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ASYMPTOTIC BEHAVIOR OF L-STATISTICS FOR A LARGE CLASS OF TIME SERIES

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Abstract. In this paper we derive the asymptotic normality of L-statistics with unbounded scores for a large class of time series. To handle the dependence structure, we use the concept of $m(n)$-decomposability as an alternative to classical mixing concepts.

Key words and phrases: Decomposable processes, mixing, empirical processes for decomposable samples, L-statistics.

1. Introduction

Linear combinations of functions of order statistics (L-statistics) are a well-established class of statistics in robust and nonparametric theory. Their usefulness does not remain restricted to the i.i.d. case but extends to other models like time series; see e.g. Chernoff *et al.* (1967), Stigler (1969), Shorack (1972), Shorack and Wellner (1986), respectively Gastwirth and Rubin (1975), Koul (1977), Portnoy (1977, 1979), Martin (1978) and Martin and Yohai (1986).

In this paper we consider asymptotic normality of L-statistics for a wide class of time series where we use a new way ($m(n)$- or *asymptotic decomposability*) to specify the dependence structure. On the one hand, this new concept allows us to avoid the usual mixing conditions that are hard to understand intuitively since they may fail to hold in very decent cases (Andrews (1984)), and that are occasionally hard to work with (Pham and Tran (1985)). On the other hand, the concept is tailor-made for dealing with linear processes or, more generally, processes with a finite order Volterra expansion (Priestley (1981)), and bilinear processes (Rao and Gabr (1984), Chanda and Ruymgaart (1990)).

We will focus on L-statistics with score functions that are allowed to grow indefinitely near the endpoints of the unit interval. This case is technically more

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interesting—and more difficult—than the case of scores that are zero in neighborhoods of the endpoints of the unit interval. For the latter simpler case and a more elaborate discussion and illustration of $m(n)$-decomposability we refer to Chanda *et al.* (1990).

For ease of reference, however, let us recall the definition of $m(n)$-decomposability for a real valued time series. Let $X_1, X_2, \ldots$ be a stochastic process where the X_i are real valued with a common d.f. F. Suppose that for each $n \in \mathbb{N}$ we have a decomposition $X_i = X_{i,m(n)} + \bar{X}_{i,m(n)}$, for some $m(n) \in \{0, 1, \ldots, n\}$, where the $X_{i,m(n)}$ have a common d.f. $F_{m(n)}$, and where

$$\text{the } X_{i,m(n)} \text{ are } m(n)\text{-dependent,} \tag{1.1}$$

$$\max_{1 \le i \le n} P(|\bar{X}_{i,m(n)}| \ge \epsilon(n)) \le \delta(n), \tag{1.2}$$

$$\text{for some } \epsilon(n), \delta(n) \to 0 \text{ as } n \to \infty.$$

Such a process is defined as $m(n)$-decomposable. It should be noted that the $m(n)$-dependence mentioned in (1.1) is the ordinary concept for fixed n. For different values of the sample size n, however, we allow the order $m(n)$ of the dependence to be different. In many cases $m(n)$ will increase as a power of n. Working with decomposable processes for large n typically requires a specification of the orders of magnitude of $m(n)$, $\epsilon(n)$ and $\delta(n)$. In interesting examples of time series it is inevitable that $m(n) \to \infty$, as $n \to \infty$; yet in many cases suitable rates of $\epsilon(n)$ and $\delta(n)$ can be obtained for $m(n) \ll n$.

To prove the asymptotic normality we use the classical Chernoff-Savage approach which was first applied by Moore (1968) to L-statistics. In Section 2 we formulate the assumptions and give the main result. In passing we give a first outline of the proof. The necessary tools from empirical process theory that might be of independent interest are summarized in Section 3, and the asymptotic normality of the first order terms is dealt with in Section 4. In Section 5 we derive the asymptotic negligibility of the remainder term and, finally, in Section 6 we simplify the expression for the variance under an additional weak-stationarity condition and give some examples.

2. Assumptions and main result

ASSUMPTION 2.1. The underlying process $X_1, \ldots, X_n$ is decomposable and hence by definition satisfies (1.1) and (1.2). The common d.f. F of the X_i is continuously differentiable on $\mathbb{R}$ with derivative f satisfying $\|f\|_\infty < \infty$, and $\{x \in \mathbb{R} : f(x) > 0\}$ convex.

Let $\hat{F}_n$ be the empirical d.f. of the X_i at stage n and $\hat{\Gamma}_n$ the empirical d.f. of the transformed random variables $\xi_i = F(X_i)$. Assumption 2.1 entails the decomposability of the ξ_i, since by the mean value theorem we have

$$\xi_i = F(X_{i,m}) + \bar{X}_{i,m} f(X_{i,m} + \theta \bar{X}_{i,m}) = \xi_{i,m} + \bar{\xi}_{i,m}, \tag{2.1}$$

for random $\theta \in (0, 1)$. The common d.f. of the ξ_i is uniform $(0, 1)$. As usual we define the corresponding reduced empirical process by

$$U_n = \{U_n(t) = n^{1/2}(\hat{\Gamma}_n(t) - t), t \in [0, 1]\}. \tag{2.2}$$

Throughout this paper the numbers

$$A, B, C \in (0, \infty), \tag{2.3}$$

will be used as generic constants that are independent of all the relevant parameters, as in particular the sample size n, and that may differ from line to line.

For each $q > 0$ and $n \in \mathbb{N}$ there exists a continuously differentiable increasing function $\ell_q : [0,1] \to [0,1]$ with

$$\begin{aligned} &\ell_q([0,1]) \subset \left[\frac{1}{2}n^{-q}, 1 - \frac{1}{2}n^{-q}\right]; \\ &\ell_q(t) = t, \;\; t \in [n^{-q}, 1 - n^{-q}]; \qquad 0 \le \ell_q^{(1)} \le 1. \end{aligned} \tag{2.4}$$

The dependence on n is suppressed in the notation. Given a function $K : (0,1) \to \mathbb{R}$ we define

$$K_q(t) = K(\ell_q(t)), \qquad t \in [0,1]. \tag{2.5}$$

When K is differentiable we have according to the chain rule that $K_q^{(1)} = (d/dt)K(\ell_q(t)) = K^{(1)}(\ell_q(t))\ell_q^{(1)}(t)$, $t \in (0,1)$, and hence

$$|K_q^{(1)}(t)| \le |K^{(1)}(\ell_q(t))|, \qquad t \in (0,1). \tag{2.6}$$

A particularly useful function is

$$R(t) = \{t(1-t)\}^{-1}, \qquad t \in (0,1). \tag{2.7}$$

Let us observe that

$$0 < \max_{0 \le t \le 1} R_q^{\gamma}(t) \le Cn^{\gamma q}. \tag{2.8}$$

To describe the class of L-statistics that we are going to consider, let $J : (0,1) \to \mathbb{R}$ and $\Psi : (0,1) \to \mathbb{R}$ be given functions. Assumption 2.1 implies the continuity of F^{-1} on $\{x \in \mathbb{R} : 0 < F(x) < 1\}$; we write $\Psi_F = \Psi(F^{-1}) : (0,1) \to \mathbb{R}$. We will focus on statistics of the type

$$T_n = \int_{-\infty}^{\infty} \Psi(x) J_\zeta(\hat{F}_n(x)) d\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} c_{ni}\Psi(X_{i:n}), \tag{2.9}$$

where J_ζ is obtained from J via (2.5) with $q = \zeta$ and $c_{ni} = J_\zeta(i/n)$. It is obvious that

$$T_n \stackrel{d}{=} \int_0^1 \Psi_F(t) J_\zeta(\hat{\Gamma}_n(t)) d\hat{\Gamma}_n(t) = \frac{1}{n}\sum_{i=1}^{n} c_{ni}\Psi_F(\xi_{i:n}). \tag{2.10}$$

The latter representation will be used throughout.

ASSUMPTION 2.2. The function $J : (0,1) \to \mathbb{R}$ is twice and the function $\Psi_F : (0,1) \to \mathbb{R}$ is once continuously differentiable. For numbers

$$\alpha, \beta > 0 \text{ such that } \alpha + \beta < \Delta, \qquad \text{for some } 0 < \Delta < \frac{1}{2}, \tag{2.11}$$

they satisfy ($J^{(0)} = J$, $\Psi_F^{(0)} = \Psi_F$)

$$\begin{aligned} &|J^{(j)}| \le CR^{\alpha+j} \text{ on } (0,1), \quad j \in \{0,1,2\}; \\ &|\Psi_F^{(j)}| \le CR^{\beta+j} \text{ on } (0,1), \quad j \in \{0,1\}. \end{aligned} \tag{2.12}$$

The parameter ζ in (2.10) satisfies

$$0 < \zeta < 1/\{5(1-\Delta)\}. \tag{2.13}$$

Because we use J_ζ rather than J as a score function, it should be noted that the scores stay bounded for each n, but are allowed to tend to ∞ (near 0 and 1) at a certain rate when $n \to \infty$. Since we don't want to impose any further condition on the distribution of the $\xi_{i,m(n)}$, this additional control of the scores is technically very convenient. As it will turn out, under Assumption 2.1 we may replace Ψ_F by $\Psi_{F,r}$ (obtained via (2.5) for a suitable $q = r$) without affecting the weak limiting behavior of the statistics in (2.10). This replacement has similar technical advantages. Let us choose

$$r = 1/\{2(1-\Delta)\}, \tag{2.14}$$

and consider $\Psi_{F,r}$ obtained from Ψ_F via (2.5) with $q = r$. Assumption 2.1 guarantees the finiteness of the numbers

$$\mu_n = \int_0^1 \Psi_F(t) J_\zeta(t) dt, \qquad \tilde{\mu}_n = \int_0^1 \Psi_{F,r}(t) J_\zeta(t) dt. \tag{2.15}$$

Let us introduce

$$\tilde{T}_n = \int_0^1 \Psi_{F,r}(t) J_\zeta(\hat{\Gamma}_n(t)) d\hat{\Gamma}_n(t). \tag{2.16}$$

Jointly with (2.14) it can easily be shown that

$$n^{1/2}(T_n - \mu_n) - n^{1/2}(\tilde{T}_n - \tilde{\mu}_n) \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{2.17}$$

Because $\Psi_{F,r}$ is bounded, the statistics $\tilde{T}_n$ are easier to deal with.

It follows from the mean value theorem that for $t \in [0,1]$ and $\theta_n(t) \in (0,1)$ random, we have

$$J_\zeta(\hat{\Gamma}_n(t)) = J_\zeta(t) + n^{-1/2} U_n(t) J_\zeta^{(1)}(t + \theta_n(t) n^{-1/2} U_n(t)). \tag{2.18}$$

By substitution in (2.16) we find

$$n^{1/2}(\tilde{T}_n - \tilde{\mu}_n) = \tilde{A}_{0n} + \tilde{A}_{1,n} + \tilde{B}_n, \tag{2.19}$$

where the $\tilde{A}$- and $\tilde{B}$-terms are first and second order terms respectively given by

$$\tilde{A}_{0n} = \int_0^1 \Psi_{F,r}(t) J_\zeta(t) dU_n(t), \tag{2.20}$$

$$\tilde{A}_{1n} = \int_0^1 U_n(t)\Psi_{F,r}(t) J_\zeta^{(1)}(t)dt, \tag{2.21}$$

$$\tilde{B}_n = \int_0^1 U_n(t)\Psi_{F,r}(t) J_\zeta^{(1)}(t+\theta_n(t)n^{-1/2}U_n(t))d\hat{\Gamma}_n(t) - \tilde{A}_{1n}. \tag{2.22}$$

The next assumption prescribes suitable orders of magnitude for the sequences of parameters in (1.2), depending on Δ in (2.11).

ASSUMPTION 2.3. Let ζ satisfy (2.13). There exist $m(n) = O(n^\rho)$, $\epsilon(n) = O(n^{-\tau})$, as $n \to \infty$, with

$$0 \le \rho < \zeta(1-2\Delta), \qquad \tau > 1 + 1/\{2(1-\Delta)\}, \tag{2.23}$$

such that $\delta(n) = n^{-\sigma}$, as $n \to \infty$, with

$$\sigma > 1 + 1/\{2(1-\Delta)\}. \tag{2.24}$$

Let us write, for brevity,

$$\tilde{K}_n(t) = \Psi_{F,r}(t)J_\zeta(t) + \int_0^1 \left\{1_{[0,s]}(t) - s\right\} \Psi_{F,r}(s) J_\zeta^{(1)}(s) ds, \qquad t \in [0,1], \tag{2.25}$$

so that $\tilde{A}_{0n} + \tilde{A}_{1n} = \sum_{i=1}^n \{\tilde{K}_n(\xi_i) - \tilde{\mu}_n\}$, and consider the triangular array of $m(n)$-dependent centered random variables

$$\tilde{Z}_{ni} = \tilde{K}_n(\xi_{i,m(n)}) - E\tilde{K}_n(\xi_{i,m(n)}) = \tilde{K}_n(\xi_{i,m(n)}) - \tilde{\tilde{\mu}}_n. \tag{2.26}$$

Under the assumptions made above, we will prove (Sections 3 and 4)

$$\begin{aligned} &\tilde{A}_{0n} + \tilde{A}_{1n} - n^{-1/2}\sum_{i=1}^n \tilde{Z}_{ni} \\ &\quad = n^{-1/2}\sum_{i=1}^n \{(\tilde{K}_n(\xi_i) - \tilde{\mu}_n) - (\tilde{K}_n(\xi_{i,m(n)}) - \tilde{\tilde{\mu}}_n)\} \xrightarrow{p} 0 \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{2.27}$$

$$\tilde{B}_n \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{2.28}$$

To prove the asymptotic normality of $n^{-1/2}\sum_{i=1}^n \tilde{Z}_{ni}$ we may apply Berk (1973) (see also Rao (1984)), provided that the next assumption is satisfied.

ASSUMPTION 2.4. The following conditions are fulfilled:

$$\operatorname{Var}\left(\sum_{i=j+1}^{k} \tilde{Z}_{ni}\right) \leq C(k-j) \quad \text{for all} \quad j,k \in \{0,1,\ldots,n\}. \tag{2.29}$$

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Var}\left(\sum_{i=1}^{n} \tilde{Z}_{ni}\right) = \tilde{v}^2 \in (0,\infty). \tag{2.30}$$

THEOREM 2.1. *When Assumptions* 2.1–2.4 *are satisfied we have*

$$n^{1/2}(T_n - \tilde{\mu}_n) \xrightarrow{d} \mathcal{N}(0,\tilde{v}^2) \quad \text{as} \quad n \to \infty, \tag{2.31}$$

where $\tilde{\mu}_n$ *may be replaced by* μ_n *(see* (2.15)*) and* $\tilde{v}^2$ *is defined in* (2.31).

3. Tools from empirical process theory

Some properties of empirical processes based on decomposable models will be reviewed here. Proofs will not be given since they are very similar to those in Nieuwenhuis and Ruymgaart (1990); see also Chanda and Ruymgaart (1990). Since these properties hold true for arbitrary n there is no need to specify the dependence on n of the parameters; in particular we will write m, ϵ, δ for the parameters in (1.2). In applications, however, these parameters will usually depend on n in a suitable way. We will then e.g. choose $\epsilon = \epsilon(n)$ in such a way that $P(\Omega^c_{\epsilon(n)}) \leq n\delta(n) = O(n^{1-\sigma\wedge\tau}) \to 0$, as $n \to \infty$, where $\sigma \wedge \tau = \min\{\sigma,\tau\}$. To formulate the results we need the function

$$\psi(\lambda) = 2\lambda^{-2} \int_0^\lambda \log(1+x)dx, \quad \lambda > 0; \qquad \psi(0) = 1. \tag{3.1}$$

It should be noted that $\psi(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$. For further properties and the role this function plays in precise asymptotic considerations the reader is referred to e.g. Shorack and Wellner (1986). We start with a basic fluctuation inequality that might be of independent interest.

THEOREM 3.1. *Suppose that Assumption* 2.1 *is fulfilled. For all* $n \in \mathbb{N}$ *we have*

$$\begin{aligned} P\left(\Omega_\epsilon \cap \left\{\sup_{a\leq s<t\leq b} |U_n(t) - U_n(s)| \geq \lambda\right\}\right) \\ \leq Cm \exp\left(\frac{-A\lambda^2}{m(b-a)}\psi\left(\frac{B\lambda}{n^{1/2}(b-a)}\right)\right), \end{aligned} \tag{3.2}$$

where $\Omega_\epsilon = \{\max_{1\leq i\leq n} |\bar{X}_{i,m}| < \epsilon\}$, *provided that*

$$\lambda \geq Cn^{1/2}(\epsilon+\delta), \qquad b-a \geq C(\epsilon+\delta). \tag{3.3}$$

Because Nieuwenhuis and Ruymgaart (1990) only exploit the decomposability property of the linear process the proof for the present Theorem 3.1 can be just copied. The proof of the next theorem requires only a minor modification of the slightly different weight functions that will be used. For arbitrary $0 \le \gamma \le 1/2$ and $q > 0$ consider the function R_q^γ (see (2.4)–(2.8)). For the proof of the next theorem the pattern of the proof of Nieuwenhuis and Ruymgaart ((1990), Theorem 3.1) may be followed by using (2.8) in particular and by partitioning $[0,1]$ into intervals of length $[n^q]^{-1}$.

THEOREM 3.2. *Suppose that Assumption* 2.1 *is satisfied. For any* $0 \le \gamma \le 1/2$ *and* $q > 0$ *we have*

$$P\left(\Omega_\epsilon \cap \left\{\sup_{0\le t\le 1} R_q^\gamma(t)|U_n(t)| \ge \lambda\right\}\right) \le Cmn^q \exp\left(\frac{-A\lambda^2}{m}\psi\left(\frac{B\lambda n^{q(1-\gamma)}}{n^{1/2}}\right)\right), \tag{3.4}$$

where Ω_ϵ *is as above, provided that*

$$\lambda \ge Cn^{1/2}n^{\gamma q}(\epsilon+\delta), \qquad n^{-q} \ge C(\epsilon+\delta). \tag{3.5}$$

As a corollary to Theorem 3.2 let us prove the useful property that, for any $0 < \beta < 1$,

$$P\left(\Omega_\epsilon \cap \left\{\hat{\Gamma}_n(t) \ge \beta t, \text{ for } n^{-q} \le t \le 1\right\}\right) \ge 1 - Cmn^q \exp\left(\frac{-An^{1-q}}{m}\right), \tag{3.6}$$

provided that (3.5) is satisfied with $\lambda = An^{(1-q)/2}$. As a matter of fact we have $P(\hat{\Gamma}_n(t) \ge \beta t, \text{ for } n^{-q} \le t \le 1) = P(\inf_{n^{-q}\le t\le 1} \hat{\Gamma}_n(t)/t \ge \beta) = P(\sup_{n^{-q}\le t\le 1}(t - \hat{\Gamma}_n(t))/t \le 1-\beta) \ge 1 - P(\sup_{n^{-q}\le t\le 1} |U_n(t)|/t \ge n^{1/2}(1-\beta)) \ge 1 - P(\sup_{n^{-q}\le t\le 1} |U_n(t)|/t^{1/2} \ge n^{(1-q)/2}(1-\beta)) \ge 1 - P(\sup_{0\le t\le 1} R_q^{1/2}|U_n(t)| \ge An^{(1-q)/2})$. The lower bound in (3.6) follows by applying (3.4) with $\gamma = 1/2$ and $\lambda = An^{(1-q)/2}$.

4. The leading terms

In this section we will be concerned with proving (2.27) and the asymptotic normality of the standarized sum of the $\tilde{Z}_{ni}$. It follows from Assumptions 2.1–2.3, (2.1) and (2.23) that

$$\left\{\max_{1\le i\le n} \bar{\xi}_{i,m(n)} \le \|f\|_\infty \epsilon(n)\right\} \supset \Omega_{\epsilon(n)}, \qquad E\bar{\xi}_{i,m(n)} \le Cn^{-\sigma\wedge\tau}. \tag{4.1}$$

The definition of $\tilde{K}_n$ in (2.25) implies that, for $i = 0, 1$,

$$|\tilde{K}_n^{(i)}| \le CR_\zeta^{-\alpha}R_r^{-\beta-i} \quad \text{on } (0,1), \tag{4.2}$$

$$\max_{0\le t\le 1} |K_n^{(i)}(t)| \le Cn^{\zeta\alpha + r(\beta+i)}, \tag{4.3}$$

where we use the fact that $0 < \zeta < r$.

Employing (4.1)–(4.3) and the mean value theorem we see that

$$(4.4)\quad n^{-1/2}\sum_{i=1}^{n} E|\tilde{K}_n(\xi_i) - \tilde{K}_n(\xi_{i,m(n)})| \leq n^{-1/2} n^{\zeta\alpha + r(\beta+1)} \sum_{i=1}^{n} E(\bar{\xi}_{i,m(n)})$$
$$\leq n^{1/2+\zeta\alpha+r(\beta+1)-\sigma\wedge\tau}.$$

According to the asssumptions and (2.14) we have $\sigma \wedge \tau - 1/2 - \zeta\alpha - r(\beta+1) > 1/2 + 1/2(1-\Delta) - r\Delta - 1/2(1-\Delta) = 1/2 - r\Delta > 0$, so that the upper bound in (4.4) converges to 0, as $n \to \infty$. Because $|E\tilde{K}_n(\xi_i) - E\tilde{K}_n(\xi_{i,m(n)})| \leq E|\tilde{K}_n(\xi_i) - \tilde{K}_n(\xi_{i,m(n)})|$ we have proved (2.27).

Let us now choose an incidental parameter

$$(4.5)\qquad \gamma = 1/\Delta - 2.$$

For such a γ we have, again using $0 < \zeta < r$,

$$(4.6)\qquad \sup_{i,n} E|\tilde{Z}_{ni}|^{2+\gamma} \leq C \sup_{i,n} E|\tilde{K}_n(\xi_{i,m(n)})|^{2+\gamma}$$
$$\leq C \sup_{i,n} E(R_\zeta^{-\alpha(2+\gamma)} \cdot R_r^{-\beta(2+\gamma)})(\xi_i - \bar{\xi}_{i,m(n)})$$
$$\leq C \int_0^1 R^{-(\alpha+\beta)(2+\gamma)}(t)dt$$
$$+ Cn^{-\sigma\wedge\tau} n^{\zeta\alpha(2+\gamma)+r\beta(2+\gamma)+r}.$$

Since $\sigma\wedge\tau - r - (\zeta\alpha + r\beta)(2+\gamma) > \sigma\wedge\tau - r - r\Delta(2 + 1/\Delta - 2) = \sigma\wedge\tau - 2r > 1 + 1/2(1-\Delta) - 1/(1-\Delta) = 1 - 1/2(1-\Delta) > 0$, it follows that the upper bound in (4.6) is bounded by a finite number because, moreover, $(\alpha+\beta)(2+\gamma) < \Delta(2+\gamma) = 1$.

Finally, let us note that the choice of ρ in (2.23) implies

$$(4.7)\qquad n^{-1}\{m(n)\}^{2+2/\gamma} \to 0 \quad \text{as} \quad n \to \infty,$$

because $\rho(2+2/\gamma) - 1 < \{(1-2\Delta)/5(1-\Delta)\}\{2(1-\Delta)/(1-2\Delta)\} - 1 = 2/5 - 1 < 0$.

Assumption 2.4 jointly with (4.6) and (4.7) suffice for application of Berk (1973), which yields $n^{-1/2}\sum_{i=1}^{n} \tilde{Z}_{ni} \xrightarrow{d} \mathcal{N}(0, \tilde{v}^2)$ as $n \to \infty$. Since we have also proved (2.27), it follows that $\tilde{A}_{0n} + \tilde{A}_{1n} \xrightarrow{d} \mathcal{N}(0, \tilde{v}^2)$ as $n \to \infty$, which settles the asymptotic normality of the leading terms.

5. The remainder term

This section is devoted to a proof of (2.28). This may be done with the help of the properties in Section 3 in a way similar to that in the i.i.d. case. As in Nieuwenhuis and Ruymgaart (1990), however, the present situation of a decomposable process is more complicated and requires control over a larger set of parameters.

Let us introduce the intervals

$$I(1) = [0, 2n^{-\zeta}], \qquad I(2) = [2n^{-\zeta}, 1 - 2n^{-\zeta}], \qquad I(3) = [1 - 2n^{-\zeta}, 1], \tag{5.1}$$

and write, for brevity,

$$t + \theta_n(t) n^{-1/2} U_n(t) = t + \theta_n(t)(\hat{\Gamma}_n(t) - t) = \tilde{\Gamma}_n(t), \tag{5.2}$$

and note that $\tilde{\Gamma}_n(t)$ is a random point between t and $\hat{\Gamma}_n(t)$. Now the remainder term will be decomposed into $\tilde{B}_n = \sum_{j=1}^{4} \tilde{B}_{nj}$, where

$$\tilde{B}_{nj} = \int_{I(j)} U_n(t) \Psi_{F,r}(t) \left\{ J_\zeta^{(1)}(\tilde{\Gamma}_n(t)) - J_\zeta^{(1)}(t) \right\} d\hat{\Gamma}_n(t), \qquad j \in \{1, 2, 3\}; \tag{5.3}$$

$$\tilde{B}_{n4} = \int_0^1 U_n(t) \Psi_{F,r}(t) J_\zeta^{(1)}(t) d\{\hat{\Gamma}_n(t) - t\}. \tag{5.4}$$

Let us single out the subsets

$$\Omega_{n1} = \left\{ \sup_{0 \le t \le 1} R_r^{1/2}(t) |U_n(t)| \le n^{\nu/2} \right\}, \tag{5.5}$$

$$\Omega_{n2} = \left\{ \hat{\Gamma}_n(t) \ge \frac{1}{2} t \ \forall t \in [n^{-\zeta}, 1] \right\}, \tag{5.6}$$

where the incidental parameter ν satisfies

$$\nu = \zeta(1 - 2\Delta). \tag{5.7}$$

Jointly with (2.14) and (5.7) conditions (2.23) and (2.24) imply that the conditions for application of (3.4) and (3.6) are satisfied and that

$$P(\Omega_n) \to 1 \ \text{ as } \ n \to \infty, \qquad \text{with } \ \Omega_n = \Omega_{\epsilon(n)} \cap \Omega_{n1} \cap \Omega_{n2}. \tag{5.8}$$

Consequently it suffices to prove that

$$C_{nj} = 1_{\Omega_n} \tilde{B}_{nj} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty, \qquad \text{for} \quad j = 1, 2, 3, 4. \tag{5.9}$$

Our assumptions entail that (note that $\zeta < r$)

$$\begin{aligned} E|C_{n1}| &\le C n^{\nu/2} \int_{I(1)} \frac{R_r^\beta(t) R_\zeta^{\alpha+1}(t)}{R_r^{1/2}(t)} dt \le C n^{\nu/2} \int_{I(1)} R_\zeta^{\alpha+\beta+1/2}(t) dt \\ &\le C n^{\nu/2 + \zeta(\alpha+\beta) - \zeta/2} \to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{5.10}$$

because $\nu/2 + \zeta(\alpha + \beta) - \zeta/2 = \zeta(\alpha + \beta) - \Delta < 0$. Since $J_\zeta = J$ on $I(2)$, we may apply the mean value theorem once more, which yields

$$J_\zeta^{(1)}(\tilde{\Gamma}_n(t)) - J_\zeta^{(1)}(t) = (\tilde{\Gamma}_n(t) - t) J_\zeta^{(2)}(\tilde{\tilde{\Gamma}}_n(t)), \tag{5.11}$$

where $\tilde{\tilde{\Gamma}}_n(t)$ is also a random point between t and $\hat{\Gamma}_n(t)$. It follows that

$$\begin{aligned}(5.12)\qquad 1_{\Omega_n}|J_\zeta^{(1)}(\tilde{\Gamma}_n(t)) - J_\zeta^{(1)}(t)| &\leq C1_{\Omega_n}|\tilde{\Gamma}_n(t) - t|R_\zeta^{\alpha+2}(\tilde{\tilde{\Gamma}}_n(t)) \\ &\leq Cn^{-1/2}|U_n(t)|R_\zeta^{\alpha+2}(t), \qquad t \in I(2).\end{aligned}$$

Hence we find by substitution (note that $\zeta < r$)

$$\begin{aligned}(5.13)\qquad E|C_{n2}| &\leq Cn^{-1/2}n^\nu \int_{I(2)} \frac{R_r^\beta(t)R_\zeta^{\alpha+2}(t)}{R_r(t)}dt \\ &\leq Cn^{-1/2+\nu}\int_{I(2)} R_\zeta^{\alpha+\beta+1}(t)dt \\ &\leq Cn^{-1/2+\nu+\zeta(\alpha+\beta)} \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

because $-1/2 + \nu + \zeta(\alpha + \beta) < -1/2 + \zeta(1 - 2\Delta) + \zeta\Delta = -1/2 + \zeta(1 - \Delta) < 0$ by (2.13). The random variable C_{n3} may be dealt with in a way similar to that of C_{n1}.

Let us now consider C_{n4} and introduce the parameter

$$(5.14)\qquad \phi = 1 - \frac{5}{2}\zeta(1 - 2\Delta) + \frac{1}{2}\rho > 1 - \frac{1 - 2\Delta}{2(1 - \Delta)} + \frac{1}{2}\rho > 0.$$

Partitioning the unit interval into subintervals of equal length $[n^\phi]^{-1}$, we define the left continuous step functions

$$(5.15)\qquad S_n(t) = \begin{cases} \dfrac{1}{[n^\phi]}, & \text{for } t = 0 = t_0; \\ \dfrac{k}{[n^\phi]}, & \text{for } t_{k-1} = \dfrac{k-1}{[n^\phi]} < t \leq \dfrac{k}{[n^\phi]},\ 1 \leq k \leq \dfrac{1}{2}[n^\phi]; \\ \dfrac{k-1}{[n^\phi]}, & \text{for } \dfrac{k-1}{[n^\phi]} < t \leq \dfrac{k}{[n^\phi]},\ \dfrac{1}{2}[n^\phi] < k \leq [n^\phi]. \end{cases}$$

We need to decompose C_{n4} into

$$(5.16)\qquad D_{n1} = 1_{\Omega_n}\int_0^1 U_n(S_n(t))\Psi_{F,r}(S_n(t))J_\zeta^{(1)}(S_n(t))d\{\hat{\Gamma}_n(t) - t\},$$

$$\begin{aligned}(5.17)\qquad D_{n2} = 1_{\Omega_n}\int_0^1 &\{U_n(t)\Psi_{F,r}(t)J_\zeta^{(1)}(t) \\ &- U_n(S_n(t))\Psi_{F,r}(S_n(t))J_\zeta^{(1)}(S_n(t))\}d\hat{\Gamma}_n(t),\end{aligned}$$

$$\begin{aligned}(5.18)\qquad D_{n3} = 1_{\Omega_n}\int_0^1 &\{U_n(S_n(t))\Psi_{F,r}(S_n(t))J_\zeta^{(1)}(S_n(t)) \\ &- U_n(t)\Psi_{F,r}(t)J_\zeta^{(1)}(t)\}dt.\end{aligned}$$

Let us note that

$$|D_{n1}| = 1_{\Omega_n} n^{-1/2} \left| \sum_{k=1}^{[n^\phi]} \{U_n(S_n(t_k))\Psi_{F,r}(S_n(t_k)) \cdot J_\zeta^{(1)}(S_n(t_k))\}\{U_n(t_k) - U_n(t_{k-1})\} \right| \tag{5.19}$$

$$\leq Cn^{-1/2} n^{\nu/2} n^\phi \left\{ 1_{\Omega_n} \max_k |U_n(t_k) - U_n(t_{k-1})| \right\} \cdot \left\{ \frac{1}{[n^\phi]} \sum_{k=1}^{[n^\phi]} \frac{R_r^\beta(S_n(t_k)) R_\zeta^{\alpha+1}(S_n(t_k))}{R_r^{1/2}(S_n(t_k))} \right\}$$

$$\leq Cn^{-1/2+\nu/2+\phi} \left\{ 1_{\Omega_n} \max_k |U_n(t_k) - U_n(t_{k-1})| \right\} \cdot \left\{ \int_0^1 R^{\alpha+\beta+1/2}(t) dt \right\}.$$

Choosing another parameter

$$\lambda = \frac{1}{2} - \frac{3}{2}\zeta(1 - 2\Delta) > \frac{1}{2} - \frac{3(1 - 2\Delta)}{5(1 - \Delta)} > 0, \tag{5.20}$$

it follows from (3.2) that

$$P\left(\Omega_n \cap \left\{ \max_k |U_n(t_k) - U_n(t_{k-1})| > n^{-\lambda} \right\}\right) \tag{5.21}$$

$$\leq P\left(\max_k \sup_{t_{k-1} \leq t \leq t_k} |U_n(S_n(t)) - U_n(t)| \geq n^{-\lambda} \right)$$

$$\leq \sum_{k=1}^{[n^\phi]} P\left(\sup_{t_{k-1} \leq s < t \leq t_k} |U_n(t) - U_n(s)| \geq n^{-\lambda} \right)$$

$$\leq Cn^{\phi+\rho} \exp\left(\frac{-An^{-2\lambda}}{n^{\rho-\phi}} \phi\left(\frac{Bn^{-\lambda}}{n^{1/2-\phi}} \right) \right) + n^\phi n n^{-\sigma}.$$

Condition (3.3) is indeed fulfilled, since $-\lambda - 1/2 + \sigma \wedge \tau > 0$ and $-\phi + \sigma \wedge \tau > 0$ as follows easily from the choices we made. In order to show that the upper bound in (5.21) converges to 0 as $n \to \infty$ note that $\phi - \rho - 2\lambda = 1 - (5/2)\zeta(1 - 2\Delta) - \rho/2 - 1 + 3\zeta(1 - 2\Delta) > 0$ by (2.23), and that

$$\phi - \frac{1}{2} - \lambda = 1 - \frac{5}{2}\zeta(1 - 2\Delta) - \frac{1}{2} - \frac{1}{2} + \frac{3}{2}\zeta(1 - 2\Delta) < 0.$$

Combining these results we may finally conclude that

$$|D_{n1}| = O_p(n^{-1/2+\nu/2+\phi-\lambda}) = o_p(1) \quad \text{as} \quad n \to \infty, \tag{5.22}$$

because

$$\begin{aligned}\frac{1}{2}\nu+\phi-\frac{1}{2}-\lambda &= \frac{1}{2}\zeta(1-2\Delta)+1-\frac{5}{2}\zeta(1-2\Delta)+\frac{1}{2}\rho-\frac{1}{2}-\frac{1}{2}+\frac{3}{2}\zeta(1-2\Delta)\\ &< \frac{1}{2}\zeta(1-2\Delta)-\frac{5}{2}\zeta(1-2\Delta)+\frac{1}{2}\zeta(1-2\Delta)+\frac{3}{2}\zeta(1-2\Delta)=0.\end{aligned}$$

By taking expectations of a suitable upper bound it follows that D_{n2} may be dealt with in a similar way as D_{n3}, so that we restrict attention to the latter random variable. With the intervals $I(j)$ as in (5.1) it can be shown as in the proof of (5.10) that the parts of D_{n3} corresonding to a restriction of the integral to either $I(1)$ or $I(3)$ converge to 0 in probability as $n \to \infty$. Hence it remains to consider the part of D_{n3} corresonding to a restriction of the integral to $I(2)$; this part will be decomposed into the sum of the random variables

$$H_{n1} = 1_{\Omega_n} \int_{I(2)} \{U_n(S_n(t)) - U_n(t)\} L_n(S_n(t))dt, \tag{5.23}$$

$$H_{n2} = 1_{\Omega_n} \int_{I(2)} U_n(t) \{L_n(S_n(t)) - L_n(t)\} dt, \tag{5.24}$$

where $L_n(t) = \Psi_{F,r}(t) J_\zeta^{(1)}(t)$, $t \in [0,1]$.

Due to the way in which we have defined S_n in (5.15) it follows that $|L_n(S_n(t))| \le CR_r^\beta(S_n(t)) R_\zeta^{\alpha+1}(S_n(t)) \le CR^{\alpha+\beta+1}(S_n(t)) \le CR^{\alpha+\beta+1}(t)$, for all $t \in I(2)$. Hence it is clear that

$$\begin{aligned}|H_{n1}| &\le C1_{\Omega_n}\left\{\sup_{0\le t\le 1}|U_n(S_n(t))-U_n(t)|\right\}\left\{\int_{I(2)}R^{\alpha+\beta+1}(t)dt\right\}\\ &\le C1_{\Omega_n}\left\{\sup_{0\le t\le 1}|U_n(S_n(t))-U_n(t)|\right\}n^{\zeta(\alpha+\beta)}.\end{aligned} \tag{5.25}$$

Reasoning as in (5.21) we see that

$$|H_{n1}| = O_p(n^{\zeta(\alpha+\beta)-\lambda}) = o_p(1) \quad \text{as} \quad n \to \infty, \tag{5.26}$$

because $\lambda - \zeta(\alpha+\beta) > 1/2 - (3/2)\zeta(1-2\Delta) - \zeta\Delta > (2-\Delta)/\{10(1-\Delta)\} > 0$ by substituting the upper bound for ζ in (2.13).

For H_{n2} let us apply the mean value theorem and employ the special construction of S_n to find

$$|L_n(S_n(t)) - L_n(t)| \le Cn^{-\phi} n^{\zeta(\alpha+\beta+2)}. \tag{5.27}$$

This entails

$$|H_{n2}| = O_p(n^{\nu/2-\phi+\zeta(\alpha+\beta+2)}) = o_p(1) \quad \text{as} \quad n \to \infty, \tag{5.28}$$

since $\phi - \zeta(\alpha+\beta+2) - \nu/2 \ge 1 - (5/2)\zeta(1-2\Delta) - \zeta(\Delta+2) - (1/2)\zeta(1-2\Delta) > 0$ by substituting the upper bound for ζ in (2.13).

6. Concluding remarks

A. *Sinplification variance*

In many special cases (see, e.g., Chanda *et al.* (1990)) the process $X_{1,m(n)}, \ldots, X_{n,m(n)}$ and hence the process

$$\xi_{1,m(n)}, \ldots, \xi_{n,m(n)} \text{ is strictly stationary.} \tag{6.1}$$

Note that this condition which is not implied by (1.1) and (1.2) will be needed below.

THEOREM 6.1. *Let* (6.1) *be satisfied. Under the conditions of Theorem* 2.1 *we have*

$$\tilde{v}^2 = \mathrm{Var}(\tilde{K}_n(\xi_1)) + 2 \lim_{n\to\infty} \sum_{j=2}^{m(n)} \mathrm{Cov}(\tilde{K}_n(\xi_1), \tilde{K}_n(\xi_j)). \tag{6.2}$$

PROOF. Relation (4.6) entails that $E|\tilde{Z}_{n1}\tilde{Z}_{nj}| \le C$, for some $C \in (0,\infty)$ independent of n, where the $\tilde{Z}_{nj}$ are defined in (2.26). Due to the $m(n)$-dependence $E\tilde{Z}_{nk}\tilde{Z}_{nj} = 0$ for $|k-j| > m(n)$. Writing $\tilde{v}_n^2 = E\tilde{Z}_{n1}^2 + 2\sum_{j=2}^{m(n)} E\tilde{Z}_{n1}\tilde{Z}_{nj}$ we see that

$$\left| \frac{1}{n} \mathrm{Var}\left(\sum_{i=1}^{n} \tilde{Z}_{ni} \right) - \tilde{v}_n^2 \right| = O(n^{\rho-1}) \to 0 \quad \text{as} \quad n \to \infty, \tag{6.3}$$

because $1-\rho > 1-(1-2\Delta)/\{5(1-\Delta)\} > 0$.

The next step is to show that we may replace $\tilde{Z}_{ni} = \tilde{K}_n(\xi_{i,m(n)}) - \tilde{\tilde{\mu}}_n$ by $\tilde{K}_n(\xi_i) - \tilde{\mu}_n$. For arbitrary $1 \le j \le n$, we have

$$\begin{aligned} &|E\{(\tilde{K}_n(\xi_1) - \tilde{\mu}_n)(\tilde{K}_n(\xi_j) - \tilde{\mu}_n) \\ &\quad - (\tilde{K}_n(\xi_{1,m(n)}) - \tilde{\tilde{\mu}}_n)(\tilde{K}_n(\xi_{j,m(n)}) - \tilde{\tilde{\mu}}_n)\}| \\ &\quad \le |E\{\tilde{K}_n(\xi_1)\tilde{K}_n(\xi_j) - \tilde{K}_n(\xi_{1,m(n)})\tilde{K}_n(\xi_{j,m(n)})\}| \\ &\qquad + |\tilde{\mu}_n - \tilde{\tilde{\mu}}_n||\tilde{\mu}_n + \tilde{\tilde{\mu}}_n| \\ &\quad \le E|\tilde{K}_n(\xi_j)||\tilde{K}_n(\xi_1) - \tilde{K}_n(\xi_{1,m(n)})| \\ &\qquad + E|\tilde{K}_n(\xi_{1,m(n)})||\tilde{K}_n(\xi_j) - \tilde{K}_n(\xi_{j,m(n)})| \\ &\qquad + |\tilde{\mu}_n - \tilde{\tilde{\mu}}_n||\tilde{\mu}_n + \tilde{\tilde{\mu}}_n|. \end{aligned} \tag{6.4}$$

By application of the mean value theorem we find that $|\tilde{\mu}_n - \tilde{\tilde{\mu}}_n||\tilde{\mu}_n + \tilde{\tilde{\mu}}_n| \le Cn^{-\sigma\wedge\tau + r(\beta+1)+\zeta\alpha}$, and that both expectations on the right in (6.4) are bounded by $Cn^{-\sigma\wedge\tau+2(r\beta+\zeta\alpha)+\tau}$. It is clear that we may carry out the replacement mentioned above, provided that

$$n^{-\sigma\wedge\tau+2(r\beta+\zeta\alpha)+r+\rho} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.5}$$

The assumed restrictions on the parameters entail that $\sigma \wedge \tau - 2(r\beta + \zeta\alpha) - r - \rho > 1 + 1/\{2(1-\Delta)\} - 2\beta/\{2(1-\Delta)\} - 2\alpha/\{5(1-\Delta)\} - 1/\{2(1-\Delta)\} - (1-2\Delta)/\{5(1-\Delta)\} > (4-8\Delta)/\{5(1-\Delta)\} > 0$ indeed. □

B. *Relations among parameters*

The parameter ρ measures how strong the dependence is among the first order parts of the sample elements, the i.i.d. case being covered by $\rho = 0$. Even in the i.i.d. case the value $\alpha + \beta$ or Δ should remain strictly below $1/2$ in order to make sure that the limiting variance exists. It is clear that this asymptotic variance will increase with ρ for constant $\alpha + \beta$. Therefore we expect that a large ρ requires a small Δ, a relationship which is indeed expressed by (2.23). The parameters τ, σ control the amount of noise around the tractable $m(n)$-dependent first order components of the sample elements. These noise components have a bearing on the values of random functions like $J_\zeta(\hat{F}_n(x))$ in (2.9). The larger τ and σ, the smaller the influence of the noisy parts and the larger the value of Δ that we can afford.

C. *Examples*

In the examples below we consider L-estimation of the symmetry point $\vartheta \in \mathbb{R}$, assuming that the X_i have the d.f. $F(\cdot - \vartheta)$ for some F symmetric around 0. We take for simplicity

$$\Psi(x) = x, \quad x \in \mathbb{R}, \qquad \text{so that} \quad \Psi_F(t) = F^{-1}(t), \quad t \in (0,1). \tag{6.6}$$

Due to symmetry we may focus on the left-hand tail of the distributions and on values of t near 0. Let α and β be the parameters in Assumption 2.2 that control the growth of $|J|$ and $|F^{-1}|$ near 0.

First let us take $f(x) \sim a|x|^{-1-1/\beta}$, as $x \to -\infty$, for some $0 < a < \infty$, which yields $|F^{-1}(t)| \sim bt^{-\beta}$, as $t \downarrow 0$, for some $0 < b < \infty$. For densities of this type the efficient score remains bounded, so that we may take any $0 < \zeta < 2/5$ and α arbitrarily close to 0 and hence β in the entire range $(0, \Delta)$.

Now let us take $f(x) \sim a \exp(-|x|^\gamma)$, as $x \to -\infty$, for some $0 < a < \infty$ and $\gamma > 1$. In this case both F^{-1} and the efficient scores are of logarithmic order, as $t \downarrow 0$, and hence any small but strictly positive value of α and any $0 < \zeta < 2/5$ are suitable where also a small strictly positive value of β suffices. We can exploit this freedom by allowing a strong dependence i.e. a large value for ρ in the range $[0, 2/5)$, see (2.23).

Finally let f be concentrated on a finite interval $[-A, A]$, so that the density has zero tails. More specifically let $f(x) \sim a(x+A)^{-1+1/\alpha}$. Then F^{-1} is bounded, of course, with $F^{-1}(t) \sim -A + (t/a)^\alpha$, as $t \downarrow 0$. The efficient score function is of order $t^{-\alpha}$, as $t \downarrow 0$. Since we can take β arbitrarily small positive, for α the range $(0, \Delta)$ is available.

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