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STATISTICAL TESTS INVOLVING SEVERAL INDEPENDENT GAMMA DISTRIBUTIONS

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Abstract. Statistical tests are developed regarding linear combinations of the parameters of several independent gamma populations. The tests are based on a generalized minimum chi-square procedure. On utilizing these, one can test hypotheses regarding the means or the scale parameters when the shape parameters are unknown. In these tests there is no need to assume the equality of the shape parameters of the underlying populations. Tests for comparing coefficients of variation of several gamma populations have also been developed. For the two population case, a power comparison of these tests with some existing tests is also presented. Two examples are provided to explain the procedure.

Key words and phrases: Gamma distribution, test of hypotheses, scale parameters, coefficient of variation, minimum chi-square, power of a test, test statistic.

1. Introduction

The gamma distribution is a generalization of the exponential distribution which has been widely used as a model in reliability studies and life testing experiments. The gamma distribution fits a wide variety of lifetime data adequately and there are failure processes that lead to it. The gamma distribution also arises mathematically in certain situations in which the exponential distribution is being used, in consequence of the well known result that the sum of independent, identically distributed exponential random variables has a gamma distribution. This fact justifies its use in waiting time problems and in the theory of queues with Poisson arrivals and exponential service times. Other applications of the gamma distribution are found in industrial engineering and quality control, see Drenick (1960), Gupta and Groll (1961), in cloud seeding experiment, see Crow (1977) and in survival analysis, see Gross and Clark (1975) and Lawless (1982). For further details on gamma distribution see Johnson and Kotz (1970).

The parameters of the gamma distribution are often estimated by the method of maximum likelihood which involves solving nonlinear equations, see Bowman and Shenton (1983, 1988) and Greenwood and Durand (1960). Other methods of estimating parameters from gamma distribution include (i) modified moment and maximum likelihood estimators of Cohen and Whitten (1982), see also a recent monograph by Cohen and Whitten (1988) and (ii) minimum chi-square estimators which are obtainable by solving linear equations, see Dahiya and Gurland (1978).

For tests of hypotheses regarding the parameters of the gamma distribution some procedures have become available recently. Engelhardt and Bain (1977) developed a conditional test for testing hypotheses regarding the scale parameter of a gamma distribution with unknown shape parameter. Grice and Bain (1980) presented an approximate test for the mean of a gamma distribution with both the parameters unknown. Recently Keating *et al.* (1990) have studied the testing of hypothesis about the shape parameter of the gamma distribution.

For the two sample problems Shiue and Bain (1983) proposed an approximate test for the equality of the scale parameters with a common unknown shape parameter. The statistic for this test turns out to be the ratio of the sample means of the two independent samples selected from the underlying gamma populations. In some practical situations, occasions arise when one may like to compare survivals of m groups of patients, each group being treated by a different treatment. To our knowledge no tests are available in the literature for comparing more than two gamma populations.

For the exponential case however, some tests do exist, see for example Nagaresenker (1980) and Kambo and Awad (1985). In the two sample case, Shiue and Bain's (1983) test has an underlying restriction in that it assumes the equality of the shape parameters of the underlying gamma populations. This restriction was removed by Shiue *et al.* in a later paper (1988).

In the present paper, in Section 2, we develop an approximate test for testing hypotheses regarding linear combinations of the parameters of m independent gamma populations. The test is based on the generalized minimum chi-square estimators. The test is asymptotic in nature and, therefore performs well for large samples. However, in some cases, even for small samples its performance is sometimes reasonable. The asymptotic null distribution of the test-statistic is that of a chi-square random variable with appropriate degrees of freedom. The general methodology developed in Section 2 is used to derive tests for (i) the equality of means, (ii) the equality of scale parameters, and (iii) the equality of coefficients of variation in Section 3. The procedure works for both the bell shaped ($\gamma \geq 1$) and the reverse J shaped ($\gamma < 1$) distributions. Two examples are provided in Section 5 to explain the procedure. The non-null distribution of the test-statistic is also given. A power study of the proposed test for both bell shaped ($\gamma \geq 1$) and reverse J shaped ($\gamma < 1$) distributions is presented in Section 4.

2. Development of the test

Consider m (≥ 2) gamma populations with the probability density function (p.d.f.)

$$f_i(x) = \frac{1}{\Gamma(\gamma_i)\theta_i^{\gamma_i}} x^{\gamma_i - 1} e^{-x/\theta_i}, \qquad x > 0, \quad \gamma_i, \theta_i > 0 \tag{2.1}$$

for $i = 1, 2, \ldots, m$. For the gamma distribution with parameters γ and θ,

$$\begin{aligned}
&E(X) = \gamma\theta, \qquad \mathrm{Var}(X) = \gamma\theta^2, \\
&\alpha_3(\text{coefficient of skewness}) = 2/\sqrt{\gamma}, \\
&\alpha_4(\text{Kurtosis}) = 3 + 6/\gamma, \\
&M_0(\text{Mode}) = \theta(\gamma - 1), \\
&CV(\text{coefficient of variation}) = 1/\sqrt{\gamma}.
\end{aligned}$$

We develop a test-statistic for testing general linear hypotheses regarding the parameters of these m gamma populations. This test will be utilized for testing hypotheses regarding the means and the coefficients of variation of these populations. The general linear hypotheses, we wish to test can be written as

$$H_0 : C\boldsymbol{\theta} = \boldsymbol{\Phi}_0 \quad \text{against} \quad H_a : C\boldsymbol{\theta} \neq \boldsymbol{\Phi}_0 \tag{2.2}$$

where $\boldsymbol{\theta}' = [\boldsymbol{\theta}_1^{*\prime}, \boldsymbol{\theta}_2^{*\prime}, \ldots, \boldsymbol{\theta}_m^{*\prime}]$ with

$$\boldsymbol{\theta}_i^{*\prime} = [\theta_{i1}^*, \theta_{i2}^*] = [\gamma_i\theta_i, \theta_i], \qquad i = 1, 2, \ldots, m, \tag{2.3}$$

where C is an $r \times 2m$ known matrix and $\boldsymbol{\Phi}_0$ is an $r \times 1$ vector of known constants, both suitably chosen for the desired hypotheses to be tested. We now develop a statistic for testing the above hypotheses.

2.1 *Formulation of the generalized minimum chi-square procedure*

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ denote a random sample from the i-th population, $i = 1, 2, \ldots, m$, and assume that these m random samples are independent. Denote the j-th cumulant of the i-th population by κ_{ij} and its sample counterpart by k_{ij}. Then the cumulants for the i-th population are

$$\kappa_{i1} = \gamma_i\theta_i, \quad \kappa_{ij} = (j-1)!\gamma_i\theta_i^j, \quad j > 1, \qquad \text{and} \qquad \kappa_{i,j+1}/\kappa_{ij} = j\theta_i.$$

Let

$$\begin{aligned}
&\eta_{i0} = \kappa_{i1}, \qquad \eta_{ij} = \kappa_{i,j+1}/\kappa_{i,j}, \qquad j \geq 1, \\
&\boldsymbol{\eta}_i' = (\eta_{i0}, \eta_{i1}, \eta_{i2}, \eta_{i3}), \qquad i = 1, 2, \ldots, m
\end{aligned}$$

and

$$\boldsymbol{\eta}' = [\boldsymbol{\eta}_1', \boldsymbol{\eta}_2', \ldots, \boldsymbol{\eta}_m'].$$

Then, it can be seen that $\boldsymbol{\eta}_i = w^*\boldsymbol{\theta}_i^*$ with

$$w^{*\prime} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \qquad i = 1, 2, \ldots, m.$$

If we let $w = \mathrm{diag}(w^*, w^*, \ldots, w^*)$, we get a linear relationship $\boldsymbol{\eta} = w\boldsymbol{\theta}$. Note that w is a $4m \times 2m$ matrix of known constants. To develop the statistic based on the generalized minimum chi-square procedure (Hinz and Gurland (1968), Tripathi and Gurland (1978)), let $h_{i0} = k_{i1}$, $h_{ij} = k_{i,j+1}/k_{ij}$, where $\boldsymbol{h}_i$ and $\boldsymbol{h}$ denote the sample counterparts of η_{i0}, η_{ij}, $\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}$ respectively. If Σ be the asymptotic covariance matrix of $\boldsymbol{h}$ then $\Sigma = J_2 J_1 V J_1' J_2'$ with V, J_1 and J_2 given below:

$$V = \operatorname{diag}(V_1, V_2, \ldots, V_m) \qquad \text{where}$$

$$V_i = \frac{1}{n_i}\begin{bmatrix} \mu'_{i2} - \mu'^2_{i1} & & & \\ \mu'_{i3} - \mu'_{i2}\mu'_{i1} & \mu'_{i4} - \mu'^2_{i2} & & \\ \mu'_{i4} - \mu'_{i3}\mu'_{i1} & \mu'_{i5} - \mu'_{i3}\mu'_{i2} & \mu'_{i6} - \mu'^2_{i3} & \\ \mu'_{i5} - \mu'_{i4}\mu'_{i1} & \mu'_{i6} - \mu'_{i4}\mu'_{i2} & \mu'_{i7} - \mu'_{i4}\mu'_{i3} & \mu'_{i8} - \mu'^2_{i4} \end{bmatrix}$$

is asymptotic covariance matrix of $(m'_{i1}, m'_{i2}, m'_{i3}, m'_{i4})$, a vector of the sample moments from the i-th sample, and μ'_{ij} is the j-th raw moment of the i-th population. J_1 and J_2 are the Jacobians

$$J_1 = \operatorname{diag}(J_{11} J_{12} \cdots J_{1m}), \qquad J_2 = \operatorname{diag}(J_{21} J_{22} \cdots J_{2m})$$

where J_{1i} and J_{2i} correspond to the following transformations:

$$J_{1i} : (\mu'_{i1}, \mu'_{i2}, \mu'_{i3}, \mu'_{i4}) \to (\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4}),$$
$$J_{2i} : (\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4}) \to (\eta_{i0}, \eta_{i1}, \eta_{i2}, \eta_{i3}).$$

The elements of J_{1i} and J_{2i} are

$$J_{1i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2\mu'_{i1} & 1 & 0 & 0 \\ -3(\mu'_{i2} - 2\mu'^2_{i1}) & -3\mu'_{i1} & 1 & 0 \\ -4(\mu'_{i3} - 6\mu'_{i1}\mu'_{i2} + 6\mu'^3_{i1}) & -6(\mu'_{i2} - 2\mu'^2_{i1}) & -4\mu'_{i1} & 1 \end{bmatrix},$$

$$J_{2i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\kappa_{i2}/\kappa^2_{i1} & 1/\kappa_{i1} & 0 & 0 \\ 0 & -\kappa_{i3}/\kappa^2_{i2} & 1/\kappa_{i2} & 0 \\ 0 & 0 & -\kappa_{i4}/\kappa^2_{i3} & 1/\kappa_{i3} \end{bmatrix}.$$

Let $\hat{\Sigma}$ denote a consistent estimate of Σ. Now consider the quadratic form

$$Q = (\boldsymbol{h} - w\boldsymbol{\theta})'\hat{\Sigma}^{-1}(\boldsymbol{h} - w\boldsymbol{\theta}). \tag{2.4}$$

Minimize Q with respect to $\boldsymbol{\theta}$ and let $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ denote the values of $\boldsymbol{\theta}$ which minimize Q under H_0 and under no restrictions respectively. Then $\hat{\boldsymbol{\theta}} = (w'\hat{\Sigma}^{-1}w)^{-1}w'\hat{\Sigma}^{-1}\boldsymbol{h}$ and

$$\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - (w'\hat{\Sigma}^{-1}w)^{-1}C'(C(w'\hat{\Sigma}^{-1}w)^{-1}C')^{-1}(C\hat{\boldsymbol{\theta}} - \boldsymbol{\Phi}_0).$$

Let Q_0 and Q_1 be the values of Q evaluated at $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$. Then, the statistic for testing H_0 in (2.2) is given by $Q = Q_0 - Q_1$. It has been shown that (Tripathi and Gurland (1978)) the asymptotic distribution of Q under H_0 is that of a chi-square random variable with r degrees of freedom (d.f.) where $r = \operatorname{rank}(C)$. Thus, a test of approximate size α for (2.1) is to reject H_0 if $Q \geq \chi^2_{r,\alpha}$ where $\chi^2_{r,\alpha}$ is the upper α-th percentage point of χ^2 distribution with r d.f.

3. Derivation of the tests

3.1 *Test for equality of means*

In order to test the equality of means of the m gamma populations, we need to test (see equation (2.3))

$$H_0 : \theta^*_{11} = \theta^*_{21} = \cdots = \theta^*_{m1}. \tag{3.1}$$

We can write the H_0 in (3.1) as $m-1$ independent contrasts given by

$$H_0 : \begin{cases} \theta^*_{11} - \theta^*_{m1} = 0 \\ \theta^*_{21} - \theta^*_{m1} = 0 \\ \qquad\vdots \\ \theta^*_{m-1,1} - \theta^*_{m1} = 0. \end{cases} \tag{3.2}$$

By choosing

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 \end{bmatrix}$$

and

$$\underset{1\times m-1}{\mathbf{\Phi}'_0} = [0\ 0\ \cdots\ 0],$$

we can write (3.2) and hence (3.1) in the form of $H_0 : C\boldsymbol{\theta} = \mathbf{\Phi}_0$. In this case $\operatorname{rank}(C) = m-1$, hence the asymptotic distribution of Q under (3.1) is χ^2_{m-1}.

3.2 *Test for equality of the scale parameters*

On utilizing the formulation presented above it is possible to test the equality of the scale parameters of the m gamma populations without having to assume the equality of the shape parameters as in Shiue and Bain (1983). Again from equation (2.3), we wish to test

$$H_0 : \theta^*_{12} = \theta^*_{22} = \theta^*_{32} = \cdots = \theta^*_{m2} \tag{3.3}$$

which can be written as $m-1$ linear contrast

$$H_0 : \begin{cases} \theta^*_{12} - \theta^*_{m2} = 0 \\ \theta^*_{22} - \theta^*_{m2} = 0 \\ \qquad\vdots \\ \theta^*_{m-1,2} - \theta^*_{m2} = 0. \end{cases} \tag{3.4}$$

By choosing

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & -1 \end{bmatrix} \quad \text{and}$$

$$\underset{1\times(m-1)}{\mathbf{\Phi}'_0} = [0\ 0\ \cdots\ 0],$$

we can write (3.4) and hence (3.3) in the form of (2.2) as $H_0 : C\boldsymbol{\theta} = \boldsymbol{\Phi}_0$. The test can be carried out as stated earlier.

3.3 *Test for equality of coefficients of variation*

Tests regarding coefficients of variation (CV) of m gamma populations can be developed utilizing the formulation presented above by introducing the following reparameterization.

The coefficient of variation for a random variable Y with standard deviation σ and mean μ is defined as $CV = \sigma/\mu$. For the i-th gamma population

$$CV_i = \frac{\sqrt{\gamma_i \theta_i^2}}{\gamma_i \theta_i} = \gamma_i^{-1/2}$$

which gives $-2\ln(CV_i) = \ln(\gamma_i)$. Let us reparameterize

$$\boldsymbol{\theta}_i^{*\prime} = (\ln(\gamma_i), \ln(\theta_i)) \quad \text{and} \quad \boldsymbol{\theta}' = (\boldsymbol{\theta}_1^{*\prime}, \boldsymbol{\theta}_2^{*\prime}, \ldots, \boldsymbol{\theta}_m^{*\prime}).$$

Now tests regarding the coefficients of variation can be put in terms of the general linear hypothesis

$$H_0 : C\boldsymbol{\theta} = \boldsymbol{\Phi}_0 \tag{3.5}$$

with C and $\boldsymbol{\Phi}_0$ suitably chosen. For example the equality of CV's of the m populations can be tested by taking the same C and $\boldsymbol{\Phi}_0$ as for the hypothesis (3.2).

To test the hypothesis (3.5), we take

$$\boldsymbol{\eta}_i' = (\ln(\eta_{i0}), \ln(\eta_{i1}), \ln(\eta_{i2}), \ln(\eta_{i3}))$$

so that $\boldsymbol{\eta}_i = w^* \boldsymbol{\theta}_i^*$ is linear in the elements of $\boldsymbol{\theta}_i^*$ where

$$w^{*\prime} = \begin{bmatrix} 1 & 0 & \ln 2 & \ln 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then, on defining

$$\begin{aligned} \boldsymbol{\eta}' &= [\boldsymbol{\eta}_1', \boldsymbol{\eta}_2', \ldots, \boldsymbol{\eta}_m'], \\ \boldsymbol{\theta}' &= [\boldsymbol{\theta}_1^{*\prime}, \boldsymbol{\theta}_2^{*\prime}, \ldots, \boldsymbol{\theta}_m^{*\prime}] \quad \text{and} \\ w &= \mathrm{diag}(w^*, w^*, \ldots, w^*) \end{aligned}$$

we get the overall linear relationship $\boldsymbol{\eta} = w\boldsymbol{\theta}$. As developed earlier, if $\boldsymbol{h}$ is the sample counterpart of $\boldsymbol{\eta}$ and $\hat{\Sigma}$ is a consistent estimate of the asymptotic covariance matrix Σ of $\boldsymbol{h}$, then a test statistic $Q = Q_0 - Q_1$ can be developed for testing (3.5). The asymptotic null distribution of Q is that of a chi-square random variable with $r = \mathrm{rank}(C)$ d.f.

4. Power comparison

In the present section, we present a power comparison between the test developed here for testing equality of scale parameters and the two-sided version of a test developed by Shiue and Bain (1983). It should be noted that the test developed by Shiue and Bain is for testing equality of the scale parameters of only two gamma distributions with common but unknown shape parameters. In order to make the comparison meaningful, we will compute the power of the test developed in this article under the same assumptions.

4.1 *Asymptotic power of a two-sided version of Shiue and Bain's test*

Suppose we wish to test

$$H_0 : \theta_1 = \theta_2 \qquad \text{vs.} \qquad H_a : \theta_1 \neq \theta_2.$$

Then an approximate α-level rejection region is given by

$$\text{reject } H_0 \text{ if } \frac{\bar{X}}{\bar{Y}} < F_L(2n_1\hat{\gamma}, 2n_2\hat{\gamma}) \quad \text{or if } \frac{\bar{X}}{\bar{Y}} > F_U(2n_1\hat{\gamma}, 2n_2\hat{\gamma})$$

where $\hat{\gamma}$ is a pooled estimate of the common shape parameter and $F_L(2n_1\gamma, 2n_2\hat{\gamma})$ and $F_U(2n_1\hat{\gamma}, 2n_2\hat{\gamma})$ are the lower and upper $\alpha/2$ cut off points of the F distribution with $2n_1\hat{\gamma}$ and $2n_2\hat{\gamma}$ degrees of freedom (d.f.). In what follows, we will assume that n_1 and n_2 are both large enough so that $\hat{\gamma}$ has converged to the true value γ. The asymptotic power at $(\theta_1, \theta_2, \gamma)$ is

$$\begin{aligned} P(\theta_1, \theta_2, \gamma) &= P\left(\frac{\bar{X}/2n_1\theta_1}{\bar{Y}/2n_2\theta_2} < \frac{n_2\theta_2}{n_1\theta_1} F_L(2n_1\gamma, 2n_2\gamma)\right) \\ &\quad + P\left(\frac{\bar{X}/2n_1\theta_1}{\bar{Y}/2n_2\theta_2} > \frac{n_2\theta_2}{n_1\theta_1} F_U(2n_1\gamma, 2n_2\gamma)\right) \\ &= 1 + F\left(\frac{n_2\theta_2}{n_1\theta_1} F_L(2n_1\gamma, 2n_2\gamma)\right) - F\left(\frac{n_2\theta_2}{n_1\theta_1} F_U(2n_1\gamma, 2n_2\gamma)\right) \end{aligned}$$

where $F(x)$ denotes the cumulative distribution function of the F distribution with $2n_1\gamma$ and $2n_2\gamma$ d.f.

4.2 *Asymptotic power of the minimum chi-square test*

Following the arguments as in Tripathi and Gurland (1978), it can be shown that when the consistent estimate of Σ in Subsection 2.1 is obtained by replacing the population moments involved in Σ with the corresponding sample moments, then the asymptotic non-null distribution of Q is that of a non-central chi-square random variable with $r = \text{rank}(C)$ d.f. and non-centrality parameter

$$\psi = (C\boldsymbol{\theta} - \boldsymbol{\Phi}_0)'(C\Sigma_{\hat{\boldsymbol{\theta}}}C')^{-1}(C\boldsymbol{\theta} - \boldsymbol{\Phi}_0),$$

where $\Sigma_{\hat{\boldsymbol{\theta}}}$ is the asymptotic covariance matrix of the minimum chi-square estimator $\hat{\boldsymbol{\theta}}$. Hence the asymptotic power is

$$P^*(\theta_1, \theta_2, \gamma) = P(Q > \chi'^2_{r,\psi}(\alpha))$$

Table 1. Asymptotic power of the minimum chi-square test for testing equality of the scale parameters of two gamma populations, $\alpha = .05$.

			$\gamma = 2.0$				$\gamma = 5.0$				$\gamma = 0.2$				$\gamma = 0.5$			
n_1	n_2	$\theta_2 \backslash \theta_1$	.5	1.5	3.5	5.5	.5	1.5	3.5	5.5	.5	1.5	3.5	5.5	.5	1.5	3.5	5.5
30	30	.5	.05	.59	.83	.88	.05	.66	.89	.92	.05	.18	.29	.32	.05	.33	.53	.59
		1.5	.59	.05	.44	.68	.66	.05	.50	.75	.18	.05	.14	.21	.33	.05	.24	.39
		3.5	.83	.44	.05	.18	.89	.50	.05	.21	.29	.14	.05	.08	.53	.24	.05	.11
		5.5	.88	.68	.18	.05	.92	.75	.21	.05	.32	.21	.08	.05	.59	.39	.11	.05
30	50	.5	.05	.78	.96	.98	.05	.84	.98	.99	.05	.26	.44	.49	.05	.48	.75	.80
		1.5	.61	.05	.60	.86	.67	.05	.68	.91	.19	.05	.19	.31	.34	.05	.34	.57
		3.5	.84	.46	.05	.24	.89	.53	.05	.28	.29	.15	.05	.09	.53	.26	.05	.14
		5.5	.88	.69	.20	.05	.93	.76	.23	.05	.32	.22	.08	.05	.59	.40	.12	.05
30	75	.5	.05	.90	.99	1.00	.05	.94	1.00	1.00	.05	.34	.59	.66	.05	.61	.89	.93
		1.5	.61	.05	.73	.95	.69	.05	.80	.98	.19	.05	.24	.42	.35	.05	.44	.72
		3.5	.84	.48	.05	.29	.89	.54	.05	.33	.29	.15	.05	.10	.54	.26	.05	.16
		5.5	.88	.70	.21	.05	.93	.77	.24	.05	.33	.22	.09	.05	.59	.41	.13	.05
30	100	.5	.05	.95	1.00	1.00	.05	.97	1.00	1.00	.05	.41	.71	.78	.05	.70	.95	.98
		1.5	.62	.05	.81	.98	.69	.05	.87	.99	.19	.05	.28	.50	.35	.05	.51	.81
		3.5	.84	.48	.05	.32	.89	.55	.05	.37	.29	.15	.05	.11	.54	.27	.05	.18
		5.5	.88	.70	.22	.05	.93	.77	.25	.05	.33	.22	.09	.05	.59	.41	.13	.05
50	50	.5	.05	.80	.97	.98	.05	.87	.98	.99	.05	.27	.44	.49	.05	.50	.75	.80
		1.5	.80	.05	.65	.88	.87	.05	.72	.93	.27	.05	.20	.32	.50	.05	.37	.59
		3.5	.97	.65	.05	.28	.98	.72	.05	.32	.44	.20	.05	.10	.75	.37	.05	.16
		5.5	.98	.88	.28	.05	.99	.93	.32	.05	.49	.32	.10	.05	.80	.59	.16	.05

50	75	.5	.05	.92	1.00	1.00	.05	.96	1.00	1.00	.05	.37	.60	.66	.05	.65	.89	.93
		1.5	.82	.05	.79	.96	.88	.05	.85	.98	.28	.05	.26	.44	.51	.05	.48	.75
		3.5	.97	.67	.05	.35	.98	.74	.05	.40	.45	.21	.05	.12	.75	.39	.05	.19
		5.5	.98	.89	.30	.05	.99	.93	.35	.05	.49	.33	.11	.05	.80	.60	.17	.05
50	100	.5	.05	.97	1.00	1.00	.05	.99	1.00	1.00	.05	.45	.72	.78	.05	.75	.96	.98
		1.5	.82	.05	.87	.99	.88	.05	.92	1.00	.28	.05	.32	.54	.52	.05	.57	.85
		3.5	.97	.68	.05	.40	.99	.75	.05	.46	.45	.22	.05	.13	.75	.40	.05	.22
		5.5	.98	.89	.31	.05	.99	.93	.36	.05	.49	.33	.11	.05	.80	.60	.18	.05
75	75	.5	.05	.93	1.00	1.00	.05	.96	1.00	1.00	.05	.38	.60	.66	.05	.67	.90	.93
		1.5	.93	.05	.82	.97	.96	.05	.88	.99	.38	.05	.28	.45	.67	.05	.51	.76
		3.5	1.00	.82	.05	.39	1.00	.88	.05	.44	.60	.28	.05	.13	.90	.51	.05	.21
		5.5	1.00	.97	.39	.05	1.00	.99	.44	.05	.66	.45	.13	.05	.93	.76	.21	.05
75	100	.5	.05	.97	1.00	1.00	.05	.99	1.00	1.00	.05	.47	.73	.78	.05	.78	.96	.98
		1.5	.94	.05	.90	.99	.97	.05	.94	1.00	.39	.05	.34	.56	.68	.05	.61	.86
		3.5	1.00	.83	.05	.45	1.00	.89	.05	.52	.61	.29	.05	.14	.90	.53	.05	.25
		5.5	1.00	.97	.41	.05	1.00	.99	.47	.05	.66	.46	.13	.05	.93	.77	.23	.05
100	100	.5	.05	.98	1.00	1.00	.05	.99	1.00	1.00	.05	.48	.73	.78	.05	.79	.96	.98
		1.5	.98	.05	.91	.99	.99	.05	.91	1.00	.48	.05	.36	.57	.79	.05	.63	.87
		3.5	1.00	.91	.05	.49	1.00	.95	.05	.56	.73	.36	.05	.15	.96	.63	.05	.29
		5.5	1.00	.99	.49	.05	1.00	1.00	.56	.05	.78	.57	.15	.05	.98	.87	.27	.01

where $\chi'^2_{r,\psi}(\alpha)$ is the upper α cut off point of a non-central chi-square random variable with r d.f. and non-centrality parameter ψ.

4.3 *Comparison of the powers*

The asymptotic power of the minimum chi-square test is given in Table 1 for a grid of the parameter values $(\theta_1, \theta_2, \gamma)$, for various sample size combinations (n_1, n_2) and for $\alpha = .05$. This table contains values for both bell shaped $(\gamma \geq 1)$ and reverse J shaped $(\gamma < 1)$ distributions. From the table it is clear that as the sample sizes increase the power increases, as expected. The powers also increase as γ increases. The minimum chi-square test has somewhat low power in the vicinity of the θ-values under H_0 but as the θ-values move away from H_0, the power increases. For moderate sample sizes, the asymptotic power of the minimum chi-square test is fairly high especially when the θ-values are far from H_0. Similar results were obtained for the power of the minimum chi-square test for $\alpha = .01$.

Power calculations for the same grid of parameters and the sample size combinations were also done on the two-sided version of Shiue and Bain's test. The powers of this test were always higher than the powers of the minimum chi-square test. For this reason, a similar table for the power of their test is not included and it is recommended that for two sample case their procedure should be preferred.

5. Examples

We present two examples of the procedure presented earlier for testing the equality of scale parameters of three gamma populations without assuming that their shape parameters are equal. The data for both the examples are generated by the IMSL subroutine. For all the samples, the sample size is 50.

Example 1. For the first example, the three samples each of size 50 were generated from the gamma populations in Table 2. The three samples from the populations in Table 2 are given in Tables 3–5.

Table 2.

Population	1	2	3
γ	1.5	3.5	1.0
θ	3.0	3.0	3.0

Suppose that the three gamma populations from which the samples in Tables 3–5 have been drawn are denoted by gamma(γ_i, θ_i), $i = 1, 2, 3$. Then, we wish to test the hypothesis

$$H_0 : \theta_1 = \theta_2 = \theta_3$$

vs.

Table 3. Sample 1: shape parameter $\gamma = 1.500$, scale parameter $\theta = 3.000$.

5.104	3.255	4.047	0.597	5.260	3.557	1.733	1.451	4.562	1.602
1.434	27.560	1.747	12.951	0.105	10.563	2.563	4.928	11.948	3.789
0.748	8.654	6.571	12.275	10.011	2.647	2.266	10.864	2.642	6.338
8.474	4.306	1.955	2.886	3.694	12.853	2.356	9.563	1.132	6.817
9.771	5.729	5.110	4.637	3.248	4.520	3.509	1.985	2.179	0.453

Table 4. Sample 2: shape parameter $\gamma = 3.500$, scale parameter $\theta = 3.000$.

9.220	8.422	9.832	17.748	14.109	10.600	9.419	3.676	7.344	10.158
3.161	7.291	4.781	3.732	11.441	22.612	3.090	6.335	6.290	21.283
5.318	4.616	11.429	21.716	5.779	0.987	7.265	11.281	13.185	13.278
4.554	6.304	7.913	12.354	22.483	6.137	15.359	7.832	6.337	11.584
13.735	18.483	6.050	10.851	10.205	8.496	9.496	13.377	3.607	4.671

Table 5. Sample 3: shape parameter $\gamma = 1.00$, scale parameter $\theta = 3.00$.

1.236	1.249	1.038	3.044	1.026	4.914	4.181	2.433	2.763	3.754
4.438	10.973	1.876	0.100	1.748	4.750	3.992	13.439	1.920	0.135
0.002	2.191	1.504	2.554	10.797	1.173	0.556	0.108	1.274	2.191
7.966	0.213	1.104	1.318	2.074	0.845	1.256	0.883	0.143	7.380
0.771	0.100	1.375	1.848	2.086	2.809	3.353	0.131	0.963	0.249

Table 6.

i	1	2	3
$\hat{\gamma}_i$	1.458	3.294	0.818
$\hat{\theta}_i$	3.663	2.947	3.136

$$H_1 : \theta_i \neq \theta_j, \qquad \text{for some} \quad i \neq j.$$

A consistent estimate of the covariance $\Sigma_{\boldsymbol{h}}$ of $\boldsymbol{h}$ was obtained by putting consistent estimtes of the parameters γ_i and θ_i into the proper component of $\Sigma_{\boldsymbol{h}}$ for the i-th population. The maximum likelihood estimates of γ_i, θ_i, obtained from the i-th sample were used to obtain the consistent estimate of $\Sigma_{\boldsymbol{h}}$. The maximum likelihood estimates of γ_i and θ_i used are given in Table 6.

The value of the test-statistic is $Q = 0.1747$ based on two d.f. Since the tabulated value of $\chi^2_{2;.05} = 5.99$, we fail to reject H_0. This conclusion is supporting the fact that the three samples were generated from three gamma populations with common scale parameters.

Example 2. For the second example, the three samples, each of size 50, are generated from the gamma distributions in Table 7 with different scale parameters. The three samples generated from the populations in Table 7 are given in Tables 8–10.

Table 7.

Population	1	2	3
γ	2.5	1.5	3.5
θ	1.5	5.5	7.5

Table 8. Sample 4: shape parameter $\gamma = 2.50$, scale parameter $\theta = 1.50$.

2.916	2.607	3.517	1.144	3.815	3.357	0.958	2.190	3.644	0.652
2.169	0.991	3.776	8.969	7.544	2.218	4.196	1.330	6.157	4.227
8.567	2.204	7.729	2.561	4.079	5.990	4.533	1.479	1.787	2.773
4.135	8.911	2.312	2.741	1.800	4.455	1.494	3.553	2.740	3.088
3.023	4.576	1.548	2.001	0.967	1.254	0.442	1.037	3.158	1.640

Table 9. Sample 5: shape parameter $\gamma = 1.50$, scale parameter $\theta = 5.50$.

9.358	5.967	7.419	1.095	9.644	6.522	3.177	2.659	8.364	2.938
2.629	50.527	3.203	23.743	0.193	19.347	4.698	9.034	21.905	6.947
1.372	15.865	12.048	22.504	18.353	4.852	4.154	19.918	4.844	11.619
15.535	7.894	3.585	5.290	6.773	23.564	4.320	17.532	2.076	12.498
17.913	10.503	9.368	8.501	5.955	8.287	6.434	3.639	3.994	0.830

Table 10. Sample 6: shape parameter $\gamma = 3.50$, scale parameter $\theta = 7.50$.

23.050	21.055	24.580	44.369	35.271	26.499	23.549	9.190	18.360	25.396
7.902	18.227	11.953	9.329	28.603	56.530	7.725	15.838	15.725	53.207
13.296	11.540	28.571	54.290	14.447	2.468	18.163	28.202	32.962	33.196
11.386	15.759	19.782	30.884	56.207	15.343	38.397	19.579	15.842	28.960
34.336	46.207	15.124	27.127	25.514	21.815	23.740	33.444	9.018	11.678

Based on the samples in Tables 8–10, we tested the same hypothesis of equality of the shape parameters, as in Example 1. The maximum likelihood estimates of γ_i, θ_i, for the three populations are given in Table 11. These estimates were used to get a consistent estimate of Σ_h as described earlier. The value of the test-statistic for this example based on two d.f. is $Q = 26.19$. This value is highly significant

Table 11.

i	1	2	3
$\hat{\gamma}_i$	2.505	1.458	3.294
$\hat{\theta}_i$	1.301	6.714	7.368

as it should be because the scale parameters for the three populations from which the samples are generated are quite different.

6. Discussions

The tests based on the generalized minimum chi-square procedure presented here are asymptotic in nature, and, therefore, perform well for large samples. However, in some cases, even for small samples, their performance is sometimes reasonable. These procedures can be used to test hypotheses regarding the parameters of more than two gamma populations. The tests are general enough to include testing hypotheses regarding means, scale parameters, shape parameters, etc. In order to carry out these tests no restrictions are needed on the nuisance parameters such as the assumption of common shape parameters made for testing equality of the scale parameters. On the other hand, Shiue and Bain's test has the disadvantage that it cannot conveniently be extended for more than two gamma populations. Thus, the tests presented here have some attractive features and hence are useful for certain practical situations.

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