ge 2\}$. The following proposition presents the second order expansion of the risk $R_N^*(c)$.

PROPOSITION 2.2. *If $m \ge 4$ then we have*

$$R_N^*(c) = 2(A^{1/2}b\sigma)c^{1/2} + c(a^2 + 3b^2 - 2ab)/b^2 + o(c) \quad as \quad c \to 0 \tag{2.4}$$

and

(2.5) $R_N^*(c) - R_{n_0}(c) = 2c + o(c) \quad as \quad c \to 0.$

Let $R_N(c) = AE\{(\theta_N - \theta)^2\} + cE(N)$ which is the risk associated with θ_N given by Ghosh and Mukhopadhyay (1989). The following theorem shows that the risk $R_N^*(c)$ is asymptotically one cost less than $R_N(c)$.

THEOREM 2.1. *Let $A = 1$. If $m \geq 4$ then we have*

$$R_N(c) - R_N^*(c) = c + o(c) \quad as \quad c \to 0.$$

Remark 2.1. Ghosh and Mukhopadhyay (1989) considered the case $A = 1$.

3. Proofs

In this section, we shall give the proofs of the results in Section 2. Let $W_1, W_2, \ldots$ be iid random variables with the pdf $f_{0,1}(x)$ in (1.1). Set $S_n = \sum_{i=1}^n W_i$, $Q = \inf\{n \geq m-1 : n(n+1)/n^* \geq S_n\}$ and $\bar{W}_Q = S_Q/Q$ throughout this section.

LEMMA 3.1. *It holds that*

$$E(s_N - \sigma) = E(\sigma\bar{W}_Q - \sigma) \tag{3.1}$$

and

$$E\{(s_N - \sigma)/N\} = E\{(\sigma\bar{W}_Q - \sigma)/(Q+1)\}. \tag{3.2}$$

PROOF. Let $Y_{in} = (n - i + 1)(X_{n(i)} - X_{n(i-1)})$ for $2 \leq i \leq n$ where $X_{n(1)} < \cdots < X_{n(n)}$ are the order statistics of $X_1, \ldots, X_n$, and let $Z_n = \sigma W_n$ for $n \geq 1$. Define $N^* = \inf\{n \geq m(\geq 2) : n(n-1)/n^* \geq S_{n-1}\}$. Since $Y = \{\sum_{i=2}^n Y_{in}, n \geq 2\}$ has the same distribution as $Z = \{\sum_{i=1}^{n-1} Z_i, n \geq 2\}$ due to Lombard and Swanepoel (1978), and $s_n = (n-1)^{-1}\sum_{i=2}^n Y_{in}$, N in (2.2) has the same distribution as N^*. Thus by use of these results, the definitions of N and N^* and the fact that $N^* = Q + 1$ we can show that for each $x > 0$ and each n $(\geq m)$ with $P\{N = n\} > 0$

$$P\{s_n \leq x \mid N = n\} = P\{\sigma\bar{W}_{n-1} \leq x \mid Q = n-1\}. \tag{3.3}$$

Hence, taking (3.3) and the identity of the distributions of N and $Q + 1$ into consideration, we obtain (3.1) and (3.2). Thus the proof of the lemma is complete. □

PROOF OF PROPOSITION 2.1. It is clear that

$$E(\theta_N - \theta) = aE(T_n - N^{-1}s_N - \mu) + bE(s_N - \sigma). \tag{3.4}$$

Since T_n and $\{s_2, \ldots, s_n\}$ are independent, $E(T_n) = \mu + \sigma/n$ for every fixed $n \geq 2$ and $P\{N < \infty\} = 1$ we get that $E(T_N - \mu) = \sum_{n=m}^\infty E\{(T_n - \mu)I(N = n)\} = E(\sigma/N)$, which, together with (3.4), implies

$$E(\theta_N - \theta) = -aE\{(s_N - \sigma)/N\} + bE(s_N - \sigma). \tag{3.5}$$

By the definition of Q we can show that $E(Q) < \infty$, so it follows from Wald's lemma that $E(S_Q) = E(Q)$, which yields

$$E(\sigma \bar{W}_Q - \sigma) = \sigma E\{(S_Q - Q)(1/Q - 1/n^*)\} + (\sigma/n^*)E(S_Q - Q) = \sigma E\{(S_Q - Q)(1/Q - 1/n^*)\}. \tag{3.6}$$

Set $R_c = (Q+1)Q/n^* - S_Q$. Then by (3.1) and (3.6) we obtain

$$\begin{aligned} E(s_N - \sigma) &= \sigma E\{(S_Q - Q)(Q - Q^2/n^*)/Q^2\} \\ &= \sigma E[(S_Q - Q)\{-(S_Q - Q - Q/n^* + R_c)\}/Q^2] \\ &= (\sigma/n^*)E(Y_c) \end{aligned} \tag{3.7}$$

where $Y_c = (n^*/Q)[\{-(S_Q - Q)^2 + Q(S_Q - Q)/n^* - (S_Q - Q)R_c\}/Q]$. From Theorems 2.1 and 2.2 of Woodroofe (1977) we get $R_c \underset{d}{\rightarrow} H$ as $c \rightarrow 0$ where "$\underset{d}{\rightarrow}$" stands for convergence in distribution, the distribution of H is given in Theorem 2.1 of Woodroofe (1977) and $E(H) = 1 - D$ if we take $D = \sum_{n=1}^{\infty} n^{-1}E(S_n - 2n)^+$. Since $Q \underset{a.s.}{\longrightarrow} \infty$ as $c \rightarrow 0$ we have by the strong law of large numbers that $(S_Q - Q)/Q \underset{a.s.}{\longrightarrow} 0$ as $c \rightarrow 0$ where "$\underset{a.s.}{\longrightarrow}$" denotes almost sure convergence. On the other hand, it can be shown that $Q/n^* \underset{a.s.}{\longrightarrow} 1$ as $c \rightarrow 0$ and we apply Corollary 1.4 of Woodroofe (1982) to have $(S_Q - Q)/Q^{1/2} \underset{d}{\rightarrow} N(0,1)$ as $c \rightarrow 0$. Thus from these results we obtain

$$Y_c \underset{d}{\rightarrow} -\chi_1^2 \quad \text{as} \quad c \rightarrow 0 \tag{3.8}$$

where χ_1^2 denotes the chi-square distribution with one degree of freedom. Let $Q^* = (Q - n^*)/(n^*)^{1/2}$ and $Z_c = (n^*)^{-1/2}(S_Q - Q)(n^*/Q)Q^*$. We shall here show the uniform integrability of $\{Z_c, c \leq c_0\}$ for some $c_0 > 0$. Let $\delta > 1$ be a constant. By Hölder's inequality, we have

$$\begin{aligned} E|Z_c|^{\delta} &\leq \{E|(n^*)^{-1/2}(S_Q - Q)|^{11\delta}\}^{1/11}\{E|(n^*/Q)Q^*|^{11\delta/10}\}^{10/11} \\ &\leq \{E|(n^*)^{-1/2}(S_Q - Q)|^{11\delta}\}^{1/11} \\ &\quad \cdot \{E(n^*/Q)^{33\delta/20}\}^{20/33}\{E|Q^*|^{33\delta/10}\}^{10/33}. \end{aligned} \tag{3.9}$$

It follows from Lemma 2.1 of Ghosh and Mukhopadhyay (1989) that for any fixed $\eta > 0$ $\{(n^*/Q)^{\eta}, c \leq c_0\}$ is uniformly integrable for some $c_0 > 0$ if $m > \eta + 1$. By Theorem 2 of Chow *et al.* (1979) $\{|(n^*)^{-1/2}(S_Q - Q)|^{\eta}, c \leq c_0\}$ is uniformly integrable for any fixed $\eta > 0$ if $m \geq 2$. From Theorem 2.3 of Woodroofe (1977) $\{|Q^*|^{99/25}, c \leq c_0\}$ is uniformly integrable if $m \geq 3$. Hence from the above results and (3.9) with $\delta = 6/5$ we have the uniform integrability of $\{Z_c, c \leq c_0\}$. Since $Y_c = -Z_c$ $\{Y_c, c \leq c_0\}$ is uniformly integrable. Thus by (3.7), (3.8) and this result we get

$$\begin{aligned} bE(s_N - \sigma) &= -b\sigma/n^* + o(1/n^*) \\ &= -(c/A)^{1/2} + o(c^{1/2}) \quad \text{as} \quad c \rightarrow 0. \end{aligned} \tag{3.10}$$

It is clear that

$$E\{(\sigma \bar{W}_Q - \sigma)/(Q+1)\} = (\sigma/n^*)E[\{n^*/(Q+1)\}\{(S_Q - Q)/Q\}] \tag{3.11}$$

and

$$\{n^*/(Q+1)\}\{(S_Q - Q)/Q\} \underset{a.s.}{\longrightarrow} 0 \quad \text{as} \quad c \to 0. \tag{3.12}$$

Next we shall show the uniform integrability of $\{(n^*/(Q+1))((S_Q - Q)/Q), c \leq c_0\}$. By use of Hölder's inequality and the fact that $Q \geq 1$ we get that for $\delta > 1$

$$\begin{aligned} &E|\{n^*/(Q+1)\}\{(S_Q - Q)/Q\}|^\delta \\ &\quad \leq E|(n^*)^{-1/2}(S_Q - Q)(n^*/Q)^{3/2}|^\delta \\ &\quad \leq \{E|(n^*)^{-1/2}(S_Q - Q)|^{9\delta}\}^{1/9}\{E(n^*/Q)^{27\delta/16}\}^{8/9}. \end{aligned} \tag{3.13}$$

Hence from (3.13) with $\delta = 7/6$ $\{(n^*/(Q+1))((S_Q - Q)/Q), c \leq c_0\}$ is uniformly integrable. This result, (3.11) and (3.12) imply that $E\{(\sigma \bar{W}_Q - \sigma)/(Q+1)\} = o(1/n^*) = o(c^{1/2})$, which, together with (3.2), yields

$$E\{(s_N - \sigma)/N\} = o(c^{1/2}) \quad \text{as} \quad c \to 0. \tag{3.14}$$

Combining (3.5), (3.10) and (3.14) we obtain $E(\theta_N - \theta) = -(c/A)^{1/2} + o(c^{1/2})$ as $c \to 0$, which gives (2.3). Therefore, the proof of the proposition is complete. □

PROOF OF PROPOSITION 2.2. By the definitions of $R_N^*(c)$ and $R_N(c)$ we get

$$R_N^*(c) = R_N(c) + 2(Ac)^{1/2}E(\theta_N - \theta) + c. \tag{3.15}$$

From Theorem 2.1 of Ghosh and Mukhopadhyay (1989) one gets $R_N(c) = 2(A^{1/2}b\sigma)c^{1/2} + c(a^2 + 4b^2 - 2ab)/b^2 + o(c)$ as $c \to 0$ if $m \geq 4$, which, together with (3.15) and Proposition 2.1, concludes (2.4). Since by simple calculations $R_{n_0}(c) = 2(A^{1/2}b\sigma)c^{1/2} + c(a^2 + b^2 - 2ab)/b^2 + o(c)$ as $c \to 0$, (2.5) follows from (2.4). Thus, the proof of Proposition 2.2 is complete. □

PROOF OF THEOREM 2.1. From (3.15) with $A = 1$

$$R_N(c) - R_N^*(c) = -2c^{1/2}E(\theta_N - \theta) - c. \tag{3.16}$$

Hence, combining (3.16) and Proposition 2.1 with $A = 1$, the conclusion in Theorem 2.1 follows.

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