Ann. Inst. Statist. Math.
Vol. 45, No. 4, 665–686 (1993)

# ORDER STATISTICS FOR NONSTATIONARY TIME SERIES

LANH TAT TRAN[1] AND BERLIN WU[2]

[1] *Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.*
[2] *National Chengchi University, Taipei, Taiwan*

(Received October 23, 1991; revised October 6, 1992)

**Abstract.** Order statistics has an important role in statistical inference. The main purpose of this paper is to investigate order statistics, and also explore its applications in the analysis of nonstationary time series. Our results show that linear functions of order statistics for a large class of time series are asymptotically normal. The methods of proof involve approximations of serially dependent random variables by independent ones. The problems of testing for the existence of a linear trend and the problem of testing randomness versus serial dependence are considered as applications.

*Key words and phrases*: Nonstationary, autoregressive processes, absolute regularity, empirical distribution functions, order statistics.

## 1. Introduction

Order statistics has an important role in statistical inference. Not much work on order statistics has been done in the case where the observations are serially dependent. Our main objective is to investigate order statistics for nonstationary time series. The principal motivation for this research is that order statistics has many applications in time series analysis. Later on in Section 4, we will discuss two important examples. The first example deals with testing for the existence of a linear trend. The second example concerns testing white noises versus serial dependence. This problem has been drawing increasing attention. See, for example, Hallin *et al.* (1985, 1987) and Chan and Tran (1992).

Order statistics often exhibit desirable robustness when the population distribution is heavy-tailed, for example, Cauchy distribution or contaminated normal. Our methods depend heavily on certain properties of serially dependent random variables. Results involving the limiting behavior of the empirical distributions of dependent r.v.'s normally require assumptions about their types of dependence. Throughout the paper, we assume that $X_t$ satisfies the absolute regularity condition, the definition of which will now be given.

Let $\{X_i, -\infty < i < \infty\}$ be a time series defined on a probability space $(\Omega, \mathcal{A}, P)$. For integers $a$, $b$ with $a \le b$, let $M_a^b$ denote the $\sigma$-algebra of events generated by $X_a, \ldots, X_b$. Then $\{X_i\}$ is absolutely regular if

$$\beta(n) = \sup_k E\{\sup |P(A \mid M_{-\infty}^k) - P(A)| : A \in M_{k+n}^{\infty}\} \downarrow 0 \tag{1.1}$$

as $n \to \infty$.

We assume that $\beta(n)$ decays to zero exponentially fast. This assumption is relatively weak. In fact, many time series are absolutely regular with $\beta(n)$ decaying to zero at an exponential rate. These include a large class of autoregressive moving average time series models and bilinear models as shown, respectively, by Pham and Tran (1985) and Pham (1986). For some statistical applications involving absolutely regular time series when an exponential rate of decay for $\beta(n)$ is assumed, see Chan and Tran (1992).

The absolute regularity condition is weaker than many other dependence conditions, e.g. $m$-dependence, $\phi$-mixing, $\Psi$-mixing, but is stronger than the strong mixing condition. Under more restrictive assumptions, the results can be extended to the strong mixing case with mixing coefficients decaying to zero at polynomial rates. However, the proofs are more technically involved and harder to follow. In fact, Volkonskii and Rozanov (1959) have pointed out that the condition of absolute regularity is more suitable for research than the strong mixing condition. The absolutely regular case with exponential rate is thus chosen as a compromise between generality and simplicity.

In the independent case, the literature on order statistics is extensive. See, for example, Wellner (1977*a*, 1977*b*) and the references therein. Order statistics for stationary mixing processes have been investigated by Mehra and Rao (1975) and Puri and Tran (1980). The technique in Puri and Tran is based on approximations of dependent r.v.'s by supermartingales, whereas more tractable approximations by independent r.v.'s are employed in the present paper. Approximations by supermartingales are too complicated for the nonstationary setting considered here.

Assume $X_i$ has a continuous distribution function $F_i$. Denote $\overline{F}_n = (F_1 + F_2 + \cdots + F_n)/n$. Let $X_{i:n}$ be the $i$-th order statistic of $X_1, \ldots, X_n$. Let $c_{1n}, \ldots, c_{nn}$ be arbitrary given constants and let $g_n : R \to R$ be a measurable function. Consider the statistic

$$T_n = \sum_{i=1}^{n} c_{in} g_n(X_{i:n}). \tag{1.2}$$

Our main results show that $T_n$ is asymptotically normal under the four Assumptions A1–A4 stated in Section 2. Assumptions A1–A3 are standard in the study of order statistics. A4 is needed to handle the nonstationary aspects of the time series. This assumption is satisfied by a large class of time series as shown by the examples in Section 4. An interesting open question is whether this assumption can be omitted. The derivation of the asymptotic distribution of $T_n$ employs a decomposition of (1.2) into a leading term and a remainder term. The asymptotic normality of the leading term is shown in Section 2. The remainder term is shown to be asymptotically negligible in Section 3. The main argument relies on some important properties of empirical distributions, which are also of independent interest. For background material on empirical distributions, the reader is referred to

Zuijlen (1978) and Alexander (1984). Our results extend those of Zuijlen (1976), Ruymgaart and Zuijlen (1977) for non-i.i.d r.v.'s to nonstationary time series. Applications to time series analysis are considered in Section 4. Nonparametric procedures for time series have been the subject of much recent attention. They are popular when there is evidence of nonnormality of distributions. Evidence of nonnormality appears in certain exchange rates and stock market prices. See Fama (1965), Mandelbrot (1967) and Dufour (1982). For a bibliography on nonparametric methods in time series, see, for example, Dufour (1982), Hallin *et al.* (1985, 1987), or Tran (1988). Throughout the paper, the letter $C$ will be used to denote constants whose values are unimportant and may vary.

## 2. Asymptotic normality of the leading term

Let $\Gamma_n$ denote the empirical distribution function of $X_1, \ldots, X_n$, that is,

$$\Gamma_n(t) = (1/n) \sum_{i=1}^{n} u(t - X_i), \tag{2.1}$$

where $u$, the indicator function, is defined later in (2.11). Then $T_n$ can be written as the following functional of empirical distribution function:

$$T_n = \int J_n(\Gamma_n(t)) g_n(t) d\Gamma_n(t), \tag{2.2}$$

where the range of integration is $(-\infty, \infty)$, and

$$J_n(i/n) = nc_{in}, \qquad \text{for} \quad 1 \leq i \leq n. \tag{2.3}$$

Following Ruymgaart and Zuijlen (1977), we will restrict our attention to $T_n$ of type (2.2) with $J_n(i/n) = J(i/(n+1))$, $g_n = g$ for some functions $J$, $g$. Then

$$T_n = \int J(\hat{\Gamma}_n(t)) g(t) d\Gamma_n(t),$$

where,

$$\hat{\Gamma}_n = [n/(n+1)]\Gamma_n. \tag{2.4}$$

Define

$$h_n(s) = g(\overline{F}_n^{-1}(s)), \tag{2.5}$$

where $\overline{F}_n^{-1}$ is the left continuous version of $\overline{F}_n$. We will occasionally use one or more of the following assumptions:

A1. The function $J$ may have discontinuities of the first kind at $s_j$ for $j = 1, 2, \ldots, k$ (where we take $s_0 = 0$ and $s_{k+1} = 1$), and has a continuous first derivative on $(s_{j-1}, s_j)$ for each $1 \leq j \leq k+1$.

A2. The function $g$ is left continuous and is continuous in a neighborhood of each $t_j$. Here $t_j = \overline{F}_n^{-1}(s_j)$, and $s_j$ is a discontinuity point of $J$ as stated in A1.

A3. There exist positive constants $C$ and $a$, $b$ with $a + b < 1/2$ such that $|J(s)| \le CR^a(s)$, $|J'(s)| \le CR^{a+1}(s)$, and $|h_n(s)| \le CR^b(s)$ for all large $n$, where $R(s) = [s(1-s)]^{-1}$.

A4. There exists an interval $[a, b]$ and some positive constants $C_1$, $C_2$ such that for all $1 \le i < \infty$, $1 \le n \le \infty$, and for all $x \le a$ or $x \ge b$,

$$C_1 \le \frac{F_i(x)(1 - F_i(x))}{\overline{F}_n(x)(1 - \overline{F}_n(x))} \le C_2.$$

Let

$$\mu_n = \int J(\overline{F}_n(t))g(t)d\overline{F}_n(t). \tag{2.6}$$

Decompose $T_n$ as follows:

$$n^{1/2}(T_n - \mu_n) = A_n + B_n + C_n + R_n \qquad \text{w.p.1}, \tag{2.7}$$

where

$$A_n = n^{1/2} \int J(\overline{F}_n(t))g(t)d(\Gamma_n(t) - \overline{F}_n(t)), \tag{2.8}$$

$$B_n = n^{1/2} \int [\Gamma_n(t) - \overline{F}_n(t)]J'(\overline{F}_n(t))g(t)d\overline{F}_n(t), \tag{2.9}$$

$$C_n = n^{1/2} \sum_{j=1}^{k} \alpha_j g(t_j)[\Gamma_n(t_j) - \overline{F}_n(t_j)]. \tag{2.10}$$

Let

$$u(x) = 0 \text{ or } 1 \qquad \text{as} \quad x < 0 \text{ or } x \ge 0, \tag{2.11}$$

$$\mu_{in} = \int J(\overline{F}_n(t))g(t)dF_i(t), \tag{2.12}$$

$$A_{in} = J(\overline{F}_n(X_i))g(X_i) - \mu_{in}, \tag{2.13}$$

$$B_{in} = \int [u(\overline{F}_n(t) - \overline{F}_n(X_i)) - \overline{F}_n(t)]J'(\overline{F}_n(t))g(t)d\overline{F}_n(t), \tag{2.14}$$

$$C_{in} = \sum_{j=1}^{k} \alpha_j g(t_j)[u(\overline{F}_n(t_j) - \overline{F}_n(X_i)) - \overline{F}_n(t_j)]. \tag{2.15}$$

Then

$$A_n + B_n + C_n = n^{-1/2} \sum_{i=1}^{n} (A_{in} + B_{in} + C_{in}), \tag{2.16}$$

$$n^{1/2}(T_n - \mu_n) = n^{-1/2} \sum_{i=1}^{n} (A_{in} + B_{in} + C_{in}) + R_n. \tag{2.17}$$

LEMMA 2.1. *For some* $\delta > 0$, $E|A_{in}|^{2+\delta} \le C(\delta) E|R^{(a+b)(2+\delta)}(\overline{F}_n(X_i))|$.

PROOF. Note that

$$\begin{aligned} E|A_{in}|^{2+\delta} &\le E|J(\overline{F}_n(X_i))g(X_i) - \mu_{in}|^{2+\delta} \\ &\le C(\delta)(E|J(\overline{F}_n(X_i))g(X_i)|^{2+\delta} + |\mu_{in}|^{2+\delta}) \\ &\le C(\delta)E|J(\overline{F}_n(X_i))g(X_i)|^{2+\delta} \\ &\le C(\delta)E|R^{(a+b)(2+\delta)}(\overline{F}_n(X_i))|. \end{aligned} \tag{2.18}$$

□

LEMMA 2.2. *For some* $\delta > 0$, $E|B_{in}|^{2+\delta} \le CE|R^{((1/2)-\delta)(2+\delta)}(\overline{F}_n(X_i))|$.

PROOF. For any $s, t \in (0,1)$,

$$|u(t-s) - t| \le CR^{(1/2)-\delta}(s)R^{(-1/2)+\delta}(t), \tag{2.19}$$

where $C$ is a positive constant independent of $\delta$ (see Lemma 2.2.1 of Ruymgaart (1973)). Employing (2.19),

$$\begin{aligned} &E|B_{in}|^{2+\delta} \\ &\quad\le CE\left|\int R^{(1/2)-\delta}(\overline{F}_n(X_i))R^{(-1/2)+\delta}(\overline{F}_n(t)) \right. \\ &\qquad\qquad \left. J'(\overline{F}_n(t))g(t)d\overline{F}_n(t)\right|^{2+\delta} \\ &\quad\le CE|R^{(1/2)-\delta}(\overline{F}_n(X_i))|^{2+\delta} \\ &\qquad \cdot\left|\int R^{(-1/2)+\delta}(\overline{F}_n(t))J'(\overline{F}_n(t))g(t)d\overline{F}_n(t)\right|^{2+\delta} \\ &\quad\le CE|R^{(1/2)-\delta}(\overline{F}_n(X_i))|^{2+\delta} \\ &\qquad \cdot\left|\int R^{(-1/2)+\delta}(s)J'(s)g(\overline{F}_n^{-1}(s))ds\right|^{2+\delta} \\ &\quad\le CE|R^{(1/2)-\delta}(\overline{F}_n(X_i))|^{2+\delta}\left|\int R^{-(1/2)+\delta+a+1+b}(s)ds\right|^{2+\delta} \\ &\quad\le CE|R^{(1/2)-\delta}(\overline{F}_n(X_i))|^{2+\delta}, \end{aligned} \tag{2.20}$$

since $|\int R^{-(1/2)+\delta+a+1+b}(s)ds|^{2+\delta}$ is bounded by a constant for $\delta < (1/2)-(a+b)$.

□

LEMMA 2.3. *For some* $\delta > 0$, $E|C_{in}|^{2+\delta} \le C\sum_{j=1}^k |\alpha_j g(t_j)|^{2+\delta}$.

PROOF. Note that

$$E|C_{in}|^{2+\delta} \le E\left|\sum_{j=1}^k \alpha_j g(t_j)[u(\overline{F}_n(t_j) - \overline{F}_n(X_i)) - \overline{F}_n(t_j)]\right|^{2+\delta} \tag{2.21}$$

$$\leq C \left| \sum_{j=1}^{k} E\alpha_j g(t_j)[u(\overline{F}_n(t_j) - \overline{F}_n(X_i)) - \overline{F}_n(t_j)] \right|^{2+\delta}$$

$$\leq C \sum_{j=1}^{k} |\alpha_j g(t_j)|^{2+\delta},$$

hence the proof of Lemma 2.3 is complete. □

Define $\sigma_n^2 = \mathrm{Var}(A_n + B_n + C_n)$.

THEOREM 2.1. *Let $\{X_i, -\infty < i < \infty\}$ be a sequence of absolutely regular random variables with $\beta(n) = O(e^{-\zeta n})$ for some $\zeta > 0$. Suppose* $\liminf \sigma_n^2 > 0$ *as $n \to \infty$, and* A1–A4 *hold. Then*

$$n^{1/2}(T_n - \mu_n)/\sigma_n \xrightarrow{L} N(0,1). \tag{2.22}$$

PROOF. Let

$$S_n = \sum_{i=1}^{n} (A_{in} + B_{in} + C_{in})$$

and

$$V_{in} = A_{in} + B_{in} + C_{in} - E(A_{in} + B_{in} + C_{in}).$$

We will occasionally drop the subscript $n$ in $V_{in}$ for simplicity when there is no fear of confusion. It is not hard to show that

$$E(S_n) = 0. \tag{2.23}$$

Thus

$$n^{1/2}(T_n - \mu_n) = n^{-1/2} \sum_{i=1}^{n} V_{in} + R_n.$$

Let $p$ and $q$ be positive integers with $p = [n^{\delta/(2(2+\delta))}/\log n]$ and $q = [c \log n]$ (with $c$ to be prescribed later on). We now set the random variables $V_{in}$ into alternate blocks of size $p$ and $q$. Let $m$ be the number of blocks of size $p$. Note that $m = [n/(p+q)]$. The sum of the $V_{in}$ is decomposed into three sums, the sum in $m$ blocks of size $p$, the sum in $m-1$ blocks of size $q$ and the sum of the remaining random variables. The sum for the large blocks of size $p$ is

$$S_{1n} = \sum_{i=1}^{m} \sum_{s=1}^{p} V_{(i-1)(p+q)+s}. \tag{2.24}$$

The sum for the small blocks of size $q$ is

$$S_{2n} = \sum_{i=1}^{m-1} \sum_{t=1}^{q} V_{(i-1)(p+q)+p+t}, \tag{2.25}$$

and the sum of the remaining $V_{in}$'s is

$$\sum_{s=(m-1)(p+q)+p+1}^{n} V_{sn}. \tag{2.26}$$

Hence

$$n^{-1/2}\sum_{i=1}^{n} V_{in} = n^{-1/2}(S_{1n} + S_{2n} + S_{3n}). \tag{2.27}$$

Then, following the argument in Theorem 2 of Yoshihara (1978), for $\epsilon > 0$ and $n$ sufficiently large, it is not hard to have the following inequalities

$$\begin{aligned} &P[n^{-1/2}(S_{1n}^*/\sigma_n) \le x-\epsilon] - P[n^{-1/2}(S_{2n}+S_{3n})/\sigma_n > \epsilon] - 2m\beta(q) \\ &\le P[n^{-1/2}S_n/\sigma_n \le x] \\ &\le P[n^{-1/2}(S_{1n}^*/\sigma_n) \le x+\epsilon] + P[n^{-1/2}(S_{2n}+S_{3n})/\sigma_n > \epsilon] \\ &\quad + 2m\beta(q), \end{aligned} \tag{2.28}$$

where $S_{1n}^*$ is a sum of $m$ independent random variables $Z_1, \ldots, Z_m$ with

$$Z_i \overset{L}{\sim} \sum_{s=1}^{p} V_{(i-1)(p+q)+s}. \tag{2.29}$$

By Theorem 3.2 to be proved later in Section 3, $R_n$ tends to zero in probability as $n \to \infty$. We next show the following:

(i) $P[n^{-1/2}(S_{1n}^*/\sigma_n) \le x-\epsilon] \to \Phi(x-\epsilon)$,
(ii) $P[n^{-1/2}(S_{1n}^*/\sigma_n) \le x+\epsilon] \to \Phi(x+\epsilon)$,
(iii) $P[n^{-1/2}(S_{2n}+S_{3n})/\sigma_n > \epsilon] \to 0$.

The proof of Theorem 2.1 will then follow by letting $\epsilon \to 0$ and using the fact that $n\beta(q) = ne^{-\zeta[c\log n]} \to 0$ (for sufficiently large $c$). The proof of (i) and (ii) are similar. We will just prove (i) and (iii).

PROOF OF (i). Let $(\sigma_n^*)^2$ be the variance of $n^{-1/2}S_{1n}^*$.

*Claim* 1. $[\sigma_n^2/(\sigma_n^*)^2] \to 1$ as $n \to \infty$.

Assume for the moment that Claim 1 is true. By the Lyapounov central limit theorem,

$$P[n^{-1/2}(S_{1n}^*/\sigma_n^*) \le x-\epsilon] \to \Phi(x-\epsilon)$$

if

$$\lim_{n\to\infty} n^{-1-(\delta/2)}(\sigma_n^*)^{-(2+\delta)}\sum_{i=1}^{m} E|Z_i|^{2+\delta} = 0. \tag{2.30}$$

By Claim 1 and the assumption that $\liminf_{n\to\infty}\sigma_n^2 > 0$, to prove (2.30), it is sufficient to show

$$\lim_{n\to\infty} n^{-1-(\delta/2)}\sum_{i=1}^{m} E|Z_i|^{2+\delta} = 0. \tag{2.31}$$

Note that $Z_i \overset{L}{\sim} \sum_{s=1}^{p} V_{(i-1)(p+q)+s}$. Thus we need to show

$$\lim_{n\to\infty} n^{-1-(\delta/2)} \sum_{i=1}^{m} E\left|\sum_{s=1}^{p} V_{(i-1)(p+q)+s}\right|^{2+\delta} = 0, \tag{2.32}$$

where

$$V_i = V_{in} = A_{in} + B_{in} + C_{in} - E(A_{in} + B_{in} + C_{in}).$$

Clearly, by the $C_r$-inequality

$$\begin{aligned}\sum_{i=1}^{m} E\left|\sum_{s=1}^{p} V_{(i-1)(p+q)+s}\right|^{2+\delta} \\ \leq \sum_{i=1}^{m} p^{2+\delta} \sum_{s=1}^{p} E|V_{(i-1)(p+q)+s}|^{2+\delta} \leq p^{2+\delta} \sum_{i=1}^{n} E|V_{in}|^{2+\delta}.\end{aligned} \tag{2.33}$$

Now

$$\begin{aligned} E|V_{in}|^{2+\delta} &= E|A_{in} + B_{in} + C_{in} - E(A_{in} + B_{in} + C_{in})|^{2+\delta} \\ &\leq C(\delta)(E|A_{in} - EA_{in}|^{2+\delta} + E|B_{in} - EB_{in}|^{2+\delta} \\ &\quad + E|C_{in} - EC_{in}|^{2+\delta}).\end{aligned} \tag{2.34}$$

Again by the $C_r$-inequality, for any random variable $X$ with finite $(2+\delta)$ moment,

$$E|X - EX|^{2+\delta} \leq C(\delta)(E|X|^{2+\delta} + |EX|^{2+\delta}) \leq C(\delta)E|X|^{2+\delta}. \tag{2.35}$$

Thus by (2.34) and (2.35),

$$E|V_{in}|^{2+\delta} \leq C(\delta)(E|A_{in}|^{2+\delta} + E|B_{in}|^{2+\delta} + E|C_{in}|^{2+\delta}). \tag{2.36}$$

It follows from (2.35), (2.36), Lemmas 2.1, 2.2 and 2.3 that

$$\begin{aligned} p^{2+\delta} \sum_{i=1}^{n} E|V_{in}|^{2+\delta} &\leq C(\delta)p^{2+\delta} \sum_{i=1}^{n} \int |J(\overline{F}_n(t))g(t)|^{2+\delta} dF_i(t) \\ &\quad + C(\delta)p^{2+\delta} \sum_{i=1}^{n} E|R^{(1/2)-\delta}(\overline{F}_n(X_i))|^{2+\delta} \\ &\quad + C(\delta)p^{2+\delta} \sum_{i=1}^{n} \left|\sum_{j=1}^{k} \alpha_j g(t_j)\right|^{2+\delta} \\ &\leq C(\delta)p^{2+\delta} n \int |J(\overline{F}_n(t))g(t)|^{2+\delta} d\overline{F}_n(t) \\ &\quad + C(\delta)p^{2+\delta} n \int |R^{(1/2)-\delta}(\overline{F}_n(t))|^{2+\delta} d\overline{F}_n(t) \\ &\quad + C(\delta)p^{2+\delta} n \\ &\leq C(\delta)p^{2+\delta} n \int_0^1 |J(s)g(\overline{F}_n^{-1}(s))|^{2+\delta} ds \\ &\quad + C(\delta)p^{2+\delta} n + C(\delta)p^{2+\delta} n \\ &\leq C(\delta)p^{2+\delta} n.\end{aligned} \tag{2.37}$$

We finally have

$$\lim n^{-1-(\delta/2)} \sum_{i=1}^{m} E \left| \sum_{s=1}^{p} V_{(i-1)(p+q)+s} \right|^{2+\delta} \leq C(\delta) \lim n^{-(\delta/2)} p^{2+\delta},$$

which tends to zero since $p = [n^{\delta/(2(2+\delta))}/\log n]$. The proof of (2.32) is now completed.

We now turn to the proof of Claim 1. Note that

$$\begin{aligned}(2.38)\quad n(\sigma_n^2 - (\sigma_n^*)^2) = {} & \operatorname{Var}(S_{1n}) - \operatorname{Var}(S_{1n}^*) + \operatorname{Var}(S_{2n}) + \operatorname{Var}(S_{3n}) \\ & + \operatorname{Cov}(S_{1n}, S_{2n}) + \operatorname{Cov}(S_{1n}, S_{3n}) + \operatorname{Cov}(S_{2n}, S_{3n}).\end{aligned}$$

We will show

(a) $n^{-1}(\operatorname{Var}(S_{1n}) - \operatorname{Var}(S_{1n}^*)) \to 0$,
(b) $n^{-1}\operatorname{Var}(S_{2n}) \to 0$,
(c) $n^{-1}\operatorname{Cov}(S_{1n}, S_{2n}) \to 0$,
(d) $n^{-1}(\operatorname{Var}(S_{3n}) + \operatorname{Cov}(S_{1n}, S_{3n}) + \operatorname{Cov}(S_{2n}, S_{3n})) \to 0$.

Claim 1 will then follow. The proof of (d) is similar to the proofs of (a), (b), (c) and is omitted. Consider first the proof of (a). Note that by A4 and Lemmas 2.1, 2.2 and 2.3,

$$\sup_{1 \leq i \leq n} \|V_i\|_{2+\delta} \leq C.$$

By the Davydov inequality (e.g. see Lemma 3.2 in Yoshihara (1984)), for $i \neq j$,

$$|\operatorname{Cov}(V_i, V_j)| \leq 10 \|V_i\|_{2+\delta} \|V_j\|_{2+\delta} \{\beta(|i-j|)\}^{\delta/(2+\delta)} \leq C\{\beta(|i-j|)\}^{\delta/(2+\delta)}.$$

Also for $1 \leq t,\ s \leq p$, $|(k-i)(p+q) + (t-s)| \geq (|k-i|-1)(p+q) + q$. Hence, for any two distinct blocks $i$ and $k$ of size $p$,

$$\begin{aligned}(2.39)\quad & \sum_{s=1}^{p} \sum_{t=1}^{p} \operatorname{Cov}(V_{(i-1)(p+q)+s}, V_{(k-1)(p+q)+t}) \\ & \leq \sum_{s=1}^{p} \sum_{t=1}^{p} C(\delta) \{\beta((|k-i|-1)(p+q)+q)\}^{\delta/(2+\delta)} \\ & \leq C(\delta) p^2 \{\beta(q)\}^{\delta/(2+\delta)}.\end{aligned}$$

Let $h = \zeta\delta/(2+\delta)$. Employing (2.39),

$$\begin{aligned}(2.40)\quad n^{-1}(\operatorname{Var}(S_{1n}) - \operatorname{Var}(S_{1n}^*)) & \leq n^{-1} \sum_{\substack{i=1 \\ i \neq k}}^{m} \sum_{k=1}^{m} C(\delta) p^2 \{\beta(q)\}^{\delta/(2+\delta)} \\ & \leq n^{-1} C(\delta) p^2 m^2 \{\beta(q)\}^{\delta/(2+\delta)} \\ & \leq n^{-1} C(\delta) p^2 m^2 e^{-hq} \\ & \leq C(\delta) n^{-1} p^2 m^2 e^{-h[c \log n]},\end{aligned}$$

which goes to zero for sufficiently large $c$.

We next prove (b). By the same argument used in the proof of (2.39), clearly for any two distinct $q$ blocks $i$ and $k$,

$$\begin{aligned}(2.41)\qquad &\sum_{s=1}^{q}\sum_{t=1}^{q}\operatorname{Cov}(V_{(i-1)(p+q)+p+s}, V_{(k-1)(p+q)+p+t})\\ &\qquad \le C(\delta)q^2\{\beta((|k-i|-1)(p+q)+p)\}^{\delta/(2+\delta)}.\end{aligned}$$

Also for the $i$-th short block of size $q$,

$$(2.42)\qquad \operatorname{Var}\left(\sum_{s=1}^{q} V_{(i-1)(p+q)+p+s}\right) \le C(\delta)q^2.$$

From (2.41) and (2.42), we have

$$\begin{aligned}&n^{-1}\operatorname{Var}(S_{2n})\\ &\quad \le n^{-1}\left(\sum_{\substack{i=1\\ i\neq k}}^{m}\sum_{k=1}^{m} C(\delta)q^2\{\beta((|k-i|-1)(p+q)+p)\}^{\delta/(2+\delta)} + \sum_{i=1}^{m-1} C(\delta)q^2\right).\end{aligned}$$

Following a similar computation as in (2.40), the proof of (b) follows. Note that $n^{-1}(m-1)q^2 \le q^2/(p+q)$, which converges to zero as $n \to \infty$ since $p = [n^{\delta/(2(2+\delta))}/\log n]$ and $q = [c\log n]$. Turning now to the proof of (c), for a long block (of size $p$) $i$ and a short block (of size $q$) $k$, we have

$$\begin{aligned}(2.43)\quad &\sum_{s=1}^{p}\sum_{t=1}^{q}\operatorname{Cov}(V_{(i-1)(p+q)+s}, V_{(k-1)(p+q)+p+t})\\ &\qquad \le \sum_{s=1}^{p}\sum_{t=1}^{q} C(\delta)\{\beta(t+p-s)\}^{\delta/(2+\delta)} \le C(\delta)\sum_{s=1}^{p}\sum_{t=1}^{q} e^{-h(t+p-s)},\end{aligned}$$

if $k = i$ or $k = i+1$; otherwise

$$\begin{aligned}(2.44)\quad &\sum_{s=1}^{p}\sum_{t=1}^{q}\operatorname{Cov}(V_{(i-1)(p+q)+s}, V_{(k-1)(p+q)+p+t})\\ &\qquad \le \sum_{s=1}^{p}\sum_{t=1}^{q} C(\delta)\{\beta((|k-i|+1)(p+q))\}^{\delta/(2+\delta)} \le pqC(\delta)e^{-h(p+q)}.\end{aligned}$$

Finally

$$\begin{aligned}n^{-1}\operatorname{Cov}(S_{1n}, S_{2n}) &\le n^{-1}\left(\sum_{i=1}^{m} 2C(\delta) + \sum_{\substack{i=1\\ i\neq k}}^{m}\ \sum_{\substack{k=1\\ i\neq k+1}}^{m-1} pqC(\delta)e^{-h(p+q)}\right)\\ &\le n^{-1}(mC(\delta) + m^2pqC(\delta)e^{-h(p+q)})\\ &\le C(\delta)(1/(p+q) + m^2pqe^{-h(p+q)}),\end{aligned}$$

which converges to zero as $n \to \infty$.

PROOF OF (iii). Observe that

$$(2.45) \qquad P[n^{-1/2}(S_{2n}+S_{3n})/\sigma_n > \epsilon] \\ \le P[n^{-1/2}S_{2n}/\sigma_n > (\epsilon/2)] + P[n^{-1/2}S_{3n}/\sigma_n > (\epsilon/2)].$$

We will only show that the first term on the right hand side of (2.45) goes to zero, since the proof of the second term is similar. Using an argument similar to that of (2.28) and (2.29), we have

$$(2.46) \qquad P[n^{-1/2}S_{2n}/\sigma_n > (\epsilon/2)] \le P\left[n^{-1/2}\sum_{i=1}^{m-1} Y_i/\sigma_n > (\epsilon/2)\right] + 2m\beta(p),$$

where the $Y_1, Y_2, \ldots, Y_{m-1}$ are independent random variables and $Y_i$ has the same distribution as $\sum_{t=1}^{q} V_{(i-1)(p+q)+t}$.

Finally, by Markov's inequality and an argument similar to (2.33), (2.37), we have

$$(2.47) \quad P\left[n^{-1/2}\sum_{i=1}^{m-1} Y_i/\sigma_n > (\epsilon/2)\right] + 2m\beta(p) \\ \le C(\delta)n^{-1}E\left|\sum_{i=1}^{m-1} Y_i\right|^2 + 2m\beta(p) \\ \le C(\delta)n^{-1}q\sum_{i=1}^{m-1} E|V_i|^2 + 2m\beta(p) \le C(\delta)n^{-1}qmn^{\delta/6} + 2m\beta(p),$$

which converges to zero as $n \to \infty$. □

## 3. Asymptotic negligibility of the remainder term

The following result will be needed in the sequel.

LEMMA 3.1. (Yoshihara (1978)) *Let $\{\xi_i\}$ be an (not necessarily stationary) absolutely regular sequence of random variables. Let $S_n = \xi_1 + \cdots + \xi_n$, $S_0 = 0$. For any $z > 0$ and any positive integer $r$ ($\le n$),*

$$(3.1) \qquad P[|S_n| \ge z] \le \sum_{j=1}^{r} P\left[|Y_j + Y_{j+r} + \cdots + Y_{j+k_j r}| \ge \frac{z}{r}\right] + 4n\beta(r).$$

*Here, for each $j$ ($1 \le j \le r$), $k_j$ is the largest integer for which $j + k_j r \le n$ and $\{Y_j\}$ are independent random variables defined on the probability space $(\Omega, \mathcal{A}, P)$ such that each $Y_j$ has the same d.f. as that of $\xi_j$.*

Assume $X_i$ has a continuous distribution function $F_i$. For $i = 1, \ldots, n$, define $X_i^* = \overline{F}_n(X_i)$. Denote by $F_i^*$ the distribution function of $X_i^*$ and by $\Gamma_n^*$ the empirical distribution function based on $X_1^*, \ldots, X_n^*$. Note that $F_1^*, \ldots, F_n^*$ are continuous distribution functions on $(0,1)$. Define the inverse function of $\overline{F}_n$ by

$$\overline{F}_n^{-1}(s) = \inf\{t : s \le \overline{F}_n(t)\}, \qquad 0 < s \le 1.$$

Then,

$$F_i^*(s) = F_i(\overline{F}_n^{-1}(s)) \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Let $\overline{F}_n^*(s) = (1/n)\sum_{i=1}^n F_i^*(s)$. It is easy to see that

$$\overline{F}_n^*(s) = \frac{1}{n}\sum_{i=1}^n F_i(\overline{F}_n^{-1}(s)) = \overline{F}_n(\overline{F}_n^{-1}(s)) = s. \tag{3.3}$$

Take $r = [c\log n]$, the integer part of $c\log n$, where $c$ is a large constant to be specified later on.

The following result due to van Zuijlen (1978) is needed:

LEMMA 3.2. *Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with $P[Z_i = 1] = 1 - P[Z_i = 0] = p_i$. Let $\overline{p} = (1/n)\sum_{i=1}^n p_i$. Then, for any $\alpha > 1/2$, there exists a constant $C$ independent of $n$ and $\overline{p}$ such that for all $n$*

$$E\left|\sum_{i=1}^n Z_i - n\overline{p}\right|^{2\alpha} \le C\{(n\overline{p}(1-\overline{p}))^\alpha + n\overline{p}(1-\overline{p})\}. \tag{3.4}$$

THEOREM 3.1. *Let $\{X_i, -\infty < i < \infty\}$ be an absolutely regular sequence of random variables. Let $\epsilon > 0$. Assume $\beta(n) = O(e^{-\zeta n})$ for some $\zeta > 0$. Then,*

(i) *for every $\nu \in (1, 3/2)$, there exists a $\theta \in (0,1)$ such that for all $n$,*

$$P[1 - ((1 - \overline{F}_n(x))/\theta)^{1/\nu} \le \Gamma_n(x) \le (\overline{F}_n(x)/\theta)^{1/\nu}, x \in (-\infty, \infty)] \ge 1 - \epsilon; \tag{3.5}$$

(ii) *for every $\nu \in (1, 2)$, there exists a $\theta \in (0,1)$ such that for all $n$,*

$$P[1 - \theta(1 - \overline{F}_n(x))^\nu \ge \Gamma_n(x) \ge \theta(\overline{F}_n(x))^\nu, x \in [X_{1:n}, X_{n:n}]] \ge 1 - \epsilon. \tag{3.6}$$

PROOF.

(i) Note that

$$\begin{aligned} &P[\Gamma_n(x) \le (\overline{F}_n(x)/\theta)^{1/\nu}, x \in (-\infty, \infty)] \\ &\quad = P[\Gamma_n(X_{i:n}) \le (\overline{F}_n(X_{i:n})/\theta)^{1/\nu}, 1 \le i \le n] \\ &\quad \ge 1 - \sum_{i=1}^n P[X_{i:n} \le \overline{F}_n^{-1}(\theta(i/n)^\nu)] \\ &\quad \ge 1 - \sum_{i=1}^n P[X_i \le \overline{F}_n^{-1}(\theta(i/n)^\nu)] = 1 - \sum_{i=1}^n P\left[\sum_{j=1}^n Z_{ij} \ge i\right], \end{aligned} \tag{3.7}$$

where for each $i$, $Z_{ij}$ ($j = 1, \dots, n$) are Bernoulli random variables with $p_{ij} = F_j(\overline{F}_n^{-1}(\theta(i/n)^\nu))$. The subscripts $i$ of $Z_{ij}$ and $p_{ij}$ will be dropped for simplicity. By Lemma 3.1, for $r = [c \log n]$, where $c$ is a positive number to be specified later,

$$(3.8) \qquad P\left[\sum_{j=1}^n Z_j \geq i\right] = P\left[\sum_{j=1}^n Z_j \geq \theta i + (1-\theta)i\right] \leq \sum_{j=1}^r P\left[S_j \geq s_j + \frac{(1-\theta)i}{r}\right] + 4n\beta(r),$$

where $s_j = p_j + p_{j+r} + \cdots + p_{j+k_j r}$ ($k_j$ is the largest integer such that $j + k_j r \leq n$), $S_j = Y_j + Y_{j+r} + \cdots + Y_{j+k_j r}$ and $Y_j$, $Y_{j+r}, \dots, Y_{j+k_j r}$ are independent random variables with the same distributions as that of $Z_j, Z_{j+r}, \dots, Z_{j+k_j r}$. To justify (3.8) we need to show $\sum_{j=1}^r s_j \leq \theta i$. But, it is easy to see that

$$\sum_{j=1}^r s_j \leq p_1 + \cdots + p_n = n\theta\left(\frac{i}{n}\right)^\nu \leq \theta i.$$

By Lemma 3.1 (for $\alpha = 2$), and (3.8)

$$\begin{aligned}(3.9) \quad P\left[\sum_{j=1}^n Z_j \geq i\right] &\leq \sum_{j=1}^r C((1-\theta)i/r)^{-4}(s_j^2 + s_j) + 4n\beta(r) \\ &\leq \sum_{j=1}^r C((1-\theta)i/r)^{-4}\left\{\left(n\theta\left(\frac{i}{n}\right)^\nu\right)^2 + n\theta\left(\frac{i}{n}\right)^\nu\right\} \\ &\quad + 4n\beta(r) \\ &\leq C\theta r^5 n^{1-\nu} i^{-4+2\nu} + 4n\beta(r).\end{aligned}$$

Note that $s_j \leq p_1 + p_2 + \cdots + p_n = n\theta(i/n)^\nu$. Finally, since $1 < \nu < 3/2$,

$$(3.10) \qquad \sum_{i=1}^n P\left[\sum_{j=1}^n Z_j \geq i\right] \leq C\theta(\log n)^5 n^{1-\nu} + 4n^2\beta(r),$$

which tends to zero as $n \to \infty$.

The proof of (3.5) then follows from (3.8), (3.10) and the symmetric result that comes from replacing $X_{i:n}$ by $-X_{i:n}$.

(ii) Obviously,

$$\begin{aligned}(3.11) \qquad & P[\Gamma_n(x) \geq \theta(\overline{F}_n(x))^\nu, x \in [X_{i:n}, X_{n:n})] \\ &= P[\Gamma_n(X_{i:n}) \geq \theta(\overline{F}_n(X_{i:n}))^\nu, 1 \leq i \leq n-1] \\ &\geq 1 - \sum_{i=1}^{n-1} P\left[X_{i:n} > \overline{F}_n^{-1}\left(\frac{i}{\theta n}\right)^{1/\nu}\right] \\ &\leq 1 - \sum_{i=1}^n P\left[\sum_{j=1}^n Z_j \geq n - i\right],\end{aligned}$$

where for each $i$, $Z_j$ $(1 \le j \le n)$ are Bernoulli r.v.'s with $p_j = 1 - F_j(\overline{F}_n^{-1}((i/\theta n)^{1/\nu}))$. Note that $\overline{p} = (p_1 + \cdots + p_i)/n = 1 - (i/\theta n)^{1/\nu}$ and that $n(i/n)^{1/\nu} > i$.

Clearly,

$$n - i - n\overline{p} = n(i/\theta n)^{1/\nu} - i = [(1/\theta)^{1/\nu} - 1]n(i/n)^{1/\nu}. \tag{3.12}$$

Employing (3.12) and the same argument in the proof of (i),

$$\begin{aligned} P\left[\sum_{j=1}^{n} Z_j \ge n - i\right] &\le P\left[\sum_{j=1}^{r}(Z_j + \cdots + Z_{j+k_j r} - s_j) \ge n - i - n\overline{p}\right] \\ &\le \sum_{j=1}^{r} P[S_j - s_j \ge (n - i - n\overline{p})/r] + 4n\beta(r) \\ &\le \sum_{j=1}^{r} C\{r^4(n - i - n\overline{p})^{-4}[n^2(1-\overline{p})^2 + n(1-\overline{p})]\} \\ &\quad + 4n\beta(r) \\ &\le C[n(i/n)^{1/\nu}]^{-4}[n^2(i/(n\theta))^{2/\nu} + n(i/(n\theta))^{1/\nu}] \\ &\le C\theta^{2/\nu} r^5 n^{-2+(2/\nu)} i^{-2/\nu} + 4n\beta(r). \end{aligned} \tag{3.13}$$

From (3.12)

$$\sum_{i=1}^{n} P\left[\sum_{j=1}^{n} Z_j \ge n - i\right] \le C\theta^{2/\nu}(\log n)^5 n^{-2+(2/\nu)} + 4n^2\beta(r) \tag{3.14}$$

for $1 < \nu < 2$ and large $C$. The proof of (3.6) is completed by (3.11), (3.13), (3.14) and symmetry.

We will need the following result of Yoshihara (1978). □

LEMMA 3.3. *Let $\{\eta_i\}$ be an absolutely regular sequence of random variables with $\beta(n)$. Let $g(x_1, \ldots, x_k)$ be a Borel function such that $|g(x_1, \ldots, x_k)| \le C$ for some constant $C$, where $(x_1, \ldots, x_k)$ is a point of $R^k$. Let $F^{(1)}$ and $F^{(2)}$ be distribution functions of random vectors $(\eta_{i_1}, \ldots, \eta_{i_j})$ and $(\eta_{i_{j+1}}, \ldots, \eta_{i_k})$, respectively, and $i_1 < i_2 < \cdots < i_k$. Then*

$$\Bigg| Eg(\eta_{i_1}, \ldots, \eta_{i_k}) - \int \underset{R^k}{\cdots} \int g(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) \\ dF^{(1)}(x_1, \ldots, x_j) dF^{(2)}(x_{j+1}, \ldots, x_k) \Bigg| \le 2C\beta(i_{j+1} - i_j).$$

COROLLARY 3.1. *Let $\{X_i, -\infty < i < \infty\}$ be an absolutely regular sequence of random variables satisfying the conditions of Theorem 3.1. Assume in addition that Assumption A4 is satisfied. Given $\epsilon > 0$,*

(i) *for every* $\nu \in (1, 3/2)$ *there exists* $\theta \in (0,1)$ *such that for all* $n$

$$[1-((1-\overline{F}_n(x))/\theta)^{1/\nu} \le \hat{\Gamma}_n(x) \le (\overline{F}_n(x)/\theta)^{1/\nu},\ x \in (-\infty, \infty)] \ge 1-\epsilon, \tag{3.15}$$

(ii) *for every* $\nu \in (1,2)$ *there exists* $\theta \in (0,1)$ *such that for all* $n \ge 1/\theta$,

$$P[1-\theta(1-\overline{F}_n(x))^{\nu} \ge \hat{\Gamma}_n(x) \ge \theta(\overline{F}_n(x))^{\nu}, x \in [X_{1:n}, X_{n:n}]] \ge 1-\epsilon. \tag{3.16}$$

PROOF. The proof of (3.15) is immediate from the definition of $\hat{\Gamma}_n$ and (3.1). From (3.6) of Theorem 3.1, we know that (3.15) holds if the interval $[X_{1:n}, X_{n:n}]$ is replaced by $[X_{1:n}, X_{n:n})$. It remains to show that

$$P\left[1-\theta(1-\overline{F}_n(X_{n:n}))^{\nu} \ge \frac{n}{n+1} \ge \theta(\overline{F}_n(X_{n:n}))^{\nu}\right] \ge 1-\epsilon.$$

But, for $0 < \theta \le n/(n+1)$,

$$P\left[\frac{n}{n+1} \ge \theta(\overline{F}_n(X_{n:n}))^{\nu}\right] \ge P\left[\frac{n}{n+1} \ge \frac{n}{n+1}(\overline{F}_n(X_{n:n}))^{\nu}\right] = 1.$$

On the other hand, we have

$$\begin{aligned} &P\left[1-\theta(1-\overline{F}_n(X_{n:n}))^{\nu} \ge \frac{n}{n+1}\right] \\ &= P\left[\overline{F}_n(X_{n:n}) \ge 1-\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right] \\ &= 1-P\left[X_{n:n} < \overline{F}_n^{-1}\left(1-\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right)\right] \\ &= 1-P\left[X_{n:n} < \overline{F}_n^{-1}\left(1-\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right)\right] \\ &= 1-P\left[\bigcap_{i=1}^{n}\left\{X_i < \overline{F}_n^{-1}\left(1-\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right)\right\}\right]. \end{aligned} \tag{3.17}$$

Denote $s_n = 1-(1/(n+1)\theta)^{1/\nu}$. By Lemma 3.3,

$$\begin{aligned} &P\left[\bigcap_{i=1}^{n}\left\{X_i < \overline{F}_n^{-1}(s_n)\right\}\right] \\ &\le P[X_1 < \overline{F}_n^{-1}(s_n), X_{1+r} < \overline{F}_n^{-1}(s_n), \ldots, X_{1+k_1 r} < \overline{F}_n^{-1}(s_n)] \\ &\le P[X_1 < \overline{F}_n^{-1}(s_n)]P[X_{1+r} < \overline{F}_n^{-1}(s_n), \ldots, X_{1+k_1 r} < \overline{F}_n^{-1}(s_n)] \\ &\quad + 2\beta(r) \\ &\le \cdots \le \prod_{i=1}^{k_1} P[X_{1+ir} < \overline{F}_n^{-1}(s_n)] + 2k_1\beta(r) \\ &= \prod_{i=1}^{k_1} F_{1+ir}\overline{F}_n^{-1}(s_n) + 2k_1\beta(r). \end{aligned} \tag{3.18}$$

Let $x_0$ be such that $\overline{F}_n(x_0) = s_n$. Then $(1/(n+1)\theta)^{1/\nu} = 1 - \overline{F}_n(x_0)$. Since $s_n \to 1$ as $n \to \infty$, for sufficiently large $n$,

$$1 - F_{1+ir}(x_0) \geq C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu},$$

by A4. Hence

$$F_{1+ir}(x_0) = 1 - (1 - F_{1+ir}(x_0)) \leq 1 - C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu},$$

implying

$$F_{1+ir}\overline{F}_n^{-1}(s_n) \leq 1 - C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}. \tag{3.19}$$

By (3.18) and (3.19),

$$P\left[\bigcap_{i=1}^{n}\left\{X_i < \overline{F}_n^{-1}(s_n)\right\}\right] \leq \left(1 - C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right)^{k_1} + 2k_1\beta(r). \tag{3.20}$$

Recall that $r = [c\log n]$. Thus

$$k_1 > Cn/\log n. \tag{3.21}$$

Hence,

$$\begin{aligned}\log\left(1 - C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right)^{k_1} &= k\log\left(1 - C\left(\frac{1}{(n+1)\theta}\right)^{1/\nu}\right) \\ &\leq -C(n/\log n)((n+1)\theta)^{-1/\nu},\end{aligned} \tag{3.22}$$

which tends to $-\infty$ since $\nu > 1$. Again, note that $2k_1\beta(r) \to 0$ as $n \to \infty$. By (3.18), (3.20), (3.22), the corollary follows. □

COROLLARY 3.2. *Assume the conditions of Corollary* 3.1 *is satisfied. Then, for every* $\epsilon > 0$ *and* $\nu \in (1,2)$ *there exists a constant* $C = C(\theta) > 0$, *such that for all* $n$

$$P\left[R(\Gamma_n(x)) \leq CR^{\nu}(\overline{F}_n(x)), x \in [X_{1:n}, X_{n:n})\right] \geq 1 - \epsilon, \tag{3.23}$$

*and*

$$P\left[R(\hat{\Gamma}_n(x)) \leq CR^{\nu}(\overline{F}_n(x)), x \in [X_{1:n}, X_{n:n}]\right] \geq 1 - \epsilon, \tag{3.24}$$

*where* $R(s)$ *is defined in Assumption* A3.

PROOF. Given $\epsilon > 0$, (3.6) of Theorem 3.1 ensures that for $\nu \in (1,2)$ there exists $\theta \in (0,1)$ such that for all $n$, the event

$$\Omega_\nu = \{\omega \in \Omega : \theta(\overline{F}_n(x))^\nu \leq \Gamma_n(x) \leq 1 - \theta(1-\overline{F}_n(x))^\nu, x \in [X_{1:n}, X_{n:n})\}$$

has a probability $P[\Omega_\nu] \geq 1-\epsilon$. Then, for each $\omega$ in $\Omega_\nu$,

$$R(\Gamma_n) = \frac{1}{\Gamma_n(1-\Gamma_n)} \leq \theta^{-2}(\overline{F}_n(1-\overline{F}_n))^{-\nu} = CR^\nu(\overline{F}_n). \tag{3.25}$$

Hence, the proof of (3.23) is completed. □

Finally, by using (3.15) of Corollary 3.1, the proof of (3.24) follows along the line of proof of (3.23).

Lemma 3.4 below is a generalization of Lemma 1.1.1 of Zuijlen ((1977), p. 12).

LEMMA 3.4. *Let $\{Z_i\}$ be an absolutely regular sequence of random variables with $P(Z_i = 1) = 1 - P(Z_i = 0) = p_i$. Assume $\beta(n) = O(e^{-\zeta n})$ for some $\zeta > 0$. For every $\alpha > 1/2$, there exists $C = C(\alpha) \in (0,\infty)$ such that for every $n$ and for $1/n \leq \overline{p} \leq 1 - 1/n$,*

$$E\left|\sum_{i=1}^n Z_i - n\overline{p}\right|^{2\alpha} \leq C(\log n)^{2\alpha+1}(n\overline{p}(1-\overline{p}))^\alpha. \tag{3.26}$$

PROOF. Let $G(t)$ be the distribution function of

$$\left|\sum_{i=1}^n Z_i - n\overline{p}\right|(n\overline{p}(-\overline{p}))^{-1/2}.$$

For each $j$ $(1 \leq j \leq r)$, denote $\overline{p}_j = (p_j + \cdots + p_{j+k_jr})/(k_j+1)$. Applying Lemma 3.1,

$$\begin{aligned} 1 - G(t) &= P\left[\left|\sum_{i=1}^n Z_i - n\overline{p}\right| > t(n\overline{p}(1-\overline{p}))^{1/2}\right] \\ &\leq \sum_{j=1}^r P\left[|Y_j + \cdots + Y_{j+k_jr} - (k_j+1)\overline{p}_j| \geq \frac{t(n\overline{p}(1-\overline{p}))^{1/2}}{r}\right] \\ &\quad + 4n\beta(r) \\ &= \sum_{j=1}^r P[|Y_j + \cdots + Y_{j+k_jr} - (k_j+1)\overline{p}_j| \geq (k_j+1)s] \\ &\quad + 4n\beta(r), \end{aligned} \tag{3.27}$$

where $Y_i$'s are independent random variables such that each $Y_i$ has the same distribution function as that of $Z_i$ and $s = t(n\overline{p}(1-\overline{p}))^{1/2}/(k_j+1)r$.

Lemma 3.2 ensures that it is sufficient to prove Lemma 3.4 when $p_j = \cdots = p_{j+k_j r} = \overline{p}_j$ and $1 \leq j \leq r$. By Bernstein's inequality (see Serfling ((1980), p. 95)), we have

$$
\begin{aligned}
(3.28)\quad & 1 - G(t) \\
&\leq \sum_{j=1}^{r} 2\exp\left(-\frac{(k_j+1)s^2}{2\overline{p}_j(1-\overline{p}_j)+s}\right) + 4n\beta(r) \\
&= \sum_{j=1}^{r} 2\exp\left(\frac{-t^2}{2r^2(k_j+1)\overline{p}_j(1-\overline{p}_j)(n\overline{p}(1-\overline{p}))^{-1} + rt(n\overline{p}(1-\overline{p}))^{-1/2}}\right) \\
&\quad + 4n\beta(r),
\end{aligned}
$$

which, by Lemma 3.1, is bounded by $2r\exp(-t^2/r(4r+t)) + 4n\beta(r)$.

Now, since $1/n \leq \overline{p} \leq 1 - 1/n$, we have

$$
\left|\sum_{i=1}^{n} Z_i - n\overline{p}\right| (n\overline{p}(1-\overline{p}))^{-1/2} \leq |n-1|\left(1-\frac{1}{n}\right)^{-1/2} \leq n.
$$

Then

$$
\begin{aligned}
(3.29)\qquad & E\left|\left(\sum_{i=1}^{n} Z_i - n\overline{p}\right)(n\overline{p}(1-\overline{p}))^{-1/2}\right|^{2\alpha} \\
&= 2\alpha\int_0^n t^{2\alpha-1}[1-G(t)]dt \\
&\leq 4\alpha r\int_0^1 dt + 4\alpha r\int_1^r t^{2\alpha-1}\cdot\exp\left(\frac{-t^2}{r(4r+t)}\right)dt \\
&\quad + 4\alpha r\int_r^n t^{2\alpha-1}\exp\left(-\frac{t^2}{r(4r+t)}\right)dt + 4n^{2\alpha+1}\beta(r).
\end{aligned}
$$

Since

$$
\int_1^r t^{2\alpha-1}\exp\left(-\frac{t^2}{r(4r+t)}\right)dt \leq \int_0^\infty t^{2\alpha-1}\exp(-t^2/9r^2)dt \leq \frac{\Gamma(\alpha)(3r)^{2\alpha}}{2},
$$

and

$$
\int_r^n t^{2\alpha-1}\exp\left(\frac{-t^2}{r(4r+t)}\right)dt \leq \int_0^\infty t^{2\alpha-1}\exp(-t/4r)dt \leq \Gamma(2\alpha)(5r)^{2\alpha},
$$

we obtain from (3.29),

$$
\begin{aligned}
& E\left|\left(\sum_{i=1}^{n} Z_i - n\overline{p}\right)(n\overline{p}(1-\overline{p}))^{-1/2}\right|^{2\alpha} \\
&\leq 4\alpha r + 4\alpha r\frac{\Gamma(\alpha)(3r)^{2\alpha}}{2} + 4\alpha r\Gamma(2\alpha)(5r)^{2\alpha} + 4n^{2\alpha+1}\beta(r) \\
&\leq Cr^{2\alpha+1} + Cn^{2\alpha+1}\beta(r) = C[c\log n]^{2\alpha+1} + Cn^{2\alpha+1}e^{-\zeta[c\log n]} \\
&\leq C(\log n)^{2\alpha+1}.
\end{aligned}
$$

□

THEOREM 3.2. *Assume the conditions of Theorem* 3.1 *are satisfied. Then* $R_n \to 0$ *in probability as* $n \to \infty$.

With the results of Section 3 presented in the present paper, the proof of Theorem 3.2 can then be obtained by a long but straightforward generalization of the results of Section 4 of Ruymgaart and Zuijlen (1977). The details are therefore omitted.

## 4. Applications to time series analysis

*Example* 1. (Testing for a linear trend.) Let $m$ be an arbitrary number. Consider the general time series model

$$Y_t = Z_t + mt + e_t, \qquad -\infty < t < \infty, \tag{4.1}$$

where $Z_t$ is a strictly stationary time series, $mt$ is a linear trend term; and $\{e_t\}$ is a sequence of white noises independent of $\{Z_t\}$. We assume that $Z_t$ is a strictly stationary times series satisfying the absolute regularity condition with $\beta(n) = O(e^{-\zeta n})$ for some $\zeta > 0$. Thus $Z_t$ might be a bilinear time series model or an autoregressive moving average time series model. Here $Z_t + mt$ is the actual process we are interested in. The observed process is $Y_t$. The white noises $\{e_t\}$ account for errors which may occur as a result of recording the data due to, for example, key punching or errors in measurement. Assume that $\{e_t\}$ is a sequence of independent but nonidentically distributed r.v.'s. This assumption is more realistic and general than the assumption that $\{e_t\}$ is stationary, since, the distribution of the errors caused by, for example, key punching, may vary with the days of the week. Model (4.1) is referred to as the additive effects outliers model. See Denby and Martin (1979). We assume that $e_{t+Pi}$, has the same distribution as $e_t$, for each integer $i$. Here $P$, the period, is a positive integer. We are interested in testing:

$$H_0 : m = 0 \qquad \text{v.s.} \qquad H_A : m > 0.$$

for definiteness, let us assume $Z_t = \theta Z_{t-1} + \epsilon_t$, where $0 < \theta < 1$; and where the $\epsilon_t$'s are i.i.d. r.v.'s with Cauchy density $f(x) = (a/\pi)(a^2 + x^2)^{-1}$ for some $a > 0$. For simplicity, assume $a = 1/2$ so that the distribution of $Z_t - Z_{t-1}$ has a standard Cauchy distribution as shown below.

Note that $Z_t - Z_{t-1}$ has the same distribution as $Z_2 - Z_1 = (\theta - 1)Z_1 + \epsilon_2$. The characteristic function of $Z_1$ and $\epsilon_2$ are respectively

$$\phi(u) = \exp(-a|u|/(1-\theta)) \qquad \text{and} \qquad \hat{\phi}(u) = \exp(-a|u|).$$

The characteristic function of $(\theta - 1)Z_1$ is

$$\exp(-a|u(\theta - 1)|/(1-\theta)) = \exp(-a|u|).$$

Since $Z_1$ and $\epsilon_2$ are independent, the characteristic function of $Z_2 - Z_1$ is $\exp(-2a|u|)$. Therefore the distribution of $Z_2 - Z_1$ is standard Cauchy since

$a = 1/2$. Assume that $e_i$, $1 \leq i \leq P$, has a Cauchy distribution with density $f(x) = (a_i/\pi)(a_i + x^2)^{-1}$, where $a_i$ is a positive number. Let $X_t = Y_{t+1} - Y_t$. If $m = 0$, then $X_t$ has a symmetric distribution about zero. Assume that we take $n+1$ consecutive observations of $Y_t$, so we have $n$ observations of $X_t$ after differencing as done above. The statistic $T_n$ is sensitive to a change in the values of $m$. Thus $T_n$ can be used to test $H_0$ versus $H_A$. Assumptions A1, A2, A3 are standard assumptions which are satisfied by suitable choices of the functions $J$, $g$. We now verify A4. Claim that there exists a constant $C > 0$ such that

$$C_1 < \frac{(1 - F_t(x))F_t(x)}{(1 - F_s(x))F_s(x)} < C_2, \tag{4.2}$$

for all $t$, $s$, and $x$. Note that

$$F_t(x) = P[X_t \leq x] = P[Z_{t+1} - Z_t + e_{t+1} - e_t + m \leq x].$$

Hence $X_t$ can take on at most $P$ distinct distributions. We only need to check (4.2) for pairs $(t, s)$ with $1 \leq t$, $s \leq P$. Without loss of generality, consider $s = 1$, $t = 2$. Clearly,

$$\begin{aligned} F_1(x) &= P\left[\frac{Z_2 - Z_1 + e_2 - e_1}{a_1 + a_2 + 1} \leq \frac{x - m}{a_1 + a_2 + 1}\right] \\ &= \frac{1}{\pi}[(\pi/2) + \text{Arctan}((x - m)/(a_1 + a_2 + 1))], \end{aligned}$$

and

$$F_2(x) = \frac{1}{\pi}[(\pi/2) + \text{Arctan}((x - m)/(a_2 + a_3 + 1))].$$

Using L'Hôpital's rule,

$$\begin{aligned} \lim_{x \to -\infty} \frac{(1 - F_1(x))F_1(x)}{(1 - F_2(x))F_2(x)} &= \lim_{x \to -\infty} F_1(x)/F_2(x) \\ &= \frac{(a_1 + a_2 + 1)[(a_2 + a_3 + 1)^2 + (x - m)^2]}{(a_2 + a_3 + 1)[(a_1 + a_2 + 1)^2 + (x - m)^2]} \\ &= (a_1 + a_2 + 1)/(a_2 + a_3 + 1). \end{aligned} \tag{4.3}$$

Similarly

$$\lim_{x \to \infty} \frac{(1 - F_1(x))F_1(x)}{(1 - F_2(x))F_2(x)} = (a_1 + a_2 + 1)/(a_2 + a_3 + 1). \tag{4.4}$$

Relations similar to (4.3) and (4.4) hold for all $(t, s)$ with $1 \leq t$, $s \leq P$. It is now clear that (4.2) holds for some constants $C_1$ and $C_2$. Assumption A4 can be easily verified using (4.2).

*Example* 2. Let $Y_t = Z_t + e_t$, $-\infty < t < \infty$, be the nonstationary time series of example 1. The only difference is that we assume that the model has been "detrended". We are interested in testing the null hypothesis that $Y_t$ is a series of white noises versus the alternatives that $Y_t$ is positively serially dependent at lag 1. One possibility is to reduce the hypothesis to the problem of testing whether the location parameter of a nonstationary time series is zero. Let $X_t = Y_{t+1}Y_t$ and assume that we have $n$ observations $X_1, \ldots, X_n$. Under the null hypothesis, $X_t$ is symmetrically distributed about zero. A test can be constructed by rejecting the null hypothesis for high values of $T_n$. If $Z_t$ is absolutely regular under alternatives, then $T_n$ is asymptotically normal under A1–A4.

## Acknowledgement

The authors would like to thank the referee for his valuable comments.

## References

Alexander, K. S. (1984). Probability inequalities for empirical processes and a law of the iterated logarithm, *Ann. Probab.*, **12**, 1041–1067.

Chan, N. H. and Tran, L. T. (1992). Nonparametric tests for serial dependence, *J. Time Ser. Anal.*, **13**, 19–28.

Denby, L. and Martin, R. D. (1979). Robust estimation of the first-order autoregressive parameter, *J. Amer. Statist. Assoc.*, **74**, 140–146.

Dufour, J. M. (1982). Rank tests for serial dependence, *J. Time Ser. Anal.*, **3**, 117–128.

Fama, E. (1965). The behavior of stock market prices, *Journal of Business*, **38**, 34–105.

Hallin, M., Inglebleek and Puri, M. L. (1985). Linear serial rank tests for randomness against ARMA alternatives, *Ann. Statist.*, **13**, 1156–1181.

Hallin, M., Inglebleek and Puri, M. L. (1987). Linear and quadratic rank tests for randomness against serial dependence, *J. Time Ser. Anal.*, **8**, 499–424.

Mandelbrot, B. (1967). The variation of some other speculative prices, *Journal of Business*, **36**, 393–413.

Mehra, K. L. and Rao, M. S. (1975). On functions of order statistics for mixing processes, *Ann. Statist.*, **3**, 874–883.

Pham, D. T. (1986). The mixing property of bilinear and generalized random coefficient autoregressive models, *Stochastic Process. Appl.*, **23**, 291–300.

Pham, D. T. and Tran, L. T. (1985). Some mixing properties of time series models, *Stochastic Process. Appl.*, **19**, 297–303.

Puri, M. L. and Tran, L. T. (1980). Empirical distribution functions and functions of order statistics for mixing random variables, *J. Multivariate Anal.*, **10**, 405–425.

Ruymgaart, F. H. (1973). Asymptotic theory of rank tests for independence, *Mathematical Centre Tracts*, **43**, Amsterdam.

Ruymgaart, F. H. and Zuijlen, M. C. A. Van (1977). Asymptotic normality for linear combinations of functions of order statistics in the non-i.i.d. case, *Indag. Math.*, **39**, 432–447.

Serfling, R. J. (1980). *Approximation Theorems of Mathematics*, Wiley, New York.

Tran, L. T. (1988). Rank order statistics for time series models, *Ann. Inst. Statist. Math.*, **40**, 247–260.

Volkonskii, V. A. and Rozanov, Yu A. (1959). Some limit theorems for random functions. I, *Theory Probab. Appl.*, **4**, 178–197.

Wellner, R. J. (1977*a*). A Glivenko-Cantelli theorem and strong laws of large numbers for functions of order statistics, *Ann. Statist.*, **5**, 573–480.

Wellner, R. J. (1977*b*). A law of iterated logarithm for functions of order statistics, *Ann. Statist.*, **5**, 481–494.

Yoshihara, K. (1978). Probability inequalities for sums of absolutely regular processes and their applications, *Z. Wahrsch. Verw. Gebiete*, **43**, 319–330.

Yoshihara, K. (1984). Density estimation for sample satisfying a certain absolute regularity condition, *J. Statist. Plann. Inference*, **9**, 19–32.

Zuijlen, M. C. A. Van (1976). Some properties of the empirical distribution function in the non-i.i.d. case, *Ann. Statist.*, **5**, 406–408.

Zuijlen, M. C. A. Van (1977). Empirical Distributions and Rank Statistics, *Mathematical Centre Tracts*, **79**, Amsterdam.

Zuijlen, M. C. A. Van (1978). Properties of the empirical distribution function for independent nonidentically distributed random variables, *Ann. Probab.*, **6**, 250–266.