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ORTHOGONALLY INVARIANT ESTIMATION OF THE SKEW-SYMMETRIC NORMAL MEAN MATRIX

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Abstract. The unbiased estimator of risk of the orthogonally invariant estimator of the skew-symmetric normal mean matrix is obtained, and a class of minimax estimators and their order-preserving modification are proposed. The estimators have applications in paired comparisons model. A Monte Carlo study to compare the risks of the estimators is given.

Key words and phrases: Isotonic regression, minimax estimator, order-preserving, paired comparisons, singular value.

1. Introduction

Let $\boldsymbol{X} = (x_{ij})$ be a $p \times p$ skew-symmetric random matrix whose upper triangle element x_{ij} $(i < j)$ is independently and normally distributed with known variance $\sigma^2 = 1$. We consider orthogonally invariant (equivariant) estimator $\hat{\boldsymbol{\Xi}} = \hat{\boldsymbol{\Xi}}(\boldsymbol{X})$ of the mean matrix $E\boldsymbol{X} = \boldsymbol{\Xi}$ such that

$$\hat{\boldsymbol{\Xi}}(\boldsymbol{X}) = \boldsymbol{Q}'\hat{\boldsymbol{\Xi}}(\boldsymbol{Q}\boldsymbol{X}\boldsymbol{Q}')\boldsymbol{Q} \qquad \text{for all } \boldsymbol{Q} \in \mathcal{O}(p),$$

where $\mathcal{O}(p)$ denotes the group of $p \times p$ orthogonal matrices. The skew-symmetric matrix $\boldsymbol{X}$ has a singular value decomposition

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{D}(\boldsymbol{l})\boldsymbol{U}' \tag{1.1}$$

with $\boldsymbol{l} = (l_1, \dots, l_t)$, $l_1 \geq \cdots \geq l_t \geq 0$, $t = [p/2]$, a vector of the nonnegative singular values,

$$\boldsymbol{D}(\boldsymbol{l}) = \begin{cases} \operatorname{diag}\left[\begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & l_2 \\ -l_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & l_t \\ -l_t & 0 \end{pmatrix}\right] & \text{if } p = 2t \\ \operatorname{diag}\left[\begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & l_2 \\ -l_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & l_t \\ -l_t & 0 \end{pmatrix}, 0\right] & \text{if } p = 2t+1, \end{cases}$$

and $\boldsymbol{U}$ a $p \times p$ orthogonal matrix. It can be easily proved that an estimator $\hat{\Xi}$ is orthogonally invariant if and only if it can be expressed as

$$\hat{\Xi} = \boldsymbol{U}\boldsymbol{D}(\boldsymbol{\psi}(\boldsymbol{l}))\boldsymbol{U}', \tag{1.2}$$

where $\boldsymbol{\psi}(\boldsymbol{l}) = (\psi_1(\boldsymbol{l}), \ldots, \psi_t(\boldsymbol{l}))$ is a vector of functions of $\boldsymbol{l}$.

In this paper, we discuss orthogonally invariant estimation of Ξ with the usual quadratic loss function

$$\|\hat{\Xi} - \Xi\|^2 = \frac{1}{2}\operatorname{tr}(\hat{\Xi} - \Xi)'(\hat{\Xi} - \Xi).$$

We shall construct a class of orthogonally invariant estimators which dominate the minimax estimator with a constant risk $\hat{\Xi}^{UB} = \boldsymbol{X}$. Since the risk depends only on the singular values $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_t)$, $\xi_1 \geq \cdots \geq \xi_t \geq 0$, of Ξ, we can put $\Xi = \boldsymbol{D}(\xi)$ without loss of generality.

Our estimation problem has following applications. Suppose that there are m 'objects' (e.g. treatments, stimuli, *etc.*) $O_1, \ldots, O_m$ and that we want to compare them. In such a case, paired comparisons method is used frequently, where the basic experimental unit is the comparison of two objects O_i and O_j $(i < j)$ and the comparisons are made for all $\binom{m}{2}$ pairs, see David (1988). Suppose that y_{ij} is observed as the degree of 'preference' of O_i over O_j. Putting $y_{ji} = -y_{ij}$, $y_{ii} = 0$, we see that the observed matrix $\boldsymbol{Y} = (y_{ij})$ is $m \times m$ and skew-symmetric. For such data Scheffé (1952) proposed an analysis of variance based on the linear model

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{M} + \boldsymbol{E}, \\ \boldsymbol{M} &= \boldsymbol{\alpha}\boldsymbol{1}_m{}' - \boldsymbol{1}_m\boldsymbol{\alpha}' + \boldsymbol{\Gamma}, \end{aligned}$$

with $\boldsymbol{M}$ an $m \times m$ skew-symmetric mean matrix, $\boldsymbol{E} = (\varepsilon_{ij})$ an $m \times m$ skew-symmetric random matrix whose upper triangle element ε_{ij} $(i < j)$ is independently distributed as $N(0, \sigma^2)$, $\boldsymbol{\alpha}$ an $m \times 1$ main effect vector, $\boldsymbol{1}_m = (1, \ldots, 1)'$ an $m \times 1$ constant vector, and $\boldsymbol{\Gamma} = \boldsymbol{Q}_m\boldsymbol{M}\boldsymbol{Q}_m$, $\boldsymbol{Q}_m = \boldsymbol{I}_m - (1/m)\boldsymbol{1}_m\boldsymbol{1}_m{}'$, an $m \times m$ skew-symmetric interaction effect matrix. The ordinary estimators of $\boldsymbol{M}$ and $\boldsymbol{\Gamma}$ are $\boldsymbol{Y}$ and $\boldsymbol{Q}_m\boldsymbol{Y}\boldsymbol{Q}_m$, respectively. Our orthogonally invariant estimators can be applied to the estimation of $\boldsymbol{M}$ and $\boldsymbol{\Gamma}$.

The outline of the paper is as follows. In Section 2 the joint density of the singular values of $\boldsymbol{X}$ is derived. Using the obtained density, the unbiased estimator of the risk of the orthogonally invariant estimator $\hat{\Xi}$ of (1.2) is given by the method developed essentially by Sheena (1991), who gave another derivation of the unbiased estimator of Stein's risk of the covariance estimation. In Section 3, we give a class of the minimax estimators of the skew-symmetric mean matrix. The estimation problem of matrix mean of Stein (1973) is contrasted with our problem here. The obtained estimators are, however, *non-order-preserving* in the sense that there exists Ξ such that

$$P_{\Xi}(\psi_1(\boldsymbol{l}) \geq \cdots \geq \psi_t(\boldsymbol{l}) \geq 0) \neq 1. \tag{1.3}$$

In Section 4 one modification method of making a non-order-preserving estimator (including the estimators given in Section 3) order-preserving is presented. The modified estimator is shown to dominate the original estimator in the same manner as Sheena and Takemura (1992). Section 5 gives a Monte Carlo study to show the performance of the estimators proposed in Sections 3 and 4.

2. Unbiased estimator of the risk difference

We start with deriving the joint density of the singular values of $\boldsymbol{X}$. The density of $\boldsymbol{X}$ is

$$\frac{1}{(2\pi)^{p(p-1)/4}} \exp\left\{-\frac{1}{4}\operatorname{tr}(\boldsymbol{X}-\Xi)'(\boldsymbol{X}-\Xi)\right\}(d\boldsymbol{X}), \tag{2.1}$$

where $(d\boldsymbol{X}) = \prod_{i<j} dx_{ij}$ is Lebesgue measure on $\boldsymbol{R}^{p(p-1)/2}$. Note that we only have to treat the case of $l_1 > \cdots > l_t > 0$, which holds true with probability 1 in our application. In this case, it is easy to show that the orthogonal matrix $\boldsymbol{U}$ in (1.1) is uniquely determined as an element of the left quotient space $\mathcal{U}_p = \mathcal{O}(p)/\mathcal{H}_p$, where

$$\mathcal{H}_p = \begin{cases} \{\operatorname{diag}(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_t) \mid \boldsymbol{H}_i \in \mathcal{O}^+(2)\} & \text{if } p = 2t \\ \{\operatorname{diag}(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_t,\pm 1) \mid \boldsymbol{H}_i \in \mathcal{O}^+(2)\} & \text{if } p = 2t+1 \end{cases}$$

is a subgroup of $\mathcal{O}(p)$, and $\mathcal{O}^+(2)$ denotes the group of 2×2 orthogonal matrix with determinant 1. The Jacobian of the transformation (1.1) is given by

$$(d\boldsymbol{X}) = \text{const}\cdot\left[\prod_{i=1}^{t} l_i^2\right]_{\text{odd}} \prod_{1\le i<j\le t}({l_i}^2-{l_j}^2)^2 d\boldsymbol{l}\cdot(d\boldsymbol{U}), \tag{2.2}$$

where $d\boldsymbol{l} = \prod_{i=1}^{t} dl_i$ and $[x]_{\text{odd}} = x$ (if p is odd), 1 (otherwise) (Khatri (1965), Lemma 2). Here $(d\boldsymbol{U})$ in (2.2) is a differential form on $\mathcal{U}_p$, defining an invariant measure on $\mathcal{U}_p$ such that

$$\int_{\boldsymbol{Q}\mathcal{D}}(d\boldsymbol{U}) = \int_{\mathcal{D}}(d\boldsymbol{U}) \quad \text{for all} \quad \mathcal{D}\subset\mathcal{U}_p,\ \ \boldsymbol{Q}\in\mathcal{O}(p).$$

See Kuriki (1992) for the details. Combining (2.1) and (2.2), and integrating it out with respect to $\boldsymbol{U}$, we get the joint density of the singular values $l_1 > \cdots > l_t > 0$ of $\boldsymbol{X}$ as

$$K_p \exp\left\{-\frac{1}{2}\sum_i {l_i}^2\right\}\left[\prod_i {l_i}^2\right]_{\text{odd}} \prod_{i<j}({l_i}^2-{l_j}^2)^2 F(\boldsymbol{l},\Xi)d\boldsymbol{l},$$

where $K_p = K_p(\Xi)$ is the normalizing constant, and

$$F(\boldsymbol{l},\Xi) = \int_{\mathcal{U}_p} \exp\left\{\sum_i l_i a_i\right\}(d\boldsymbol{U})$$

with $a_i = (\boldsymbol{U}'\Xi\boldsymbol{U})_{2i-1,2i}$. Here the method of Sheena (1991) is exploited. Noting that

$$\frac{\partial F}{\partial l_i} = \int_{\mathcal{U}_p} a_i \exp\left\{\sum_j l_j a_j\right\}(d\boldsymbol{U}),$$

we have

$$E_{\Xi}[\psi_i a_i] = K_p \int_{\mathcal{L}} \psi_i \exp\left\{-\frac{1}{2}\sum_j {l_j}^2\right\}\left[\prod_j {l_j}^2\right]_{\mathrm{odd}} \prod_{j<k}({l_j}^2 - {l_k}^2)^2 \frac{\partial F}{\partial l_i} d\boldsymbol{l} \tag{2.3}$$

with

$$\mathcal{L} = \{(l_1, \ldots, l_t) \mid l_1 > \cdots > l_t > 0\}.$$

If ψ_i is absolutely continuous as a function of l_i, by integration by parts we have

$$\begin{aligned} \text{R.H.S. of (2.3)} = {} & K_p \int_{\mathcal{L}_i} \psi_i \exp\left\{-\frac{1}{2}\sum_j {l_j}^2\right\}\left[\prod_j {l_j}^2\right]_{\mathrm{odd}} \\ & \cdot \prod_{j<k}({l_j}^2 - {l_k}^2)^2 F\Big|_{l_i=l_{i+1}}^{l_{i-1}} \prod_{j\neq i} dl_j \\ & - K_p \int_{\mathcal{L}} \frac{\partial}{\partial l_i}\left\{\psi_i \exp\left\{-\frac{1}{2}\sum_j {l_j}^2\right\}\left[\prod_j {l_j}^2\right]_{\mathrm{odd}} \right. \\ & \left. \cdot \prod_{j<k}({l_j}^2 - {l_k}^2)^2\right\} F d\boldsymbol{l} \end{aligned} \tag{2.4}$$

with

$$\mathcal{L}_i = \{(l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_t) \mid l_1 > \cdots > l_{i-1} > l_{i+1} > \cdots > l_t > 0\},$$

where we put $l_0 = \infty$ and $l_{t+1} = 0$ for simplicity. Expanding the second term of R.H.S. of (2.4) we obtain the following lemma.

LEMMA 2.1. *Suppose the conditions that*
(a) ψ_i *is absolutely continuous as a function of* l_i;
(b) *as* $l_i \to l_{i-1}$ *and* $l_i \to l_{i+1}$,

$$\lim \psi_i(\boldsymbol{l}) \exp\left\{-\frac{1}{2}{l_i}^2\right\}[{l_i}^2]_{\mathrm{odd}} \prod_{j\neq i}({l_i}^2 - {l_j}^2)^2 F(\boldsymbol{l}, \Xi) = 0.$$

Then the identity

$$E_{\Xi}[\psi_i a_i] = E_{\Xi}\left[l_i\psi_i - \frac{\partial \psi_i}{\partial l_i} - \left\{2\frac{\psi_i}{l_i}\right\}_{\mathrm{odd}} - 4\sum_{j\neq i}\frac{l_i\psi_i}{{l_i}^2 - {l_j}^2}\right]$$

holds provided that each expectation exists, where $\{x\}_{\text{odd}} = x$ (*if* p *is odd*), 0 (*otherwise*).

Noting that

$$\|\hat{\Xi} - \Xi\|^2 - \|\boldsymbol{X} - \Xi\|^2 = \sum_i ({\psi_i}^2 - {l_i}^2) - 2\sum_i (\psi_i - l_i) a_i,$$

we immediately obtain the unbiased estimator of the risk difference.

THEOREM 2.1. *If the conditions* (a), (b) *of Lemma* 2.1 *are satisfied, the identity*

$$\begin{aligned}(2.5)\qquad & E_\Xi[\|\hat{\Xi} - \Xi\|^2] - E_\Xi[\|\boldsymbol{X} - \Xi\|^2] \\ &= E_\Xi\left[\sum_i {l_i}^2{\phi_i}^2 - 2\sum_i l_i \frac{\partial \phi_i}{\partial l_i}\right. \\ &\qquad \left. -(2 + \{4\}_{\text{odd}})\sum_i \phi_i - 8\sum_{i<j} \frac{{l_i}^2\phi_i - {l_j}^2\phi_j}{{l_i}^2 - {l_j}^2}\right]\end{aligned}$$

with $\phi_i = 1 - \psi_i / l_i$ *holds provided that each expectation exists.*

3. Orthogonally invariant minimax estimator

Using Theorem 2.1 we give a class of orthogonally invariant minimax estimators.

THEOREM 3.1. *Let* α *and* β *be continuous piecewise differentiable functions on* $(0, \infty)$ *such that*

(a) $0 \leq \alpha, \beta \leq 1$;

(b) $\alpha', \beta' \geq 0$;

(c) $E_\Xi[\alpha'(\sum_j {l_j}^2)] < \infty$, $E_\Xi[\beta'(\sum_j {l_j}^2)] < \infty$.

If $p \geq 3$, *then the orthogonally invariant estimator* $\hat{\Xi}$ *of* (1.2) *with* $\psi_i = l_i(1 - \phi_i)$,

$$(3.1)\qquad \phi_i = 4\alpha\left(\sum_j {l_j}^2\right)\sum_{j \neq i} \frac{1}{{l_i}^2 - {l_j}^2} + \left\{2\beta\left(\sum_j {l_j}^2\right)\frac{1}{{l_i}^2}\right\}_{\text{odd}}$$

is minimax.

PROOF. It can be checked that the conditions (a) and (b) of Lemma 2.1 and that $E_\Xi[\|\hat{\Xi} - \Xi\|^2] < \infty$. Substituting (3.1), we see that R.H.S. of (2.5) becomes

$$\begin{aligned} E_\Xi & \left[-16\alpha(1-\alpha)\sum_{i \neq j} \frac{{l_i}^2}{({l_i}^2 - {l_j}^2)^2}\right. \\ &\qquad \left. -8t(t-1)\alpha' + \left\{-4\beta(1-\beta)\sum_i \frac{1}{{l_i}^2} - 8t\beta'\right\}_{\text{odd}}\right],\end{aligned}$$

which is finite and negative since

$$E_{\Xi}\left[\frac{{l_i}^2}{({l_i}^2-{l_j}^2)^2}\right]<\infty \quad \text{and} \quad E_{\Xi}\left[\frac{1}{{l_i}^2}\right]<\infty \qquad (\text{if } p \text{ is odd}).$$

The proof is completed. □

In the subclass that α and β are constant functions, the best choice is that $\alpha \equiv \beta \equiv 1/2$, i.e.

$$\phi_i = 2\sum_{j\neq i}\frac{1}{{l_i}^2-{l_j}^2}+\left\{\frac{1}{{l_i}^2}\right\}_{\text{odd}}. \tag{3.2}$$

We denote the estimator $\hat{\Xi}$ of (1.2) with ϕ_i (3.2) by $\hat{\Xi}^{CR}$ (Crude estimator).

When the variance σ^2 is unknown but there exists an independent estimator $\hat{\sigma}^2$ such that $\nu\hat{\sigma}^2/\sigma^2 \sim \chi^2(\nu)$, the estimator

$$\hat{\Xi} = \boldsymbol{U}\boldsymbol{D}(\boldsymbol{\psi})\boldsymbol{U}', \qquad \psi_i = l_i\left(1-\frac{\nu\hat{\sigma}^2}{\nu+2}\phi_i\right)$$

with ϕ_i (3.1) is shown to be minimax. See Stein ((1981), Section 7).

Remark 3.1. Let $\boldsymbol{Z}=(z_{ij})$ be a $p\times n$ ($n\geq p+1$) random matrix whose (i,j)-th element z_{ij} is independently and normally distributed with variance 1. Stein ((1973), Section 5) considered the orthogonally invariant estimator $\hat{\boldsymbol{\Theta}}=\hat{\boldsymbol{\Theta}}(\boldsymbol{Z})$ of the mean matrix $E\boldsymbol{Z}=\boldsymbol{\Theta}$ having the form

$$\hat{\boldsymbol{\Theta}} = \boldsymbol{B}\operatorname{diag}(\psi_i(\boldsymbol{l}))\boldsymbol{C}' = (\boldsymbol{I}-\boldsymbol{B}\operatorname{diag}(\phi_i(\boldsymbol{l}))\boldsymbol{B}')\boldsymbol{Z}, \tag{3.3}$$

where $\boldsymbol{Z}=\boldsymbol{B}\operatorname{diag}(l_i)\boldsymbol{C}'$, $\boldsymbol{l}=(l_1,\dots,l_p)$, $l_1>\cdots>l_p>0$, is the singular value decomposition of $\boldsymbol{Z}$, $\psi_i=l_i(1-\phi_i)$. The loss function is quadratic, i.e.

$$\|\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta}\|^2=\operatorname{tr}(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})'(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta}).$$

Stein ((1973), p. 377, (17)) suggested some modifications based on the estimator $\hat{\boldsymbol{\Theta}}$ with

$$\phi_i = 2\sum_{j\neq i}\frac{1}{{l_i}^2-{l_j}^2}+\frac{n-p-1}{{l_i}^2}. \tag{3.4}$$

The estimator $\hat{\boldsymbol{\Theta}}$ of (3.3) with ϕ_i (3.4) seems the same type as $\hat{\Xi}^{CR}$ defined by (3.2), however, it turns out not to have finite risk nor to satisfy the boundary condition corresponding to (b) of Lemma 2.1 because the linkage factor contained in the joint density of the singular values of $\boldsymbol{Z}$ is of the form $\prod_{i<j}({l_i}^2-{l_j}^2)$ (not of the form $\prod_{i<j}({l_i}^2-{l_j}^2)^2$).

4. Order-preserving estimator

The estimator given in Theorem 3.1 is non-order-preserving in the sense of (1.3). In this section, we show that non-order-preserving estimator is inadmissible and dominated by a modified estimator which preserves the order. We give two theorems without proofs, since a large part of them parallels the arguments of Sheena and Takemura (1992).

THEOREM 4.1. *Let $\tilde{\boldsymbol{\psi}}(\boldsymbol{l}) = (\tilde{\psi}_1(\boldsymbol{l}), \ldots, \tilde{\psi}_t(\boldsymbol{l}))$ be a vector of functions of $\boldsymbol{l} = (l_1, \ldots, l_t)$ such that*

(a) $\{\tilde{\psi}_i\}$ *majorizes weakly* $\{\psi_i\}$, *i.e.* $\sum_{i=1}^{j} \tilde{\psi}_i \geq \sum_{i=1}^{j} \psi_i$ *for* $1 \leq j \leq t$;

(b) $\sum_{i=1}^{t} \tilde{\psi}_i^2 \leq \sum_{i=1}^{t} \psi_i{}^2$.

If

(c) *there exists* $\boldsymbol{\Xi}$ *such that* $P_{\boldsymbol{\Xi}}(\tilde{\boldsymbol{\psi}} \neq \boldsymbol{\psi}) > 0$,

then the estimator $\tilde{\boldsymbol{\Xi}} = \boldsymbol{U}\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\boldsymbol{U}'$ *dominates* $\hat{\boldsymbol{\Xi}} = \boldsymbol{U}\boldsymbol{D}(\boldsymbol{\psi})\boldsymbol{U}'$.

One method of constructing $\tilde{\boldsymbol{\psi}}$ from $\boldsymbol{\psi}$ is given as follows.

THEOREM 4.2. *Let $\bar{\psi}_i$ be the unique solution of*

$$\min\left\{\sum_{i=1}^{t}(f_i - \psi_i)^2 \mid f_1 \geq \cdots \geq f_t \geq 0\right\} = \sum_{i=1}^{t}(\bar{\psi}_i - \psi_i)^2.$$

If $\boldsymbol{\psi}$ is non-order-preserving, then

$$\hat{\boldsymbol{\Xi}}^{OP} = \boldsymbol{U}\boldsymbol{D}(\bar{\boldsymbol{\psi}})\boldsymbol{U}', \qquad \bar{\boldsymbol{\psi}} = (\bar{\psi}_1, \ldots, \bar{\psi}_t),$$

dominates $\hat{\boldsymbol{\Xi}} = \boldsymbol{U}\boldsymbol{D}(\boldsymbol{\psi})\boldsymbol{U}'$.

Note that $\bar{\psi}_1 \geq \cdots \geq \bar{\psi}_t \geq 0$ can be obtained by the isotonic regression, see Robertson *et al.* ((1988), Section 1.4).

5. Monte Carlo study

We study the risk performance of several proposed estimators with a Monte Carlo study. The variance $\sigma^2 = 1$ is assumed to be known. We compare the risks of the estimators $\hat{\boldsymbol{\Xi}}^{UB}$ of Section 1, $\hat{\boldsymbol{\Xi}}^{CR}$ of Section 3, $\hat{\boldsymbol{\Xi}}^{OP}$ of Section 4 (based on $\hat{\boldsymbol{\Xi}}^{CR}$), usual James-Stein estimator

$$\hat{\boldsymbol{\Xi}}^{JS} = \left\{1 - \frac{\frac{1}{2}p(p-1) - 2}{\frac{1}{2}\operatorname{tr} \boldsymbol{X}'\boldsymbol{X}}\right\}\boldsymbol{X}$$

and positive-part James-Stein estimator $\hat{\boldsymbol{\Xi}}^{PP}$. Note that $\hat{\boldsymbol{\Xi}}^{JS}$ and $\hat{\boldsymbol{\Xi}}^{PP}$ are also orthogonally invariant, that $\hat{\boldsymbol{\Xi}}^{JS}$ is non-order-preserving, and that $\hat{\boldsymbol{\Xi}}^{PP}$ is order-preserving. The average losses of five estimators over 100000 replications are given

Table 1. Average losses of estimators.

p	ξ	UB	CR	OP	JS	PP
4	$(0,0)$	6.0	4.0	3.2	2.0	1.4
	$(1,1)$	6.0	4.4	3.9	3.1	2.6
	$(2,2)$	6.0	4.8	4.4	4.5	4.4
	$(5/\sqrt{2},5/\sqrt{2})$	6.0	5.0	4.6	5.4	5.4
	$(4,3)$	6.0	5.1	4.8	5.4	5.4
	$(5,5)$	6.0	5.0	4.6	5.7	5.7
	$(4\sqrt{2},3\sqrt{2})$	6.0	5.2	5.0	5.7	5.7
	$(6,6)$	6.0	5.0	4.6	5.8	5.8
5	$(0,0)$	10.0	6.0	4.3	2.0	1.3
	$(1,1)$	10.0	6.6	5.3	3.4	2.8
	$(2,2)$	10.0	7.8	7.0	5.7	5.6
	$(5/\sqrt{2},5/\sqrt{2})$	10.0	8.7	8.2	8.0	8.0
	$(4,3)$	10.0	8.8	8.4	8.0	8.0
	$(5,5)$	10.0	8.9	8.4	8.9	8.9
	$(4\sqrt{2},3\sqrt{2})$	10.0	9.1	8.8	8.9	8.9
	$(6,6)$	10.0	8.9	8.5	9.2	9.2

in Table 1. The result indicates that: When the nonnegative singular values $\xi_1,\ldots,\xi_t$ of Ξ are close together, the orthogonally invariant estimators $\hat{\Xi}^{CR}$ and $\hat{\Xi}^{OP}$ save much risk; Making an estimator order-preserving is effective; If the singular values of Ξ are small, $\hat{\Xi}^{JS}$ and $\hat{\Xi}^{PP}$ are better than the proposed orthogonally invariant estimators.

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